Introduction to Tensor Ranks and Tensor Invariants

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Outline

What is a tensor?		**Tensors in the wild**		**Group actions on tensors**
Outer product	$\rightarrow$	Matrix multiplication	$\rightarrow$	Diagonal action
Tensor basis		Quantum entanglement		Permutation action
Restriction		Combinatorics		(Anti)symmetric tensors

Tensors as multidimensional arrays

Starting point — Matrices

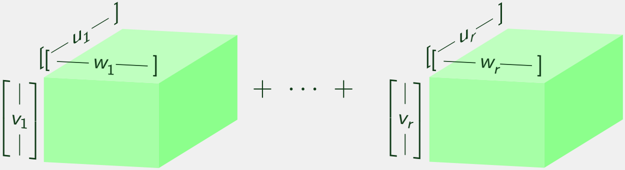
Let $\mathbb{F}$ be a field. We can write a matrix $M \in \mathbb{F}^{n\times m}$ as

$$M = \sum_{i=1}^{r} v_i \otimes w_i^{\top} = \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \begin{bmatrix} — w_1 — \end{bmatrix} + \cdots + \begin{bmatrix} | \\ v_r \\ | \end{bmatrix} \begin{bmatrix} — w_r — \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

for vectors $v_i \in \mathbb{F}^n, w_i \in \mathbb{F}^m$. *Examples:* $M = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{i,j}\, e_i e_j^{\top}$, SVD when $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Tensors as multidimensional arrays

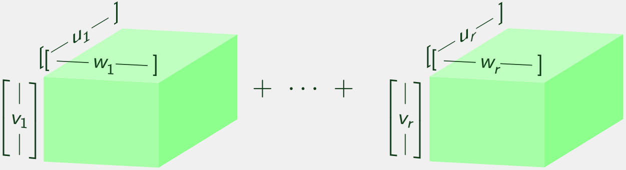
$$T = \sum_{i=1}^{r} v_i \otimes w_i \otimes u_i =$$



First examples — The W and diagonal tensors

Tensors as multidimensional arrays

$$T = \sum_{i=1}^{r} v_i \otimes w_i \otimes u_i =$$



$$\begin{aligned}
\mathrm{W} &:= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \\
&= \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \\
&= \left(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix} \right) + \left(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix} \right) + \left(\begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix} \right) \\
&= \left(\begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix} \right) \\
\langle 2 \rangle &:= e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 = \left(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right)
\end{aligned}$$

Abstract tensors

Let V, W, U be finite dimensional vector spaces with respective bases $\{v_i\}_i, \{w_j\}_j, \{u_k\}_k$.

Definition — Abstract 3-tensor space *(Straightforward to generalize to k-tensors)*

We define a **tensor vector space** $V \otimes W \otimes U$ as the linear span of the (abstract) elements

$$\{v_i \otimes w_j \otimes u_k\}_{i,j,k}$$

together with a map $V \times W \times U$: $(v, w, u) \mapsto v \otimes w \otimes u$ that is multilinear:

- **Multilinearity I:** $(v + v') \otimes w \otimes u = v \otimes w \otimes u + v' \otimes w \otimes u$
- **Multilinearity II:** $(\alpha v) \otimes w \otimes u = \alpha(v \otimes w \otimes u)$ for all $\alpha \in \mathbb{F}$.

and similarly for the other components.

It is easy to check the outer product satisfies this!

Kronecker product

You could also define:

Another example — Kronecker product

Given column vectors $v \in V$, $w \in W$. Define their **Kronecker product** by

$$v \boxtimes w = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \boxtimes w := \begin{bmatrix} a_1 w \\ \hline a_2 w \\ \hline \vdots \\ \hline a_n w \end{bmatrix} \in V \boxtimes W$$

i.e. replacing each entry of v with a scaled copy of w, resulting in one very tall vector.

This also sasisfies the abstract definition!

