Irreducible representations, Schur-Weyl duality, and explicit construction of highest weight vectors

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Seminar Mathematics and Computation: Tensor Ranks and Tensor Invariants (Summer 2024)

2024-Apr-18

- $V = \mathbb{C}^a$, $W = \mathbb{C}^b$, $U = \mathbb{C}^c$.
- The space of tensors: $X = V \otimes W \otimes U$
- Standard basis tensors $e_{i,j,k} = e_i \otimes e_j \otimes e_k$
- $Z \subseteq X$, for example $Z = \{T \in X \mid R(T) = 1\}$.
- Consider $T' = \langle 2 \rangle = e_{1,1,1} + e_{2,2,2}$
- To prove $R(T') \geq 2$, we want to find $f : X \to \mathbb{C}$ with
  - $\forall T \in Z : f(T) = 0$
  - $f(T') \neq 0$
- For example, $f = x_{2,1,1}x_{1,2,2} + x_{1,1,1}x_{2,2,2} - x_{2,2,1}x_{1,1,2} - x_{1,2,1}x_{2,1,2}$
- We want to study the polynomials f on X:
  - $\mathbb{C}[X] \simeq \mathbb{C}[x_{1,1,1}, \ldots, x_{a,b,c}]$.
  - In particular, we are interested in homogeneous degree d polynomials: $f \in \mathbb{C}[X]_d$
  - We want f to vanish on Z, i.e., $f \in I(Z)_d$ $\quad$ ($I(Z) = \{f \in \mathbb{C}[X] \mid f(Z) = \{0\}\}$ is the vanishing ideal of Z)
- In our cases: Z is closed under the action of $G = \mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$
- Define a linear action of G on $\mathbb{C}[X]$ via canonical pullback: $(gf)(T) := f(g^{-1}T)$
- For example, $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) f = x_{1,1,1}x_{2,2,2} + x_{2,1,1}x_{1,2,2} - x_{1,2,1}x_{2,1,2} - x_{2,2,1}x_{1,1,2}$
- Observe: If $f \in I(Z)_d$, then $\forall g \in G : gf \in I(Z)_d$
- Hence, $I(Z)_d$ is a G-representation
- Since G is linearly reductive, $I(Z)_d$ decomposes into irreducible G-representations: $I(Z)_d = \bigoplus_i \mathscr{V}_i$
- Decompose $f = \sum_i f_i$ with $f_i \in \mathscr{V}_i$.
- Since $f(T') \neq 0$, at least one i has $f_i(T') \neq 0$
- Hence, it is sufficient to restrict out attention to f in irreducible G-representations
- Even stronger: We can restrict to so-called highest weight vectors

Symmetric tensor powers and the connection to polynomials

- $X = V \otimes W \otimes U$
- Recall: The tensor power $\bigotimes^d(X) = X^{\otimes d}$
- The symmetric group $\mathfrak{S}_d$ acts linearly on $X^{\otimes d}$ by permutation of the tensor factors:

$$\pi(x_1 \otimes \cdots \otimes x_d) \; = \; x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(d)}$$

- Let $\mathsf{Sym}^d X \subseteq \bigotimes^d X$ denote the set of $T \in X$ that are invariant under the action of $\mathfrak{S}_d$.
- For example, $e_{1,1,2} \otimes e_{1,1,1} \in \bigotimes^2 X \setminus \mathsf{Sym}^2 X$. But $e_{1,1,2} \otimes e_{1,1,1} + e_{1,1,1} \otimes e_{1,1,2} \in \mathsf{Sym}^2 X$
- The **symmetrization** projection $\bigotimes^d X \twoheadrightarrow \mathsf{Sym}^d X$: $T \mapsto \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi(T)$
- $\mathbb{C}[X]_d \simeq \mathsf{Sym}^d(X^*)$ via explicit isomorphism φ:
  - $\varphi(x_{i,j,k}) = e^*_{i,j,k}$ $\quad \varphi(x_{i,j,k} x_{i',j',k'}) = \frac{1}{2}(e^*_{i,j,k} \otimes e^*_{i',j',k'} + e^*_{i',j',k'} \otimes e^*_{i,j,k})$ etc
  - For example, $\varphi(x_{111}x_{122} - x_{121}x_{112}) = \frac{1}{2}(e^*_{1,1,1} \otimes e^*_{1,2,2} + e^*_{1,2,2} \otimes e^*_{1,1,1} - e^*_{1,2,1} \otimes e^*_{1,1,2} - e^*_{1,1,2} \otimes e^*_{1,2,1})$
- We want to study $\mathsf{Sym}^d(X^*)$. First, we study $\bigotimes^d(X^*)$, and then we symmetrize.

