Irreducible representations, Schur-Weyl duality, and explicit construction of highest weight vectors

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\[ V = \mathbb{C}^a, \ W = \mathbb{C}^b, \ U = \mathbb{C}^c. \]

The space of tensors: \[ X = V \otimes W \otimes U \]

Standard basis tensors \[ e_{i,j,k} = e_i \otimes e_j \otimes e_k \]

\[ Z \subseteq X, \ \text{for example} \ Z = \{ T \in X \mid R(T) = 1 \}. \]

Consider \( T' = \langle 2 \rangle = e_{1,1,1} + e_{2,2,2} \)

To prove \( R(T') \geq 2 \), we want to find \( f : X \to \mathbb{C} \) with

\[ \forall T \in Z : f(T) = 0 \]

\[ f(T') \neq 0 \]

For example, \[ f = x_{2,1,1}x_{1,2,2} + x_{1,1,1}x_{2,2,2} - x_{2,2,1}x_{1,1,2} - x_{1,2,1}x_{2,1,2} \]

We want to study the polynomials \( f \) on \( X \):

\[ \mathbb{C}[X] \simeq \mathbb{C}[x_{1,1,1}, \ldots, x_{a,b,c}]. \]

In particular, we are interested in homogeneous degree \( d \) polynomials: \( f \in \mathbb{C}[X]_d \)

We want \( f \) to vanish on \( Z \), i.e., \( f \in I(Z)_d \quad (I(Z) = \{ f \in \mathbb{C}[X] \mid f(Z) = \{0\} \} \) is the vanishing ideal of \( Z \)

In our cases: \( Z \) is closed under the action of \( G = \text{GL}_a \times \text{GL}_b \times \text{GL}_c \)

Define a linear action of \( G \) on \( \mathbb{C}[X] \) via canonical pullback: \( (gf)(T) := f(g^{-1}T) \)

For example, \( ((0 \ 1 \ 0), (1 \ 0 \ 0), (0 \ 1 \ 0))f = x_{1,1,1}x_{2,2,2} + x_{2,1,1}x_{1,2,2} - x_{2,2,1}x_{1,1,2} - x_{1,2,1}x_{2,1,2} \)

Observe: If \( f \in I(Z)_d \), then \( \forall g \in G : gf \in I(Z)_d \)

Hence, \( I(Z)_d \) is a \( G \)-representation

Since \( G \) is linearly reductive, \( I(Z)_d \) decomposes into irreducible \( G \)-representations: \( I(Z)_d = \bigoplus_i \mathcal{V}_i \)

Decompose \( f = \sum_i f_i \) with \( f_i \in \mathcal{V}_i \).

Since \( f(T') \neq 0 \), at least one \( i \) has \( f_i(T') \neq 0 \)

Hence, it is sufficient to restrict out attention to \( f \) in irreducible \( G \)-representations

Even stronger: We can restrict to so-called highest weight vectors
Symmetric tensor powers and the connection to polynomials

- \( X = V \otimes W \otimes U \)
- Recall: The tensor power \( \bigotimes^d(X) = X \otimes^d \)
- The symmetric group \( S_d \) acts linearly on \( X \otimes^d \) by permutation of the tensor factors:
  \[
  \pi(x_1 \otimes \cdots \otimes x_d) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(d)}
  \]
- Let \( \text{Sym}^d X \subseteq \bigotimes^d X \) denote the set of \( T \in X \) that are invariant under the action of \( S_d \).
- For example, \( e_{1,1,2} \otimes e_{1,1,1} \in \bigotimes^2 X \setminus \text{Sym}^2 X \). But \( e_{1,1,2} \otimes e_{1,1,1} + e_{1,1,1} \otimes e_{1,1,2} \in \text{Sym}^2 X \)
- The symmetrization projection \( \bigotimes^d X \rightarrow \text{Sym}^d X: T \mapsto \frac{1}{d!} \sum_{\pi \in S_d} \pi(T) \)
- \( \mathbb{C}[X]_d \simeq \text{Sym}^d(X^*) \) via explicit isomorphism \( \varphi \):
  - \( \varphi(x_{i,j,k}) = e^*_{i,j,k} \)
  - \( \varphi(x_{i,j,k} x_{i',j',k'}) = \frac{1}{2}(e^*_{i,j,k} \otimes e^*_{i',j',k'} + e^*_{i',j',k'} \otimes e^*_{i,j,k}) \)
    etc
  - For example, \( \varphi(x_{111} x_{122} - x_{121} x_{112}) = \frac{1}{2}(e^*_{1,1,1} \otimes e^*_{1,2,2} + e^*_{1,2,2} \otimes e^*_{1,1,1} - e^*_{1,2,1} \otimes e^*_{1,1,2} - e^*_{1,1,2} \otimes e^*_{1,2,1}) \)
- We want to study \( \text{Sym}^d(X^*) \). First, we study \( \bigotimes^d(X^*) \), and then we symmetrize.
Recall, $X = V \otimes W \otimes U$.