How to transform tensors — Linear operations

Take a 3-tensor $T = \sum_i v_i \otimes w_i \otimes u_i \ \in \ V \otimes W \otimes U$. *(note: not basis elements anymore)*

Let $A\colon V \to V'$, $B\colon W \to W'$, $C\colon U \to U'$ be linear maps.

Definition — Applying linear maps

Define $A \otimes B \otimes C : \ V \otimes W \otimes U \ \to \ V' \otimes W' \otimes U'$ by

$$(A \otimes B \otimes C)(v \otimes w \otimes u) := (Av) \otimes (Bw) \otimes (Cu)$$

$$(A \otimes B \otimes C)T := \sum_i Av_i \otimes Bw_i \otimes Cu_i$$

Example: $\langle 3 \rangle := \sum_{i=1}^3 e_i \otimes e_i \otimes e_i \ \in \ \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Then

$$\begin{aligned}
\left(\begin{bmatrix} 0&1&1\\1&0&0 \end{bmatrix} \otimes \begin{bmatrix} 1&0&1\\0&1&0 \end{bmatrix} \otimes \begin{bmatrix} 1&1&0\\0&0&1 \end{bmatrix} \right) \langle 3 \rangle &= \left(\begin{bmatrix} 0&1&1\\1&0&0 \end{bmatrix} \otimes \begin{bmatrix} 1&0&1\\0&1&0 \end{bmatrix} \otimes \begin{bmatrix} 1&1&0\\0&0&1 \end{bmatrix} \right)(e_1 \otimes e_1 \otimes e_1) \\
&+ \left(\begin{bmatrix} 0&1&1\\1&0&0 \end{bmatrix} \otimes \begin{bmatrix} 1&0&1\\0&1&0 \end{bmatrix} \otimes \begin{bmatrix} 1&1&0\\0&0&1 \end{bmatrix} \right)(e_2 \otimes e_2 \otimes e_2) \\
&+ \left(\begin{bmatrix} 0&1&1\\1&0&0 \end{bmatrix} \otimes \begin{bmatrix} 1&0&1\\0&1&0 \end{bmatrix} \otimes \begin{bmatrix} 1&1&0\\0&0&1 \end{bmatrix} \right)(e_3 \otimes e_3 \otimes e_3) \\
&= \begin{bmatrix} 0\\1 \end{bmatrix} \otimes \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 1\\0 \end{bmatrix} + \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 0\\1 \end{bmatrix} \otimes \begin{bmatrix} 1\\0 \end{bmatrix} + \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 1\\0 \end{bmatrix} \otimes \begin{bmatrix} 0\\1 \end{bmatrix} = \mathrm{W}
\end{aligned}$$

Restriction

Definition — Applying linear maps

Let $A\colon V \to V'$, $B\colon W \to W'$, $C\colon U \to U'$ be linear maps. Then

$$(A \otimes B \otimes C) \sum_i v_i \otimes w_i \otimes u_i = \sum_i Av_i \otimes Bw_i \otimes Cu_i$$

Take 3-tensors $T \in V \otimes W \otimes U$ and $S \in V' \otimes W' \otimes U'$

Definition — Restriction

We say T **restricts** to S, and write $T \geq S$, whenever there exists linear maps A, B, C such that

$$(A \otimes B \otimes C)T = S$$

Example: the previous example shows $\langle 3 \rangle \geq \mathrm{W}$.

Remark: Restriction on matrices (2-tensors) is left-right multiplication, since

$$(A \otimes B)(v \otimes w) = Av \otimes Bw = Av(Bw)^\top = A(vw^\top)B^\top.$$

Matrix multiplication and Bilinear maps

$$\mathrm{MM}_n \colon \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$$

Standard algorithm: $\mathcal{O}(n^3)$. **Best bounds:** $\mathcal{O}(n^\omega)$ with $\omega \in [2,\ 2.371552]$.

Central question — Matrix multiplication

How many multiplications (between inputs) are needed to do $n \times n$ matrix multiplication?