Recall, $X = V \otimes W \otimes U$.

- $\bigotimes^d (X^*) = \bigotimes^d (V^* \otimes W^* \otimes U^*) \simeq \bigotimes^d (V^*) \otimes \bigotimes^d (W^*) \otimes \bigotimes^d (U^*)$
- We first study one factor: $\bigotimes^d (\mathbb{C}^n)$. Main tool: Schur-Weyl duality

Irreducible representations

- Let G be a group and let $\mathscr{V} = \mathbb{C}^N$.
- A group homomorphism $\varrho : G \to \mathrm{GL}(\mathscr{V})$ is called a **representation** of G.
- Short notation for $g \in G, v \in \mathscr{V}$: $gv = \varrho(g)(v)$
- A representation $\mathscr{V}$ of GL_n is called **polynomial** if all $(\dim \mathscr{V})^2$ many coordinate functions of $\varrho(g)$ are given by polynomials in the n^2 matrix entries.
- A linear subspace $\mathscr{W} \subseteq \mathscr{V}$ is called a **subrepresentation** if $\forall g \in G, w \in \mathscr{W} : gw \in \mathscr{W}$.
- A representation $\mathscr{V}$ is **irreducible** if 0 and $\mathscr{V}$ are the only subrepresentations.
- A linear map $\varphi : \mathscr{V} \to \mathscr{W}$ between two representations $\mathscr{V}$ and $\mathscr{W}$ of G is called a **morphism of representations**, if $\forall g \in G, v \in \mathscr{V} : \varphi(gv) = g\varphi(v)$.
- A bijective morphism of representations is called an **isomorphism of representations**.
- Important task in representation theory: Classify for a group G its irreducible representations up to isomorphism.
- This has been achieved for many groups, including GL_n and $\mathfrak{S}_d$.

- The orbit $Gv := \{gv \mid g \in G\}$
- $\forall v \in \mathscr{V}$: linspan(Gv) is a subrepresentation.
- In particular, is $\mathscr{V}$ is irreducible and $v \neq 0$, then linspan$(Gv) = \mathscr{V}$.

- $\mathbb{C}^n$ is an irreducible representation of GL_n
- $\mathbb{C}^2 \otimes \mathbb{C}^2$ is a GL_2-representation, but it is **not** irreducible:
  it contains the nontrivial 1-dim subrepresentation spanned by: $e_1 \wedge e_2 = \frac{1}{2}(e_{1,2} - e_{2,1})$. $e_{i,j} = e_i \otimes e_j$

$$
\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 \wedge e_2 &= \begin{pmatrix} a \\ c \end{pmatrix} \wedge \begin{pmatrix} b \\ d \end{pmatrix} \\
&= \begin{pmatrix} a \\ 0 \end{pmatrix} \wedge \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \wedge \begin{pmatrix} b \\ d \end{pmatrix} \\
&= \begin{pmatrix} a \\ 0 \end{pmatrix} \wedge \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \wedge \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \wedge \begin{pmatrix} 0 \\ d \end{pmatrix} \\
&= \begin{pmatrix} a \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \wedge \begin{pmatrix} b \\ 0 \end{pmatrix} \\
&= (ad - bc)(e_1 \wedge e_2)
\end{aligned}
$$

