Tensor rank and substitution method

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Plan

- Tensors, restrictions and tensor rank
- Direct sums and Kronecker products of tensors
- Properties of tensor rank. Asymptotic rank
- Lower bounds. Substitution method
We consider tensor products of finite-dimensional vector spaces.
Tensors: abstract definition

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- For order three tensors in $U \otimes V \otimes W$:
- We have a trilinear map

$$u \in U, v \in V, w \in W \mapsto u \otimes v \otimes w \in U \otimes V \otimes W$$
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There is a bijective correspondence:

trilinear $F : U \times V \times W \to X$ $\iff$ linear $L : U \otimes V \otimes W \to X$

$L(u \otimes v \otimes w) = F(u, v, w)$
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  trilinear $F: U \times V \times W \to X \leftrightarrow$ linear $L: U \otimes V \otimes W \to X$

  $$L(u \otimes v \otimes w) = F(u, v, w)$$

  $$L \left( \sum_i u_i \otimes v_i \otimes w_i \right) = \sum_i F(u_i, v_i, w_i)$$
Let \((u_1, \ldots, u_\ell), (v_1, \ldots, v_m), (w_1, \ldots, w_n)\) be bases of \(U, V, W\).

Then \(u_i \otimes v_j \otimes w_k\) form a basis of \(U \otimes V \otimes W\).
Tensors: concrete representation

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- Every tensor \(T \in U \otimes V \otimes W\) decomposes as

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  \]
- An order three tensor \(T\) is given by a three-way array \((t_{ijk})\)
Let $A: U \to U'$, $B: V \to V'$, $C: W \to W'$ be linear maps.

Then we have a linear map

$$(A \otimes B \otimes C): U \otimes V \otimes W \to U' \otimes V' \otimes W'$$

defined by the identity

$$(A \otimes B \otimes C)(u \otimes v \otimes w) = (Au \otimes Bv \otimes Cw)$$
Definition (Restriction preorder)

$T$ is a restriction of $S$ if $T = (A \otimes B \otimes C)S$ for some linear maps $A, B, C$

Notation: $T \leq S$

Definition (Equivalence of tensors)

Tensors $T$ and $S$ are equivalent if $T \leq S$ and $S \leq T$. 
Definition (Diagonal tensor)

\[ I_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \in \mathbb{F}^r \otimes \mathbb{F}^r \otimes \mathbb{F}^r \]

Definition (Tensor rank)

\[ R(T) = \min \{ r \mid T \leq I_r \} \]
\[ I_3 = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3 \]
\[ W = e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0 \]
Example

\[ l_3 = |1\rangle \otimes |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle \otimes |2\rangle + |3\rangle \otimes |3\rangle \otimes |3\rangle \]

\[ W = |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle \]
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\[ R(W) \leq 3 \]

\[ W = \left( \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right) \cdot I_3 \]
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**Definition**

A decomposition of the form

\[ T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \]

is called a *rank decomposition* of \( T \).

**Theorem**

\( R(T) \) is the minimal number of summands in a rank decomposition of \( T \).

- \( R(T) \leq r \iff T \leq I_r \)
Rank decompositions

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Theorem

\( R(T) \) is the minimal number of summands in a rank decomposition of \( T \).

\[ R(T) \leq r \iff T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \text{ with } u_i = A |i\rangle, v_i = B |i\rangle, w_i = C |i\rangle \]
Example

\[ A_3 = |1\rangle \wedge |2\rangle \wedge |3\rangle = \sum_{\pi \in S_3} (-1)^{\sigma} |\pi(1)\rangle \otimes |\pi(2)\rangle \otimes |\pi(3)\rangle \]

\[ R(A_3) \leq 5 \]
A_3 = |1⟩ \land |2⟩ \land |3⟩ = \sum_{\pi \in \mathcal{S}_3} (-1^\sigma) |\pi(1)⟩ \otimes |\pi(2)⟩ \otimes |\pi(3)⟩ \\
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Properties of tensor rank

\[ T \leq S \Rightarrow R(T) \leq R(S) \]
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\[ R(I_r) = r \]

