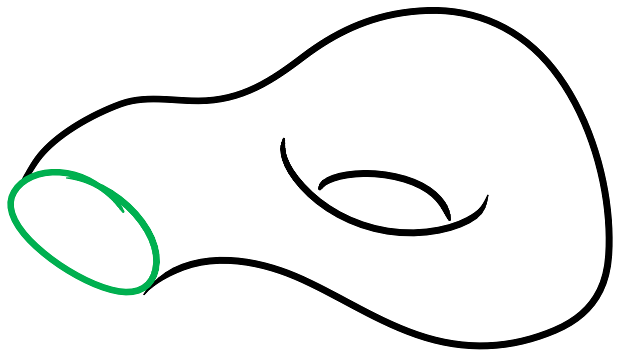
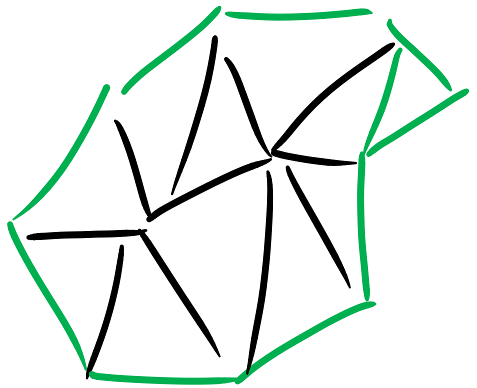
Topologically ordered models in higher dimensions



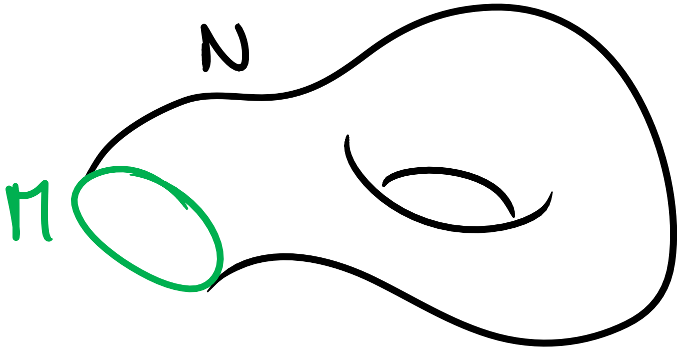
Michael Walter
Stanford University



Conference on Quantum Groups and Quantum Information Theory (July 2015)

QFT & Topology

Quantum states from space-time path integral (Feynman):



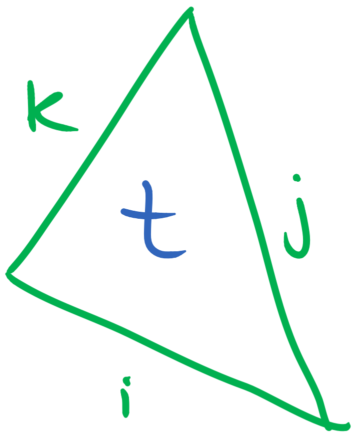
$$\langle \varphi | \Psi \rangle = \int_{\Phi|_M = \varphi} \text{amplitude}(\Phi)$$

If independent of metric: TQFT, *quantum* invariants (Witten, ...)

Effective theory for topological q. phases, "topological order" (Wen, ...) -> David's talk