- For $g \in \mathrm{GL}_n$: $g(e_1 \wedge \cdots \wedge e_n) = \mathrm{det}(g)(e_1 \wedge \cdots \wedge e_n)$

Definition **weight vector**

For a GL_n-representation $\mathscr{V}$ and $\lambda \in \mathbb{Z}^n$, a vector $v \in \mathscr{V}$ is called a **weight vector of weight** λ if

$$\forall \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \in \mathrm{GL}_n : \ \mathrm{diag}(\alpha_1, \ldots, \alpha_n) v = \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \cdots \alpha_n^{\lambda_n} v.$$

- For example, $e_1 \wedge \cdots \wedge e_i$ is a weight vector of weight $(1, 1, \ldots, 1, 0, \ldots, 0) = (1^i)$ in $\bigotimes^i \mathbb{C}^n$ for $n \geq i$.

If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, then λ is called a **partition**. Define $\lambda^t = \mu$ via $\mu_i = \max\{j \mid \lambda_j \geq i\}$.

Definition **highest weight vector (HWV)**

For a GL_n-representation $\mathscr{V}$ and a partition λ, a vector $v \in \mathscr{V}$ is called a **highest weight vector of weight** λ if

1. v is a weight vector of weight λ
2. $\forall g \in \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} : gv = v.$

- For example, $e_1 \wedge \cdots \wedge e_i$ is a **highest** weight vector of weight (1^i).
- v is an SL_n-invariant iff v is a highest weight vector of weight (k^n) for some k.

- A polynomial representation $\mathscr{V}$ of GL_n is irreducible iff there $\exists$ a unique (up to scale) HWV v in $\mathscr{V}$.
- The weight λ of v is called the **isomorphism type of** $\mathscr{V}$.
- Two irreducible representations of GL_n are isomorphic iff they have the same isomorphism type.
- For a HWV v, the linear span of the orbit $\mathrm{GL}_n \, v$ is irreducible.
- To every λ there exists an irreducible representation of isomorphism type λ, for example using the HWV h_λ:

  $$h_\lambda := (e_1 \wedge \cdots \wedge e_{\mu_1}) \otimes (e_1 \wedge \cdots \wedge e_{\mu_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\mu_{\lambda_1}})$$

$h_\lambda := (e_1 \wedge \cdots \wedge e_{\mu_1}) \otimes (e_1 \wedge \cdots \wedge e_{\mu_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\mu_{\lambda_1}})$

Let $S^\lambda(\mathbb{C}^n)$ denote the irreducible GL_n-representation of type λ, i.e., $S^\lambda(\mathbb{C}^n) = \text{linspan}(\mathrm{GL}_n\, h_\lambda)$

For example, $S^{(2,1)}(\mathbb{C}^3)$.
Highest weight vector: $(e_1 \wedge e_2) \otimes e_1$.
$\text{linspan}(\mathrm{GL}_3((e_1 \wedge e_2) \otimes e_1))$ is an irreducible GL_3-representation with basis

- $(e_1 \wedge e_2) \otimes e_1$ — weight (2,1,0)
- $(e_1 \wedge e_2) \otimes e_2$ — weight (1,2,0)
- $(e_1 \wedge e_3) \otimes e_1$ — weight (2,0,1)
- $(e_1 \wedge e_3) \otimes e_2 + (e_1 \wedge e_2) \otimes e_3$ — weight (1,1,1)
- $(e_2 \wedge e_3) \otimes e_1 + (e_1 \wedge e_3) \otimes e_2$ — weight (1,1,1)
- $(e_2 \wedge e_3) \otimes e_2$ — weight (0,2,1)
- $(e_1 \wedge e_3) \otimes e_3$ — weight (1,0,2)
- $(e_2 \wedge e_3) \otimes e_3$ — weight (0,1,2)