- I.O.U a proof
Direct sum of tensors

Let $U_1$, $V_1$, $W_1$ and $U_2$, $V_2$, $W_2$ be vector spaces

The injections $U_1 \hookrightarrow U_1 \oplus U_2$, $V_1 \hookrightarrow V_1 \oplus V_2$, $W_1 \hookrightarrow W_1 \oplus W_2$
give an injection $U_1 \otimes V_1 \otimes W_1 \hookrightarrow (U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$

Same for $U_2 \otimes V_2 \otimes W_2$

Definition (Direct sum)

For $T_1 \in U_1 \otimes V_1 \otimes W_1$ and $T_2 \in U_2 \otimes V_2 \otimes W_2$ their direct sum is the sum of their embeddings in $(U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$
Direct sum: example

- The space $(U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$ decomposes into 8 "blocks" $U_i \otimes V_j \otimes W_k$.
- Direct sums use "diagonal blocks"
Direct sum: diagonal tensors

- We have seen this diagonal placement before

![Diagram of $I_3$]

- Note $F^a \oplus F^b \cong F^{a+b}$

- Using this isomorphism on all three factors, we get

  $$I_a \oplus I_b \sim I_{a+b}$$

- This gives an alternative definition of diagonal tensors

  $$I_1 = 1 \otimes 1 \otimes 1 \in F \otimes F \otimes F; \quad I_a = I_1^{\oplus a}$$
\[
((A_1 \oplus A_2) \otimes (B_1 \oplus B_2) \otimes (C_1 \oplus C_2))(T_1 \oplus T_2) = \\
= [(A_1 \otimes B_1 \otimes C_1)T_1] \oplus [(A_2 \otimes B_2 \otimes C_2)T_2]
\]
Direct sum and restrictions

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\[
\begin{cases}
T_1 \leq S_1 \\
T_2 \leq S_2
\end{cases} \Rightarrow T_1 \oplus T_2 \leq S_1 \oplus S_2
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\[R(T_1 \oplus T_2) \leq R(T_1) + R(T_2)\]
Strassen conjectured that $R(T_1 \oplus T_2) = R(T_1) + R(T_2)$

The conjecture was disproven in 2017 by Shitov
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The proof uses generic tensors of a special form

No explicit pair of tensors with $R(T_1 \oplus T_2) < R(T_1) + R(T_2)$ is known
Tensor product and Kronecker product

- It will be useful for us to introduce two different tensor products $U \otimes V$ and $U \boxtimes V$
- Intuitively, we think of $U \otimes V \otimes \ldots$ as matrices and tensors, and $U \boxtimes V \boxtimes \ldots$ as “long vectors” composed of other vectors

\[
(u_1, u_2, u_3) \otimes (v_1, v_2) = 
\begin{bmatrix}
  u_1 v_1 & u_1 v_2 \\
  u_2 v_1 & u_2 v_2 \\
  u_3 v_1 & u_3 v_2
\end{bmatrix}
\]

\[
(u_1, u_2, u_3) \boxtimes (v_1, v_2) = (u_1 v | u_2 v | u_3 v) =

= (u_1 v_1, u_1 v_2, u_2 v_1, u_2 v_2, u_3 v_1, u_3 v_2)
\]

- Of course, $U \otimes V$ and $U \boxtimes V$ are isomorphic as vector spaces and tensor products, the difference is purely syntactic convenience
Definition (Kronecker product)

For $T_1 \in U_1 \otimes V_1 \otimes W_1$ and $T_2 \in U_2 \otimes V_2 \otimes W_2$ we define the **Kronecker product** $T_1 \boxtimes T_2 \in (U_1 \boxtimes U_2) \otimes (V_1 \boxtimes V_2) \otimes (W_1 \boxtimes W_2)$ as a bilinear function of $T_1$ and $T_2$ satisfying

$$(u_1 \otimes v_1 \otimes w_1) \boxtimes (u_2 \otimes v_2 \otimes w_2) = (u_1 \boxtimes u_2) \otimes (v_1 \boxtimes v_2) \otimes (w_1 \boxtimes w_2)$$