Consider bilinear maps $V \times W \to U$, with $\{v_i\}_i$, $\{w_j\}_j$ and $\{u_k\}_k$ bases. **Claim:**

Proposition — Bilinear map/Tensor equivalence

$$\{\text{bilinear maps } V \times W \to U\} \xrightarrow{\sim} V^* \otimes W^* \otimes U : \quad f \mapsto \sum_{i,j,k} t_{i,j,k}\ v_i^* \otimes w_j^* \otimes u_k$$

- Bilinearity gives $f(v,w) = f\Big(\sum_i (v_i^* v)\, v_i, \sum_j (w_j^* w)\, w_i\Big) = \sum_{i,j} (v_i^* v)(w_j^* w)\, f(v_i, w_i)$
- Then we write $f(v_i, w_i) = \sum_k \Big(\underbrace{u_k^* f(v_i, w_i)}_{=:\ t_{i,j,k} \in \mathbb{F}} \Big)\, u_k \quad \Longrightarrow \quad f(v,w) = \sum_{i,j,k} t_{i,j,k} (v_i^* v)(w_j^* w)\, u_k$

Matrix multiplication as a tensor

$$\mathrm{MM}_n \in (\mathbb{F}^{n\times n})^* \otimes (\mathbb{F}^{n\times n})^* \otimes \mathbb{F}^{n\times n}$$

Take double indices $(i,i'), (j,j'), (k,k')$,

and the standard matrix basis $E_{i,i'} := e_i e_{i'}^\top$.

$$\begin{array}{c|c|c|c} & & i' & \\ \hline & & & \\ \hline & & & \\ \hline i & & 1 & \end{array} = E_{3,2}$$

$$\mathrm{MM}_n(E_{i,i'}, E_{j,j'}) = E_{i,i'} E_{j,j'} = e_i (e_{i'}^\top e_j) e_{j'}^\top = \begin{cases} E_{i,j'} & \text{if } i' = j \\ 0 & \text{else} \end{cases}$$

Example $(n=2)$: $\mathrm{MM}_2\left(\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] = \mathrm{MM}_2\left(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\right)$

So

$$t_{(i,i'),(j,j'),(k,k')} := E^*_{k,k'}\Big(\mathrm{MM}_n(E_{i,i'}, E_{j,j'})\Big)$$

$$= \begin{cases} 1 & \text{if } i = k,\ i' = j,\ j' = k' \\ 0 & \text{else} \end{cases}$$

(k, k') slice

$$\left(\begin{array}{ll} (1,1): & \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \\ (1,2): & \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \\ (2,1): & \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}, \\ (2,2): & \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \end{array}\right)$$

Bilinear complexity

Question: How many multiplications do we need? $\iff$ What is the tensor rank of MM_n?
Idea: Compare with a bilinear map for which we know.

Define the **diagonal bilinear map / tensor** as

$$f_r(x,y) := \begin{bmatrix} x_1 y_1 \\ \vdots \\ x_r y_r \end{bmatrix} = \sum_{i=1}^{r} x_i y_i e_i \qquad \longleftrightarrow \qquad \langle r \rangle := \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \quad \in \; \mathbb{F}^r \otimes \mathbb{F}^r \otimes \mathbb{F}^r$$

Fact: if we have a restriction $\mathrm{MM}_n \leq \langle r \rangle$, then MM_n needs $\leq r$ multiplications.

Definition — Tensor rank

Given a 3-tensor T, we define its **(tensor) rank** as

$$\mathrm{R}(T) := \min\{ r \mid T \leq \langle r \rangle \},$$

i.e. the size of the smallest diagonal tensor that restricts to T.

Outlook — Tensor rank and matrix multiplication

Definition — Tensor rank

Given a 3-tensor T, we define its **(tensor) rank** as

$$R(T) := \min\{\, r \mid T \leq \langle r \rangle \}$$

Central question — Tensor rank of matrix multiplication

What is $R(MM_n)$?