Today: Construction of lattice models with topological order

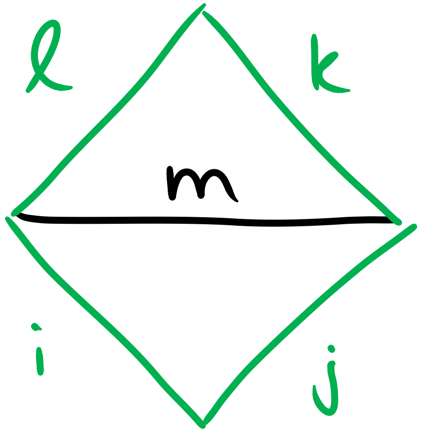
1D States from 2D Triangulations



$I \ni i, j, \dots$

t_{ijk} tensor in $(\mathbb{C}^I)^{\otimes 3}$

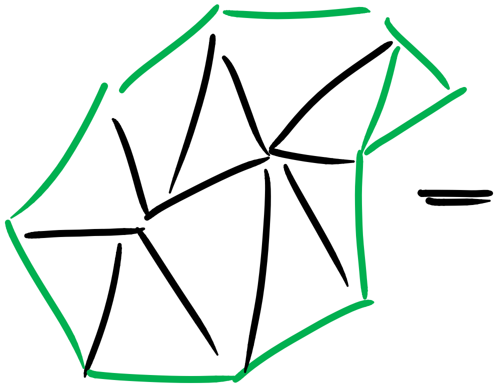
gluing $\Downarrow$ contraction



$$\sum_m t_{ijm} t_{mkl}$$

1D States from 2D Triangulations

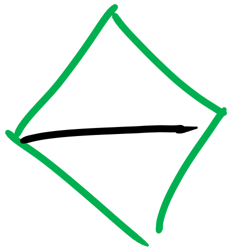
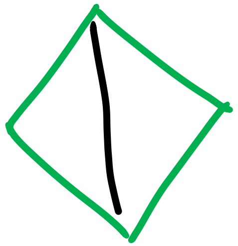
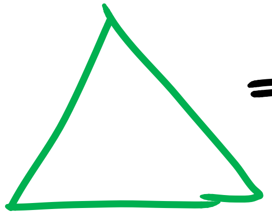
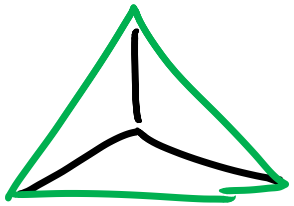
Tensor network for any triangulated surface N^2:

 $= N$

$$Z(N) \in \mathcal{H}(\partial N) = \bigotimes_{\substack{\text{edge } e \\ \in \partial N}} \mathbb{C}^I$$

...independent of the bulk triangulation iff:

[Pachner]

 =  and  = 

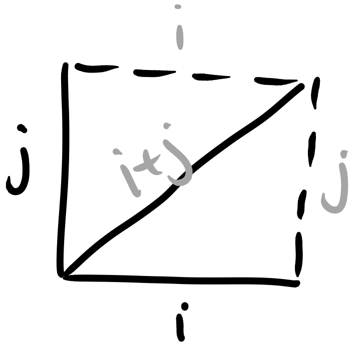
t_{ijk} is "topo. tensor"

Example: Ising Tensor

$I = \mathbb{Z}_2$ $\qquad$ $t_{ijk} = \begin{cases} \sqrt{\frac{1}{2}} & (i+j+k \equiv 0) \\ 0 & (\text{otherwise}) \end{cases}$

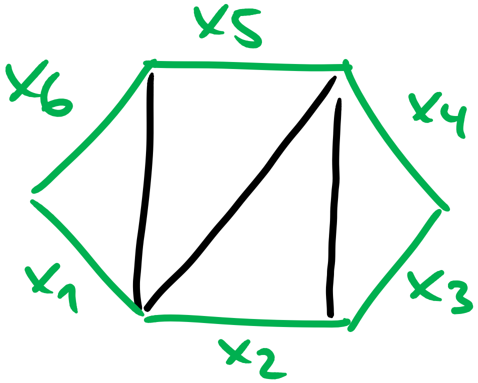
This defines a topological tensor. E.g.:

$\mathbb{T}^2$:



$Z(\mathbb{T}^2) = 2$

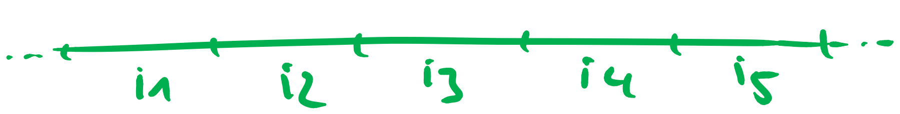
$\mathbb{D}^2$:



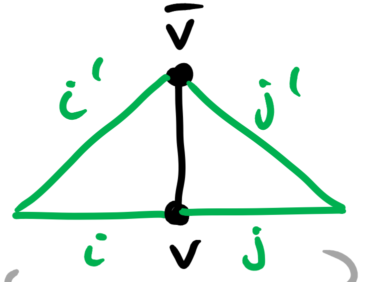
$Z(\mathbb{D}^2) \propto \sum_{x_1+\dots+x_6 \equiv 0} |x_1,\dots,x_6\rangle$

1D: Construction of Lattice Model

Given t_{ijk} and triangulated M^1...



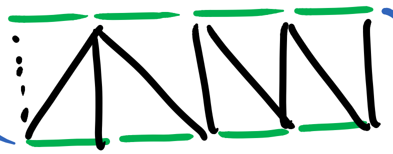
Hilbert space: $\mathcal{H}(M) = \bigotimes_{\substack{\text{edges } e \\ \in M}} \mathbb{C}^I$

Hamiltonian: $H = -\sum_{\text{vertex } v} H_v$ where $H_v = Z\Big($  $\Big)$

local operator!

The H_v are commuting projectors!