- Let $T = (t_{ijk})$. Then $T \boxtimes S = \sum_{i,j,k} (|i\rangle \otimes |j\rangle \otimes |k\rangle) \boxtimes (t_{ijk}S)$
- $T \boxtimes S$ has the “outer structure” of $T$, but instead of scalars, we have scalar multiples of $S$ as blocks
Kronecker product: example

\[ W = |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle \]

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Matrix multiplication tensors

\[ M_{abc} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} |ij\rangle \otimes |jk\rangle \otimes |ik\rangle \]
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\]

\[
M_{a11} \sim \sum_{i=1}^{a} |i\rangle \otimes 1 \otimes |i\rangle \in F^a \otimes F \otimes F^a
\]

\[
M_{1b1} \sim \sum_{j=1}^{b} |j\rangle \otimes |j\rangle \otimes 1 \in F^b \otimes F^b \otimes F
\]

\[
M_{11c} \sim \sum_{k=1}^{c} 1 \otimes |k\rangle \otimes |k\rangle \in F \otimes F^c \otimes F^c
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\[ M_{abc} \sim M_{a11} \boxtimes M_{1b1} \boxtimes M_{11c} \]
Kronecker product and restrictions

\[ I_a \boxtimes I_b = I_{ab} \]

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((A_1 \boxtimes A_2) \otimes (B_1 \boxtimes B_2) \otimes (C_1 \boxtimes C_2)) \cdot (T_1 \boxtimes T_2) = \\
= ((A_1 \otimes B_1 \otimes C_1) T_1) \boxtimes ((A_2 \otimes B_2 \otimes C_2) T_2)
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\[
\left\{ \begin{array}{l}
T_1 \leq S_1 \\
T_2 \leq S_2
\end{array} \right. \Rightarrow T_1 \boxtimes T_2 \leq S_1 \boxtimes S_2
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\[ R(T_1 \boxtimes T_2) \leq R(T_1)R(T_2) \]
$R(W \boxtimes W) \leq 7$

$W \boxtimes W =$
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Properties of rank

Monotonicity: \( T \leq S \Rightarrow R(T) \leq R(S) \)

Subadditivity: \( R(T \oplus S) \leq R(T) + R(S) \)

Submultiplicativity: \( R(T \boxtimes S) \leq R(T) \cdot R(S) \)

Normalization: \( R(I_r) = r \)
Asymptotic behaviour of rank

- Denote $\rho_n = \log R(T \boxtimes n)$
- From the properties of rank it follows that $\rho$ is subadditive

$$\rho_{n+m} \leq \rho_n + \rho_m$$

- Fekete’s lemma: $\frac{\rho_n}{n}$ converges

**Definition**

Asymptotic rank of $T$ is defined as

$$R(T) = \lim_{n \to \infty} \left( R(T \boxtimes n) \right)^{\frac{1}{n}}$$

- We have $R(T \boxtimes n) \sim R(T)^n + o(n)$
As for the diagonal tensors, we have

\[ M_{a11} \boxtimes M_{a'11} = M_{aa',1,1} \]

And the same for \( M_{1b1} \) and \( M_{11b} \)
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It follows that

\[ M_{abc} \boxtimes M_{a'b'c'} = M_{aa',bb',cc'} \]

This property has interpretation for matrix multiplication maps: block matrices can be multiplied blockwise
The question about asymptotic complexity of matrix multiplication can be stated in terms of tensor rank

$$R(M_{n\times n}) = n^{\omega + o(1)}$$
The question about asymptotic complexity of matrix multiplication can be stated in terms of tensor rank

\[ R(M_{nnn}) = n^{\omega + o(1)} \]

Note that \( M_{aaa} \otimes M_{bbb} = M_{ab,ab,ab} \)

\[ R(M_{222}) = \lim_n \sqrt[n]{R(M_{2n,2n,2n})} = \lim_n \sqrt[n]{2^{\omega n + o(n)}} = 2^\omega \]
In applications, tensor rank provides a measure of complexity for objects represented by tensors.
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Algebraic Computation: complexity of computing multilinear maps.