- $\otimes^d(X^*) = \otimes^d(V^* \otimes W^* \otimes U^*) \simeq \otimes^d(V^*) \otimes \otimes^d(W^*) \otimes \otimes^d(U^*)$

- We first study one factor: $\otimes^d(\mathbb{C}^n)$. Main tool: Schur-Weyl duality

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**Irreducible representations**

- Let $G$ be a group and let $\mathcal{V} = \mathbb{C}^N$.
- A group homomorphism $\varrho : G \to \text{GL}(\mathcal{V})$ is called a **representation** of $G$.
- Short notation for $g \in G, v \in \mathcal{V}$: $gv = \varrho(g)(v)$
- A representation $\mathcal{V}$ of $\text{GL}_n$ is called **polynomial** if all $(\dim \mathcal{V})^2$ many coordinate functions of $\varrho(g)$ are given by polynomials in the $n^2$ matrix entries.
- A linear subspace $\mathcal{W} \subseteq \mathcal{V}$ is called a **subrepresentation** if $\forall g \in G, w \in \mathcal{W} : gw \in \mathcal{W}$.
- A representation $\mathcal{V}$ is **irreducible** if 0 and $\mathcal{V}$ are the only subrepresentations.
- A linear map $\varphi : \mathcal{V} \to \mathcal{W}$ between two representations $\mathcal{V}$ and $\mathcal{W}$ of $G$ is called a **morphism of representations**, if $\forall g \in G, v \in \mathcal{V} : \varphi(gv) = g\varphi(v)$.
- A bijective morphism of representations is called an **isomorphism of representations**.
- Important task in representation theory: Classify for a group $G$ its irreducible representations up to isomorphism.
- This has been achieved for many groups, including $\text{GL}_n$ and $\mathfrak{S}_d$.

- The orbit $Gv := \{gv \mid g \in G\}$
- $\forall v \in \mathcal{V} : \text{linspan}(Gv)$ is a subrepresentation.
- In particular, is $\mathcal{V}$ is irreducible and $v \neq 0$, then $\text{linspan}(Gv) = \mathcal{V}$. 

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\( \mathbb{C}^n \) is an irreducible representation of \( \text{GL}_n \)

\( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is a \( \text{GL}_2 \)-representation, but it is not irreducible: it contains the nontrivial 1-dim subrepresentation spanned by: 
\[
e_1 \wedge e_2 = \frac{1}{2}(e_{1,2} - e_{2,1}).
\]

\( e_{i,j} = e_i \otimes e_j \)

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} e_1 \wedge e_2 = \begin{pmatrix}
a \\
c
\end{pmatrix} \wedge \begin{pmatrix}
b \\
d
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a \\
c
\end{pmatrix} \wedge \begin{pmatrix}
b \\
d
\end{pmatrix} + \begin{pmatrix}
0 \\
c
\end{pmatrix} \wedge \begin{pmatrix}
b \\
d
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a \\
c
\end{pmatrix} \wedge \begin{pmatrix}
b \\
d
\end{pmatrix} + \begin{pmatrix}
a \\
c
\end{pmatrix} \wedge \begin{pmatrix}
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
c
\end{pmatrix} \wedge \begin{pmatrix}
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
c
\end{pmatrix} \wedge \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
= (ad - bc)(e_1 \wedge e_2)
\]

\[\text{For } g \in \text{GL}_n : \quad g(e_1 \wedge \cdots \wedge e_n) = \det(g)(e_1 \wedge \cdots \wedge e_n)\]
Definition **weight vector**

For a \(GL_n\)-representation \(V\) and \(\lambda \in \mathbb{Z}^n\), a vector \(v \in V\) is called a **weight vector of weight** \(\lambda\) if

\[ \forall \text{diag}(\alpha_1, \ldots, \alpha_n) \in GL_n : \text{diag}(\alpha_1, \ldots, \alpha_n)v = \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \cdots \alpha_n^{\lambda_n}v. \]

- For example, \(e_1 \wedge \cdots \wedge e_i\) is a weight vector of weight \((1, 1, \ldots, 1, 0, \ldots, 0) = (1^i)\) in \(\otimes^i \mathbb{C}^n\) for \(n \geq i\).