Example — Naive MM_2 and [Strassen 1969]

Naive algorithm: $MM_2 \leq \langle 8 \rangle$ **Strassen:** $R(MM_2) = 7$

This is just the beginning of the story. In this seminar we will/might see:

- A session on tensor rank
- A session on border bank
- Asymptotic aspects
- *Student topic:* Schönhage's τ-theorem

Quantum states

Definition — Quantum multipartite systems and states

- We define a **(single-partite) quantum system** as a Hilbert space $\mathbb{C}^n$.
- We define a **multi-partite quantum systems** as the tensor product of such systems. E.g. a quantum system with three parties is given by $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$.
- We define a **quantum state** as an element T of a quantum system with $\|T\|_2 = 1$.

Example — Three qubits

A **qubit** is the system $\mathbb{C}^2$. Examples of states: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $|+\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (superposition).
Three parties can each have a qubit. Their shared system is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Examples of states:

$$|+\rangle \otimes |+\rangle \otimes |+\rangle \qquad \frac{\langle 2 \rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2)$$

$$\frac{W}{\sqrt{3}} = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1)$$

Quantum states — Intuition

Example — Three qubits

A **qubit** is the system $\mathbb{C}^2$. Examples of states: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ and $|+\rangle := \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ (superposition).

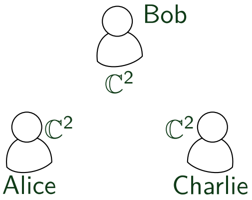
Three parties can each have a qubit. Their shared system is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Examples of states:

$$|+\rangle \otimes |+\rangle \otimes |+\rangle \qquad \frac{\langle 2\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2)$$

$$\frac{W}{\sqrt{3}} = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1)$$

Intuition:

- Three parties Alice, Bob and Charlie share state $\langle 2\rangle / \sqrt{2}$. They can "interact" only with their qubit.
- Alice "measures": the state collapses to outcome e_1 or e_2.
- If Alice outcomes is e_1. Then Bob's and Charlie's qubits are now in state e_1 too. This phenomenon is entanglement.



Entanglement

Takeaway — Quantum entanglement

Entanglement in quantum systems is modelled by tensors over $\mathbb{C}$.

- Entanglement is a vital resource for many quantum computing applications.
- Different types are possible. Example: $\langle 2\rangle/\sqrt{2}$ and $\mathrm{W}/\sqrt{3}$.

Central question — Quantum entanglement

Can we classify the different types of entanglement?
Can we classify the equivalence classes and their relations under restriction?

- **Intuition:** Entanglement cannot increase under local operations.
- In its most general form, this is restriction.
- *Physical interpretation: SLOCC (Stochastic Local Operations and Classical Communication) transformations.*

Definition — Equivalence classes under restriction

We write $T \sim S$ whenever there are restrictions $T \geq S$ and $T \leq S$.

Example — The W state and the diagonal state of order 3

- We know $\langle 3 \rangle \geq \mathrm{W}$. **Claim:** $\langle 3 \rangle \not\sim \mathrm{W}$, as $\langle 3 \rangle \not\leq \mathrm{W}$. We use a restriction monotone.
- We say: $\langle 3 \rangle / \sqrt{3}$ contains strictly more entanglement than $\mathrm{W}/\sqrt{3}$.

Definition — Restriction monotone

We say a function $f\colon \{3\text{-tensors}\} \to \mathbb{R}$ is **monotone** when $S \leq T \implies f(S) \leq f(T)$.

Theorem — Flattening ranks

Given $T \in V \otimes W \otimes U$ we can consider T as a matrix $M_T \in V \otimes (W \otimes U)$, and compute matrix rank. We call this the 1st **flattening rank** R_1. Then $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ are restriction monotones.