Upper bounds $\approx$ “algorithms”, lower bounds $\approx$ “hardness proofs”.

Restrictions $\approx$ “reductions between problems”.

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Algebraic Computation: complexity of computing multilinear maps.

Upper bounds $\approx$ “algorithms”, lower bounds $\approx$ “hardness proofs”.

Restrictions $\approx$ “reductions between problems”.

Computing tensor rank is hard; the exact value is known only for small or very restricted tensors.

Have some construction for upper bounds / explicit restrictions.

The situation with lower bounds is much worse.
Tensor rank and ranks of tensors

- There are other notions of rank for tensors
- Subrank, slice rank, flattening rank . . .
- What are the common properties of these ranks?
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**Definition**

\[ F : \{\text{tensors}\} \rightarrow \mathbb{R}_+ \] is a *rank functional* if it satisfies

- **Monotonicity:** \[ T \leq S \Rightarrow F(T) \leq F(S) \]
- **Normalization:** \[ F(I_r) = r \]
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**Theorem**

Tensor rank dominates every rank functional

- *Proof:* \( R(T) = r \)
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- What are the common properties of these ranks?

Definition

\( F : \{\text{tensors}\} \rightarrow \mathbb{R}_+ \) is a rank functional if it satisfies

- Monotonicity: \( T \leq S \Rightarrow F(T) \leq F(S) \)
- Normalization: \( F(I_r) = r \)

Theorem

Tensor rank dominates every rank functional

- \textit{Proof: } \( R(T) = r \Rightarrow T \leq I_r \Rightarrow F(T) \leq F(I_r) = r \) for all ranks \( F \).
Tensor rank and ranks of tensors

- There are other notions of rank for tensors
- Subrank, slice rank, flattening rank ...
- What are the common properties of these ranks?

**Definition**

\( F : \{ \text{tensors} \} \rightarrow \mathbb{R}^+ \) is a *rank functional* if it satisfies

- **Monotonicity:** \( T \leq S \implies F(T) \leq F(S) \)
- **Normalization:** \( F(I_r) = r \)

**Proposition**

If there exists some rank functional, then \( R(I_r) = r \)

- **Proof:** \( R(I_r) \geq F(I_r) = r \). It is obvious that \( R(I_r) \leq r \).
Flattening and flattening rank

- Flattening is a way to transform a tensor into a matrix.
- Flattening with respect to the 1st factor:

\[
T \in U \otimes V \otimes W \quad \mapsto \quad F_1(T) \in U \otimes (V \boxtimes W)
\]

\[
u \otimes v \otimes w \quad \mapsto \quad u \otimes (v \boxtimes w)
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Definition (Flattening rank)

Flattening rank of \( T \) is the rank of the flattening

\[ R_1(T) = \text{rk} F_1(T) \]

Proposition

Flattening rank is a rank functional

\[ T \leq S \]
**Flattening and flattening rank**

- **Flattening with respect to the 1st factor**

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Flattening and flattening rank

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- \( T \leq S \Rightarrow T = (A \otimes B \otimes C)S \Rightarrow \mathcal{F}_1(T) = (A \otimes (B \boxtimes C))\mathcal{F}_1(S) \)
  
  Or, in terms of matrix multiplication, \( \mathcal{F}_1(T) = A \cdot \mathcal{F}_1(S) \cdot (B \boxtimes C)^\top \)
Flattening and flattening rank

- Flattening with respect to the 1st factor

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*Flattening rank* of \( T \) is the rank of the flattening

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Proposition

Flattening rank is a rank functional

- \( \mathcal{F}_1(1_r) = \sum_{i=1}^{r} |i\rangle \otimes |ii\rangle \)
Theorem

Tensor rank dominates every rank functional
Theorem

Every rank functional is a lower bound for tensor rank

- Lower bound problem solved?
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- Not really: all known rank functionals
  - Either are very similar to tensor rank
  - Or give weak lower bounds
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- Known lower bound methods
  - Continuous methods
  - Substitution method
  - Coding theory methods over finite fields
### Definition (Conciseness)

A tensor $T \in U \otimes V \otimes W$ is *concise in the 1st factor* (or 1-concise) if

$$R_1(T) = \dim U$$
**Concise tensors**

**Definition (Conciseness)**

A tensor $T \in U \otimes V \otimes W$ is **concise in the 1st factor** (or 1-concise) if

$$R_1(T) = \dim U$$

**Proposition**

If $T = (A \otimes B \otimes C)S$ and $T$ is 1-concise, then $A$ is surjective.