If \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\), then \(\lambda\) is called a **partition**. Define \(\lambda^t = \mu\) via \(\mu_i = \max\{j | \lambda_j \geq i\}\).

Definition **highest weight vector (HWV)**

For a \(GL_n\)-representation \(V\) and a partition \(\lambda\), a vector \(v \in V\) is called a **highest weight vector of weight** \(\lambda\) if

1. \(v\) is a weight vector of weight \(\lambda\)
2. \(\forall g \in \begin{pmatrix} 1 & \ast & \cdots & \ast \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ast \\ 0 \cdots 0 & 1 \end{pmatrix}: gv = v.\)

- For example, \(e_1 \wedge \cdots \wedge e_i\) is a **highest** weight vector of weight \((1^i)\).
- \(v\) is an \(SL_n\)-invariant iff \(v\) is a highest weight vector of weight \((k^n)\) for some \(k\).

- A polynomial representation \(V\) of \(GL_n\) is irreducible iff there \(\exists\) a unique (up to scale) HWV \(v\) in \(V\).
- The weight \(\lambda\) of \(v\) is called the **isomorphism type of** \(V\).
- Two irreducible representations of \(GL_n\) are isomorphic iff they have the same isomorphism type.
- For a HWV \(v\), the linear span of the orbit \(GL_n v\) is irreducible.
- To every \(\lambda\) there exists an irreducible representation of isomorphism type \(\lambda\), for example using the HWV \(h_\lambda\):

\[ h_\lambda := (e_1 \wedge \cdots \wedge e_{\mu_1}) \otimes (e_1 \wedge \cdots \wedge e_{\mu_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\mu_{\lambda_1}}) \]
Let \( S^\lambda(\mathbb{C}^n) := \langle e_1 \wedge \cdots \wedge e_{\mu_1} \rangle \otimes \langle e_1 \wedge \cdots \wedge e_{\mu_2} \rangle \otimes \cdots \otimes \langle e_1 \wedge \cdots \wedge e_{\mu_\lambda_1} \rangle \)

Let \( S^\lambda(\mathbb{C}^n) \) denote the irreducible \( GL_n \)-representation of type \( \lambda \), i.e., \( S^\lambda(\mathbb{C}^n) = \text{linspan}(GL_n h_\lambda) \)

For example, \( S^{(2,1)}(\mathbb{C}^3) \).

Highest weight vector: \( (e_1 \wedge e_2) \otimes e_1 \).

\( \text{linspan}(GL_3(\langle e_1 \wedge e_2 \otimes e_1 \rangle)) \) is an irreducible \( GL_3 \)-representation with basis

- \( (e_1 \wedge e_2) \otimes e_1 \) \hspace{2cm} \text{weight } (2,1,0)
- \( (e_1 \wedge e_2) \otimes e_2 \) \hspace{2cm} \text{weight } (1,2,0)
- \( (e_1 \wedge e_3) \otimes e_1 \) \hspace{2cm} \text{weight } (2,0,1)
- \( (e_1 \wedge e_3) \otimes e_2 + (e_1 \wedge e_2) \otimes e_3 \) \hspace{2cm} \text{weight } (1,1,1)
- \( (e_2 \wedge e_3) \otimes e_1 + (e_1 \wedge e_3) \otimes e_2 \) \hspace{2cm} \text{weight } (1,1,1)
- \( (e_2 \wedge e_3) \otimes e_2 \) \hspace{2cm} \text{weight } (0,2,1)
- \( (e_1 \wedge e_3) \otimes e_3 \) \hspace{2cm} \text{weight } (1,0,2)
- \( (e_2 \wedge e_3) \otimes e_3 \) \hspace{2cm} \text{weight } (0,1,2)

The space of weight \((1,1,\ldots,1)\) is called the zero weight space, \( S^\lambda(\mathbb{C}^n)^0 \).

Note: The group \( S_3 \subset GL_3 \) acts on

\( \text{linspan}(\langle e_1 \wedge e_3 \otimes e_2 + (e_1 \wedge e_2) \otimes e_3 , (e_2 \wedge e_3) \otimes e_1 + (e_1 \wedge e_3) \otimes e_2 \rangle) \)

\( S^{(2,1)}(\mathbb{C}^3)^0 \) is an irreducible \( S_3 \)-representation.

The irreducible representations of \( S_n \)

Let \( \lambda \) be a partition with \( \sum \lambda_i = n \).

\( S^\lambda(\mathbb{C}^n)^0 \) is an irreducible \( S_n \)-representation, denoted by \( [\lambda] \), called the Specht module.