Proof: Restriction $(A \otimes B \otimes C)T$ becomes left-right matrix multiplication $(A)M_T(B \boxtimes C)^*$. □

$$\mathrm{R}_1(\mathrm{W}) = \mathrm{rank}\big(e_1 \otimes (e_1 \boxtimes e_2) + e_1 \otimes (e_2 \boxtimes e_1) + e_2 \otimes (e_1 \boxtimes e_1)\big)$$

$$= \mathrm{rank}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \mathrm{rank} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 2$$

$$\mathrm{R}_1(\langle 3 \rangle) = \mathrm{rank} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 3$$

Example — The W state and the diagonal state of order 2

- **Claim:** $\langle 2 \rangle \not\sim \mathrm{W}$. **In fact:** $\langle 2 \rangle \not\leq \mathrm{W}$ and $\langle 2 \rangle \not\geq \mathrm{W}$. We will use an invariant.
- Both tensors live in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$: equivalence implies restriction with *invertible* matrices.
- Thus: $\langle 2 \rangle / \sqrt{2}$ has a genuinly different type of entanglement than $\mathrm{W}/\sqrt{3}$.

Definition — Restriction semi-invariant

We say a function $f \colon V \otimes W \otimes U \to \mathbb{R}$ is an **semi-invariant** when $f(T) = 0 \iff f((A \otimes B \otimes C)T) = 0$ for all invertible $(A, B, C) \in \mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{GL}(U)$.

Proposition — Hyperdeterminant/3-tangle

There exists an semi-invariant f for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with $f(\langle 2 \rangle) \neq 0 = f(\mathrm{W})$. It is called the **hyperdeterminant** or **3-tangle**.

Proof. We might see this as part of a student topic :)

Outlook — Quantum entanglement, monotones and invariants

Central question — Quantum entanglement

Can we classify equivalence under restriction, and determine (non-)existence of restrictions?

Definition — Restriction monotone

We say a function $f\colon \{\text{3-tensors}\} \to \mathbb{R}$ is **monotone** when $S \leq T \implies f(S) \leq f(T)$.

Definition — Restriction semi-invariant

We say a function $f\colon V \otimes W \otimes U \to \mathbb{R}$ is an **semi-invariant** when $f\big((A \otimes B \otimes C)T\big) = 0 \iff f(T) = 0$ for all invertible $(A, B, C) \in \mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{GL}(U)$.

Again just the beginning of the story. In this seminar we will/might see:

- Schur–Weyl duality, covariants
- The quantum functionals
- More monotones, (semi-)invariants
- *Student topic:* classification of classes in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

The cap set problem

Definition — Cap sets

Let $\mathbb{F} = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

A **3-term progression** in $\mathbb{F}^n$ is a sequence $(a, a+b, a+2b) \in \mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}^n$.

$\mathcal{A} \subset \mathbb{F}^n$ is called a **cap set** when no 3 distinct elements of $\mathcal{A}$ form a 3-term progression.

Example $(n = 2)$: $\mathcal{A} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a cap set, $\mathcal{A}' = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is not.

Motivating problem — Maximum size of cap sets

What is the maximum size of a cap set in terms of n?

Or: does there exists a $C < 3$ such that the size is $\mathcal{O}(C^n)$?

- A bound $\mathcal{O}(3^n/n)$ was known since 1995, by Alon and Dubiner.
- Whether an exponential improvement over 3^n was possible became a big open problem.
- Settled with 2.756^n in 2016 by Ellenberg & Gijswijt, based on work by Croot, Lev & Pach.
- We can reformulate this result in terms of tensors!