The space of weight (1,1,. . . ,1) is called the **zero weight space**, $S^\lambda(\mathbb{C}^n)^0$. Remark: $S^\lambda(\mathbb{C}^n)^0 = \left(S^\lambda(\mathbb{C}^n)\right)^{\mathrm{ST}_n}$
Note: The group $\mathfrak{S}_3 \subset \mathrm{GL}_3$ acts on

$$\text{linspan}\big((e_1 \wedge e_3) \otimes e_2 + (e_1 \wedge e_2) \otimes e_3\ ,\ (e_2 \wedge e_3) \otimes e_1 + (e_1 \wedge e_3) \otimes e_2\big)$$

$S^{(2,1)}(\mathbb{C}^3)^0$ is an irreducible $\mathfrak{S}_3$-representation.

The irreducible representations of $\mathfrak{S}_n$

Let λ be a partition with $\sum_i \lambda_i = n$.
$S^\lambda(\mathbb{C}^n)^0$ is an irreducible $\mathfrak{S}_n$-representation, denoted by $[\lambda]$, called the Specht module.
The set of Specht modules is a complete set of pairwise non-isomorphic irreducible $\mathfrak{S}_n$-representations.

The irreducible representations of $\mathrm{GL}_n \times \mathfrak{S}_d$ are tensor products $S^\lambda(\mathbb{C}^n) \otimes [\mu]$.

Schur-Weyl duality

$$\bigotimes^d \mathbb{C}^n \;\simeq\; \bigoplus_\lambda S^\lambda(\mathbb{C}^n) \otimes [\lambda]$$

where λ is a partition with at most n entries, and $|\lambda| = d$.

- In particular, $\mathsf{HWV}_\lambda(\bigotimes^d \mathbb{C}^n) \;\simeq\; [\lambda]$ is irreducible.
- Hence, $\mathsf{HWV}_\lambda(\bigotimes^d \mathbb{C}^n) = \mathsf{linspan}\{\pi h_\lambda \mid \pi \in \mathfrak{S}_n\}$.
- And $\mathsf{HWV}_{\lambda,\mu,\nu}(\bigotimes^d V \otimes \bigotimes^d W \otimes \bigotimes^d U) = \mathsf{linspan}\{\pi h_\lambda \otimes \sigma h_\mu \otimes \tau h_\nu \mid \pi, \sigma, \tau \in \mathfrak{S}_n\}$.
- $\mathsf{HWV}_{\lambda\,\mu,\nu}(\mathsf{Sym}^d(V \otimes W \otimes U))$ is obtained via symmetrization of $\mathsf{HWV}_{\lambda\,\mu,\nu}(\bigotimes^d(V \otimes W \otimes U))$.

Example (Hauenstein-I-Landsberg 2013): Let $\lambda = \mu = \nu = (5,5,5,5)$. Let $d = 20$.
Define $f_1, f_2, f_3, f_4 \in \mathsf{HWV}_{\lambda\,\mu,\nu}(\mathsf{Sym}^d(V \otimes W \otimes U))$ via
$\pi^{(1)} = \mathrm{id}, \sigma^{(1)} = (10, 15, 5, 9, 13, 4, 17, 14, 7, 20, 19, 11, 2, 12, 8, 3, 16, 18, 6, 1), \tau^{(1)} = (10, 11, 6, 2, 8, 9, 4, 20, 15, 16, 13, 18, 14, 19, 7, 5, 17, 3, 12, 1)$
$\pi^{(2)} = \mathrm{id}, \sigma^{(2)} = (19, 10, 1, 5, 7, 12, 2, 13, 16, 6, 18, 9, 11, 20, 3, 17, 14, 8, 15, 4), \tau^{(2)} = (10, 5, 13, 6, 3, 16, 11, 1, 4, 18, 15, 17, 9, 2, 8, 12, 19, 7, 14, 20)$
$\pi^{(3)} = \mathrm{id}, \sigma^{(3)} = (16, 20, 9, 13, 8, 1, 4, 19, 11, 17, 7, 2, 14, 3, 6, 5, 12, 15, 18, 10), \tau^{(3)} = (1, 20, 11, 19, 5, 16, 17, 2, 18, 13, 7, 12, 14, 10, 8, 15, 6, 9, 3, 4)$
$\pi^{(4)} = \mathrm{id}, \sigma^{(4)} = (11, 5, 2, 1, 16, 10, 20, 3, 17, 19, 12, 18, 13, 9, 14, 4, 8, 6, 15, 7), \tau^{(4)} = (1, 6, 15, 13, 20, 3, 18, 11, 14, 2, 9, 5, 4, 17, 12, 8, 19, 16, 7, 10)$
$f = -266054 f_1 + 421593 f_2 + 755438 f_3 + 374660 f_4$ vanishes on all rank ≤ 6 tensors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, but not on the 2×2 matrix multiplication tensor.
Other examples: Bürgisser-I 2011, Bürgisser-I 2013, Bläser-Christandl-Zuiddam 2017