The set of Specht modules is a complete set of pairwise non-isomorphic irreducible \( S_n \)-representations.
The irreducible representations of $\text{GL}_n \times \mathfrak{S}_d$ are tensor products $S^\lambda(\mathbb{C}^n) \otimes [\mu]$.

**Schur-Weyl duality**

$$\bigotimes^d \mathbb{C}^n \simeq \bigoplus_\lambda S^\lambda(\mathbb{C}^n) \otimes [\lambda]$$

where $\lambda$ is a partition with at most $n$ entries, and $|\lambda| = d$.

- In particular, $\text{HWV}_\lambda(\bigotimes^d \mathbb{C}^n) \simeq [\lambda]$ is irreducible.
- Hence, $\text{HWV}_\lambda(\bigotimes^d \mathbb{C}^n) = \text{linspan}\{\pi h_\lambda \mid \pi \in \mathfrak{S}_n\}$.
- And $\text{HWV}_{\lambda,\mu,\nu}(\bigotimes^d V \otimes \bigotimes^d W \otimes \bigotimes^d U) = \text{linspan}\{\pi h_\lambda \otimes \sigma h_\mu \otimes \tau h_\nu \mid \pi, \sigma, \tau \in \mathfrak{S}_n\}$.
- $\text{HWV}_{\lambda,\mu,\nu}(\text{Sym}^d (V \otimes W \otimes U))$ is obtained via symmetrization of $\text{HWV}_{\lambda,\mu,\nu}(\bigotimes^d (V \otimes W \otimes U))$.

Example (Hauenstein-I-Landsberg 2013): Let $\lambda = \mu = \nu = (5, 5, 5, 5)$. Let $d = 20$. Define $f_1, f_2, f_3, f_4 \in \text{HWV}_{\lambda,\mu,\nu}(\text{Sym}^d (V \otimes W \otimes U))$ via

$$\pi^{(1)} = \text{id}, \sigma^{(1)} = (10, 15, 5, 9, 13, 4, 17, 14, 7, 20, 19, 11, 2, 12, 8, 3, 16, 18, 6, 1), \tau^{(1)} = (10, 11, 6, 2, 8, 9, 4, 20, 15, 16, 13, 18, 14, 19, 7, 5, 17, 3, 12, 1)$$

$$\pi^{(2)} = \text{id}, \sigma^{(2)} = (19, 10, 1, 5, 7, 12, 2, 13, 16, 6, 18, 9, 11, 20, 3, 17, 14, 8, 15, 4), \tau^{(2)} = (10, 5, 13, 6, 3, 16, 11, 1, 4, 18, 15, 17, 9, 2, 8, 12, 19, 7, 14, 20)$$

$$\pi^{(3)} = \text{id}, \sigma^{(3)} = (16, 20, 9, 13, 8, 1, 4, 19, 11, 17, 7, 2, 14, 3, 6, 5, 12, 15, 18, 10), \tau^{(3)} = (1, 20, 11, 19, 5, 16, 17, 2, 18, 13, 7, 12, 14, 10, 8, 15, 6, 9, 3, 4)$$

$$\pi^{(4)} = \text{id}, \sigma^{(4)} = (11, 5, 2, 1, 16, 10, 20, 3, 17, 19, 12, 18, 13, 9, 14, 4, 8, 6, 15, 7), \tau^{(4)} = (1, 6, 15, 13, 20, 3, 18, 11, 14, 2, 9, 5, 4, 17, 12, 8, 19, 16, 7, 10)$$

$f = -266054 f_1 + 421593 f_2 + 755438 f_3 + 374660 f_4$ vanishes on all rank $\leq 6$ tensors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, but not on the $2 \times 2$ matrix multiplication tensor.

Other examples: Bürgisser-I 2011, Bürgisser-I 2013, Bläser-Christandl-Zuiddam 2017
The Algebraic Peter-Weyl theorem

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), let \( \ell(\lambda) := \max\{i \mid \lambda_i \neq 0\} \). And let \( |\lambda| = \sum_i \lambda_i \).

For any group \( G \), the product \( G \times G \) acts on \( G \) via \( (g_1, g_2)g = g_1 g g_2^{-1} \).