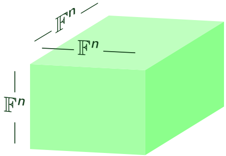
The cap set tensor

Definition — The cap set tensor *(or rather: the 3-term progression tensor)*

Let $a \in \mathbb{F}^n = \mathbb{F}_3^n$ label standard basis elements $e_a \in \mathbb{F}^{3^n}$. We define the **cap set tensor** as

$$T_{\text{capset},n} := \sum_{\substack{a,b,c \in \mathbb{F}^n \\ (a,b,c) \text{ a 3-term progression}}} e_a \otimes e_b \otimes e_c \in \mathbb{F}^{3^n} \otimes \mathbb{F}^{3^n} \otimes \mathbb{F}^{3^n}$$

Intuition: The cap set tensor encodes all 3-term progressions.



$$T_{\text{capset},1} := \langle 3 \rangle + \sum_{(i,j,k) \text{ a permutation of } (0,1,2)} e_i \otimes e_j \otimes e_k$$

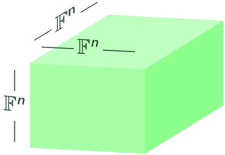
$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Subrank

Claim: A cap set $\mathcal{A} = \{a_1, \ldots, a_m\} \subset \mathbb{F}^n$ gives rise to a restriction $T_{\text{capset},n} \geq \langle m \rangle$.

Intuition: The cap set tensor encodes all 3-term progressions.

Restricting $T_{\text{capset},n}$ to indices $a, b, c \in \mathcal{A} \subset \mathbb{F}^n$ gives 1 if and only if $a = b = c$.



Example: $\mathcal{A} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$(A \otimes A \otimes A) T_{\text{capset},2} = \langle 3 \rangle$$

$$\begin{matrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Definition — Subrank

Given a 3-tensor T, we define its **subrank** as

$$\mathrm{Q}(T) := \max\{ q \mid \langle q \rangle \leq T \},$$

Central question — Subrank of $T_{\text{capset},n}$

What is $\mathrm{Q}(T_{\text{capset},n})$?

Outlook — Combinatorics

Definition — Subrank

Given a 3-tensor T, we define its **subrank** as

$$Q(T) := \max\{ q \mid \langle q \rangle \leq T \},$$

Central question — Subrank of $T_{\text{capset},n}$

What is $Q(T_{\text{capset},n})$?

- The maximum size of a cap set in $\mathbb{F}^n$ is bounded by

$$Q(T_{\text{capset},n}) \leq \text{slicerank}(T_{\text{capset},n}) \approx 2.756^n$$

- Originally proven via an equivalent formulation using polynomials $\mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$.
- There are many more problems! Other fields than $\mathbb{F}_3$, other arithmetic progressions, etc.

Once again again the beginning of the story. In this seminar we will/might see:

- A session on subrank
- More upper bounds for subrank
- A general asymptotic formulation
- *Student topic:* slice rank

Rank and subrank

Definition — Subrank

Given a 3-tensor T, we define its **subrank** as

$$Q(T) := \max\{\, q \mid \langle q \rangle \leq T \},$$

Definition — Rank

Given a 3-tensor T, we define its **rank** as

$$R(T) := \min\{\, r \mid T \leq \langle r \rangle \}$$

- $Q(T) \leq R(T)$.
  Proof: Use a flattening rank to show $\langle q \rangle \not\leq \langle r \rangle$ if $q > r$. □
- $Q(T) \neq R(T)$, since $\langle 1 \rangle \leq W \leq \langle 3 \rangle$ is the best we can do.
  Proof idea: Use the hyperdeterminant to show $W \not\leq \langle 2 \rangle$ and $\langle 2 \rangle \not\leq W$.
- For matrices, $Q(M) = \mathrm{rank}(M) = R(M)$!
  Proof: Restriction with $(A \otimes B)$ is left-right multiplication AMB^*. Set $r := \mathrm{rank}(M)$. Use Gaussian elimination to map M to I_r. Use $M = \sum_{i=1}^{r} v_i \otimes w_i$ to map I_r to M. □

Takeaway — The tensor world

The tensor world is a lot more complicated & interesting than the matrix world!
We use ranks (rank, subrank, slice rank, . . .), monotones, invariants, etc.

Group actions

Recall the definition of invariants. $\mathrm{GL}(V)\times\mathrm{GL}(W)\times\mathrm{GL}(U)$ is a group $\to$ representation theory!