The Algebraic Peter-Weyl theorem

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $\ell(\lambda) := \max\{i \mid \lambda_i \neq 0\}$. And let $|\lambda| = \sum_i \lambda_i$.
For any group G, the product $G \times G$ acts on G via $(g_1, g_2)g = g_1 g g_2^{-1}$.
Algebraic Peter-Weyl theorem for GL_n:

$$\mathbb{C}[\mathrm{GL}_n]_d \overset{\mathrm{GL}_n \times \mathrm{GL}_n}{\simeq} \bigoplus_{\substack{\lambda \in \mathbb{Z}^n \\ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \\ |\lambda| = d}} S_\lambda(\mathbb{C}^n) \otimes (S_\lambda(\mathbb{C}^n))^* \qquad \mathbb{C}[\mathbb{C}^{n\times n}]_d \overset{\mathrm{GL}_n \times \mathrm{GL}_n}{\simeq} \bigoplus_{\substack{\text{partition } \lambda \\ \ell(\lambda) \leq n \\ |\lambda| = d}} S_\lambda(\mathbb{C}^n) \otimes (S_\lambda(\mathbb{C}^n))^*$$

"Rectangular" version:

$$\mathbb{C}[\mathbb{C}^{n\times m}]_d \overset{\mathrm{GL}_n \times \mathrm{GL}_m}{\simeq} \bigoplus_{\substack{\text{partition } \lambda \\ \ell(\lambda) \leq \min(n,m) \\ |\lambda| = d}} S_\lambda(\mathbb{C}^n) \otimes (S_\lambda(\mathbb{C}^m))^*$$

For $m = d$, go to the right zero weight space:

$$(\mathbb{C}[\mathbb{C}^{n\times d}]_d)^0 \overset{\mathrm{GL}_n \times \mathfrak{S}_d}{\simeq} \bigoplus_{\substack{\text{partition } \lambda \\ \ell(\lambda) \leq \min(n,d) \\ |\lambda| = d}} S_\lambda(\mathbb{C}^n) \otimes [\lambda]^*$$

A basis for the LHS is given by degree d monomials in $x_{i,j}, i \in \{1, \dots, n\}, j \in \{1, \dots, d\}$, in which every j appears once.
Hence: $(\mathbb{C}[\mathbb{C}^{n\times d}]_d)^0 \simeq \bigotimes^d \mathbb{C}^n$ via the isomorphism $x_{i_1,1} x_{i_2,2} \cdots x_{i_d,d} \mapsto e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_d}$.

- The restrictions $\ell(\lambda) \leq \min(n, d)$ and $|\lambda| = d$ are equivalent to λ being a partition of d into at most n parts.
- The irreducibles of the symmetric group are self-dual: $[\lambda]^* \simeq [\lambda]$.