**Proof:** As matrices $F_1(T) = A \cdot F_1(S) \cdot (B \boxtimes C)^\top$

Therefore $\text{rk } A \geq \text{rk } F_1(T) = \dim U$

**Corollary**

If $T = \sum_i u_i \otimes v_i \otimes w_i$ and $T$ is 1-concise, then \{u_i\} generates $U$. 

Theorem (Substitution method)

Let $T \in U \otimes V \otimes W$ be a 1-concise tensor, and $X \subset U$. Then there exists a projection $\Pi: U \rightarrow X$ such that

$$R(T) \geq R((\Pi \otimes \text{Id} \otimes \text{Id})T) + \Delta$$

where $\Delta = \dim U - \dim X$

**Proof:** Let $R(T) = r$, so $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$
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- $T$ is 1-concise $\Rightarrow$ the vectors $u_i$ generate $U$
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- W.l.o.g \( u_1, \ldots, u_\Delta \) span a complement of \( X \)
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\((\Pi \otimes \text{Id} \otimes \text{Id}) \cdot T = \sum_{i=\Delta+1}^{r} \Pi u_i \otimes v_i \otimes w_i \)
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**Theorem (Substitution method)**

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- \( T \) is 1-concise \( \Rightarrow \) the vectors \( u_i \) generate \( U \)
- W.l.o.g \( u_1, \ldots, u_\Delta \) span a complement of \( X \)
- Let \( \Pi: U \rightarrow X \) be the projection along \( \text{Span}(u_1, \ldots, u_\Delta) \)
- \((\Pi \otimes \text{Id} \otimes \text{Id}) \cdot T = \sum_{i=\Delta+1}^{r} \Pi u_i \otimes v_i \otimes w_i \)
- Therefore \( R((\Pi \otimes \text{Id} \otimes \text{Id}) \cdot T) \leq r - \Delta \)
Example

\[ P_n = \sum_{i+j+k=n-1} |i\rangle \otimes |j\rangle \otimes |k\rangle \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n \]
Example

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- Let \( \Pi: \mathbb{F}^n \rightarrow \text{Span}(|0\rangle) \) be a projection. \( \Pi |i\rangle = \alpha_i |0\rangle \), with \( \alpha_0 = 1 \)

\[ (\Pi \otimes \text{Id} \otimes \text{Id}) \cdot P_n = \sum_{i+j+k=n-1} \Pi |i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{j+k=n-1-i} \alpha_i |0\rangle \otimes |j\rangle \otimes |k\rangle \]

- \( (\Pi \otimes \text{Id} \otimes \text{Id}) \cdot P_n = |0\rangle \otimes M \) where \( M \) is a triangular matrix with 1 on the diagonal

- \( R(M) = n \Rightarrow R(P_n) \geq n + (n - 1) = 2n - 1 \)
Generalization of the example: attempt 1

**Definition (Contraction)**
For \( f \in U^* \), denote \( Tf = (f \otimes \text{Id} \otimes \text{Id}) \cdot T \in \mathbb{F} \otimes V \otimes W \cong V \otimes W \)

**Definition (Minrank)**
Define *minrank* of \( T \) as

\[
\text{mr}(T) = \min\{\text{rk}(Tf) \mid f \neq 0\}
\]

**Theorem**
For a 1-concise tensor \( T \in U \otimes V \otimes W \)

\[
R(T) \geq \text{mr}(T) + \dim U - 1
\]

- Does not generalize the example: \( \text{mr}(P_n) = 1 \)
Questions?

Thank you!