Definition — Restriction semi-invariant

We say a function $f\colon V\otimes W\otimes U\to\mathbb{R}$ is an **semi-invariant** when $f\big((A\otimes B\otimes C)T\big)=0 \iff f(T)=0$ for all invertible $(A,B,C)\in\mathrm{GL}(V)\times\mathrm{GL}(W)\times\mathrm{GL}(U)$.

Next week: *Schur–Weyl duality*. Two group representations will be essential:

Definition — The diagonal action

Let $T\in V^{\otimes n}$. Then $g\in\mathrm{GL}(V)$ acts on T as

$$g\cdot T=(\underbrace{g\otimes\cdots\otimes g}_{n\text{ times}})T$$

Definition — The permutation action

Let $T\in V^{\otimes n}$. Then $\pi\in S_n$ acts on T by permuting the tensor factors. So as

$$\pi\cdot T=\sum_i v_{i,\pi^{-1}(1)}\otimes\cdots\otimes v_{i,\pi^{-1}(n)}$$

where $V^{\otimes n}:=\underbrace{V\otimes\cdots\otimes V}_{n\text{ times}}$ and $T=\sum_i v_{i,1}\otimes\cdots\otimes v_{i,n}$

Symmetric tensors

Definition — Symmetric tensors

We call a tensor $T \in V^{\otimes n}$ **symmetric** when $\pi \cdot T = T$ for all $\pi \in S_n$.

Most tensors are not symmetric, e.g. $e_1 \otimes e_1 \otimes e_2$, as applying (13) gives $e_2 \otimes e_1 \otimes e_1$.

Examples:

$$v \otimes \cdots \otimes v \qquad \langle r \rangle := \sum_{i=1}^{r} e_i \otimes e_i \otimes \cdots \otimes e_i$$

$$T_{\text{capset},1} := \sum_{\substack{a,b,c \in \mathbb{F}_3 \\ (a,b,c) \text{ a 3-term progression}}} e_a \otimes e_b \otimes e_c = \langle 3 \rangle + \sum_{\pi \in S_3} \pi \cdot (e_1 \otimes e_2 \otimes e_3)$$

Definition — Symmetrization

Given $T \in V^{\otimes n}$, define its **symmetrization** as $\frac{1}{n!} \sum_{\pi \in S_n} \pi \cdot T$.

Facts:
- The set of symmetric tensors in $V^{\otimes n}$ form a vector space.
- Symmetrization acts as a linear projector onto this subspace.
- The diagonal action of $\mathrm{GL}(V)$ leaves this subspace invariant.

Antisymmetric tensors

Definition — Antisymmetric tensors

We call a tensor $T \in V^{\otimes n}$ **antisymmetric** when $\pi \cdot T = \text{sgn}(\pi) T$ for all $\pi \in S_n$.

Examples: $e_1 \otimes e_2 - e_2 \otimes e_1$

$$e_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_3 \otimes e_1 - e_2 \otimes e_1 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 - e_3 \otimes e_2 \otimes e_1$$

Definition — Antisymmetrization & wedge product

Given $T \in V^{\otimes n}$, define its **antisymmetrization** as $\frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi)\ \pi \cdot T$.

Given $v_1, \ldots, v_n \in V$, define their **wedge product** as

$$v_1 \wedge \cdots \wedge v_n := \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi)\ \pi \cdot (v_1 \otimes \cdots \otimes v_n) \ \in \ V^{\otimes n}$$

Facts:
- The set of antisymmetric tensors in $V^{\otimes n}$ form a vector space.
- Antisymmetrization acts as a linear projector onto this subspace.
- $v_1 \wedge \cdots \wedge v_n = 0 \iff \{v_1, \ldots, v_n\}$ are linearly dependent. *(hint: consider first $v_i = v_j$)*

Slides will be available at the webpage: `qi.rub.de/tensors_ss24`.

That's it for today. Thanks!