Hence, we obtain Schur-Weyl duality:

$$\bigotimes^d \mathbb{C}^n \overset{\mathrm{GL}_n \times \mathfrak{S}_d}{\simeq} \bigoplus_{\lambda} S_\lambda(\mathbb{C}^n) \otimes [\lambda]$$

Kronecker coefficients

$$\begin{aligned}
\mathsf{Sym}^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) &\simeq \left(\bigotimes^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right)^{\mathfrak{S}_d} \simeq \left((\bigotimes^d \mathbb{C}^a) \otimes (\bigotimes^d \mathbb{C}^b) \otimes (\bigotimes^d \mathbb{C}^c)\right)^{\mathfrak{S}_d} \\
&\simeq \Big(\big(\bigoplus_\lambda S^\lambda(\mathbb{C}^a) \otimes [\lambda]\big) \otimes \big(\bigoplus_\mu S^\mu(\mathbb{C}^b) \otimes [\mu]\big) \otimes \big(\bigoplus_\nu S^\nu(\mathbb{C}^c) \otimes [\nu]\big)\Big)^{\mathfrak{S}_d} \\
&\simeq \Big(\bigoplus_{\lambda,\mu,\nu} S^\lambda(\mathbb{C}^a) \otimes S^\mu(\mathbb{C}^b) \otimes S^\nu(\mathbb{C}^c) \otimes [\lambda] \otimes [\mu] \otimes [\nu]\Big)^{\mathfrak{S}_d} \\
&\simeq \bigoplus_{\lambda,\mu,\nu} S^\lambda(\mathbb{C}^a) \otimes S^\mu(\mathbb{C}^b) \otimes S^\nu(\mathbb{C}^c) \otimes \Big([\lambda] \otimes [\mu] \otimes [\nu]\Big)^{\mathfrak{S}_d}
\end{aligned}$$

Hence, $\dim \mathsf{HWV}_{\lambda,\mu,\nu}(\mathsf{Sym}^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)) = \dim \Big([\lambda] \otimes [\mu] \otimes [\nu]\Big)^{\mathfrak{S}_d} = k(\lambda,\mu,\nu)$

- $k(\lambda,\mu,\nu)$ is called the **Kronecker coefficient**.
- Stanley's problem 10 (2000): Find a non-negative combinatorial interpretation for $k(\lambda,\mu,\nu)$
- Formally, we ask if the map $(\lambda,\mu,\nu) \mapsto k(\lambda,\mu,\nu)$ is in #P.
- The non-symmetrized version is understood:

$$\dim \mathsf{HWV}_{\lambda,\mu,\nu}(\bigotimes^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)) = \dim([\lambda] \otimes [\mu] \otimes [\nu]) = \dim[\lambda] \cdot \dim[\mu] \cdot \dim[\nu]$$

$\dim[\lambda]$ = number of standard tableaux of shape λ.

Example: $\lambda = (3,1,1)$

$$\begin{array}{ccc}1&4&5\\2&&\\3&&\end{array} \quad \begin{array}{ccc}1&3&5\\2&&\\4&&\end{array} \quad \begin{array}{ccc}1&2&5\\3&&\\4&&\end{array} \quad \begin{array}{ccc}1&3&4\\2&&\\5&&\end{array} \quad \begin{array}{ccc}1&2&4\\3&&\\5&&\end{array} \quad \begin{array}{ccc}1&2&3\\4&&\\5&&\end{array}$$

Summary

- $Z \subseteq X = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$
- $I(Z)_d = \{f \in \mathbb{C}[X] \mid f(Z = \{0\})\}$
- If $I(Z)_d \neq 0$, then $I(Z)_d$ contains some HWV
- $\text{HWV}_{\lambda,\mu,\nu}(\mathbb{C}[X]_d) \simeq ([\lambda] \otimes [\mu] \otimes [\nu])^{\mathfrak{S}_d}$
- Explicit HWV, given partitions λ, μ, ν and permutations π, σ, τ: $\frac{1}{d!} \sum_{\kappa \in \mathfrak{S}_d} (\kappa\pi h_\lambda) \otimes (\kappa\sigma h_\mu) \otimes (\kappa\tau h_\nu)$

Thank you for your attention!