Algebraic Peter-Weyl theorem for \( GL_n \):

\[
\mathbb{C}[GL_n]_d \cong \bigoplus_{\lambda \in \mathbb{Z}^n \atop \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \atop |\lambda| = d} S_{\lambda}(\mathbb{C}^n) \otimes (S_{\lambda}(\mathbb{C}^n))^*.
\]

“Rectangular” version:

\[
\mathbb{C}[C^n \times C^m]_d \cong \bigoplus_{\lambda \in \mathbb{Z}^n \atop \ell(\lambda) \leq \min(n, m) \atop |\lambda| = d} S_{\lambda}(\mathbb{C}^n) \otimes (S_{\lambda}(\mathbb{C}^m))^*.
\]

For \( m = d \), go to the right zero weight space:

\[
(C[C^n \times d]_d)_0 \cong \bigoplus_{\lambda \in \mathbb{Z}^n \atop \ell(\lambda) \leq \min(n, d) \atop |\lambda| = d} S_{\lambda}(\mathbb{C}^n) \otimes [\lambda]^*.
\]

A basis for the LHS is given by degree \( d \) monomials in \( x_{i,j} \), \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, d\} \), in which every \( j \) appears once.

Hence: \((C[C^n \times d]_d)_0 \cong \bigotimes^d \mathbb{C}^n \) via the isomorphism \( x_{i_1,1} x_{i_2,2} \cdots x_{i_d,d} \mapsto e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} \).

- The restrictions \( \ell(\lambda) \leq \min(n, d) \) and \( |\lambda| = d \) are equivalent to \( \lambda \) being a partition of \( d \) into at most \( n \) parts.
- The irreducibles of the symmetric group are self-dual: \([\lambda]^* \simeq [\lambda]\).

Hence, we obtain Schur-Weyl duality:

\[
\bigotimes^d \mathbb{C}^n \cong GL_n \times \mathbb{S}_d \bigoplus_{\lambda} S_{\lambda}(\mathbb{C}^n) \otimes [\lambda].
\]
Kronecker coefficients

\[ \text{Sym}^d(C^a \otimes C^b \otimes C^c) \approx (\bigotimes^d (C^a \otimes C^b \otimes C^c))^\otimes_d \approx (\bigotimes^d (C^a) \otimes (\bigotimes^d (C^b) \otimes (\bigotimes^d (C^c))))^\otimes_d \]

\[ \approx \left( \bigoplus_{\lambda} S^\lambda(C^a) \otimes (\bigoplus_{\mu} S^\mu(C^b) \otimes [\mu]) \otimes (\bigoplus_{\nu} S^\nu(C^c) \otimes [\nu]) \right)^\otimes_d \]

\[ \approx \left( \bigoplus_{\lambda, \mu, \nu} S^\lambda(C^a) \otimes S^\mu(C^b) \otimes S^\nu(C^c) \otimes [\lambda] \otimes [\mu] \otimes [\nu] \right)^\otimes_d \]

\[ \approx \bigoplus_{\lambda, \mu, \nu} S^\lambda(C^a) \otimes S^\mu(C^b) \otimes S^\nu(C^c) \otimes ([\lambda] \otimes [\mu] \otimes [\nu])^\otimes_d \]

Hence, \( \dim \text{HWV}_{\lambda, \mu, \nu}(\text{Sym}^d(C^a \otimes C^b \otimes C^c)) = \dim ([\lambda] \otimes [\mu] \otimes [\nu])^\otimes_d = k(\lambda, \mu, \nu) \)

- \( k(\lambda, \mu, \nu) \) is called the Kronecker coefficient.
- Stanley’s problem 10 (2000): Find a non-negative combinatorial interpretation for \( k(\lambda, \mu, \nu) \)
- Formally, we ask if the map \( (\lambda, \mu, \nu) \mapsto k(\lambda, \mu, \nu) \) is in \#P.
- The non-symmetrized version is understood:

\[ \dim \text{HWV}_{\lambda, \mu, \nu}(\bigotimes^d(C^a \otimes C^b \otimes C^c)) = \dim([\lambda] \otimes [\mu] \otimes [\nu]) = \dim[\lambda] \cdot \dim[\mu] \cdot \dim[\nu] \]

\[ \dim[\lambda] = \text{number of standard tableaux of shape } \lambda. \]

Example: \( \lambda = (3, 1, 1) \)

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Summary

- \( Z \subseteq X = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c \)

- \( I(Z)_d = \{ f \in \mathbb{C}[X] \mid f(Z = \{0\}) \} \)

- If \( I(Z)_d \neq 0 \), then \( I(Z)_d \) contains some HWV

- \( \text{HWV}_{\lambda,\mu,\nu}(\mathbb{C}[X]_d) \simeq ([\lambda] \otimes [\mu] \otimes [\nu])\mathcal{S}_d \)

- Explicit HWV, given partitions \( \lambda, \mu, \nu \) and permutations \( \pi, \sigma, \tau \): \( \frac{1}{d!} \sum_{\kappa \in \mathcal{S}_d} (\kappa \pi h_\lambda) \otimes (\kappa \sigma h_\mu) \otimes (\kappa \tau h_\nu) \)

Thank you for your attention!