1D: Ground Space & Renormalization

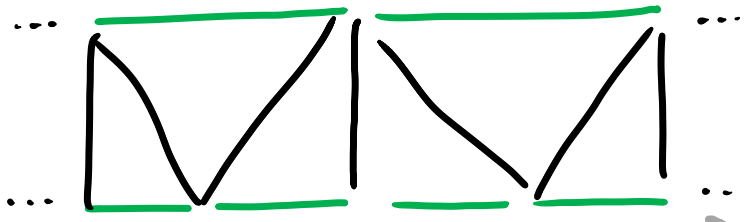
GS projector: $P_0 = \prod_v H_v = Z(M\times[0,1]) = Z\Big($  $\Big)$

tensor network

GS dimension: $\mathrm{tr}\, P_0 = Z(M\times S^1)$ ⟵ topo invariant

Renormalization:

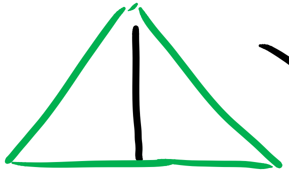
coase lattice

$Z\Big($  $\Big)$ ⟵ partial iso GS ⟼ GS

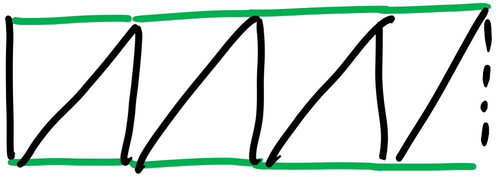
fine lattice

Example: Ising Model $\quad I = \mathbb{Z}_2 \quad t_{ijk} = \sqrt{\tfrac{1}{2}}\,(i+j+k=0)$

Hilbert space: $\quad (\mathbb{C}^2)^{\otimes \#\text{edges in } M}$

Hamiltonian: $\quad H_v = Z($  $) = \frac{1}{2}(I \otimes I + X \otimes X)$

GS projector (M = triangulated S^1):

$P_0 = Z($  $) \longrightarrow$ GS = span$\{|\text{even}\rangle, |\text{odd}\rangle\}$

Indeed, $\operatorname{tr} P_0 = Z(S^1 \times S^1) = 2$ as computed before. ✓

Example: Ising Model $\quad I = \mathbb{Z}_2 \quad t_{ijk} = \sqrt{\frac{1}{2}}\ (i+j+k \equiv 0)$

Ground space: $\quad \text{span}\{ \underbrace{|\text{even}\rangle, |\text{odd}\rangle}_{\text{locally indistinguishable}} \}$

However, superpositions can be locally distinguished!

$$|\text{even}\rangle \pm |\text{odd}\rangle = |\pm\rangle^{\otimes N} \quad \text{where} \quad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

“Topological Bit”

More generally: “No topological order in 1D.”

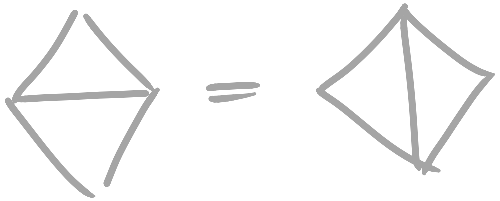
1D: Topological Tensors from Semisimple Algebras [Fukuma et al]

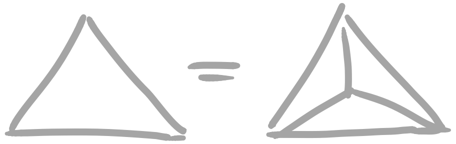
$\mathcal{A}$ fin.-dim. algebra $\longrightarrow$ basis $(a_i)_{i\in I}$

$$a_i a_j = \sum_k t_{ij}^k a_k$$

Associativity: $\quad (a_i a_j) a_k = a_i (a_j a_k)$

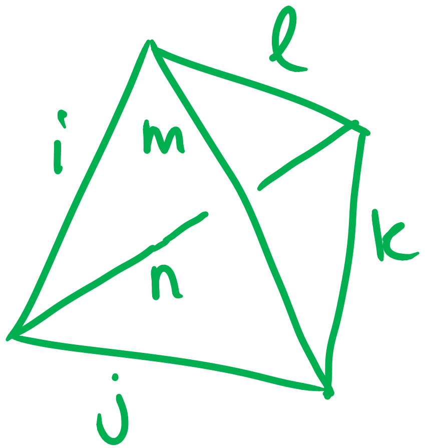
$$\sum_m t_{ij}^m t_{mk}^\ell = \sum_n t_{in}^\ell t_{jk}^n$$



Semisimplicity $\rightsquigarrow$  .

Ising tensor: $\mathbb{C}[\mathbb{Z}_2]$

Going Up: 2D States from 3D Triangulations



$i, j, \ldots \in I$ on edges

$\varphi_{ijk} \in H(ijk)$ on triangles

Today: Fusion rules only!

$\# H(ijk) \in \{0,1\}$

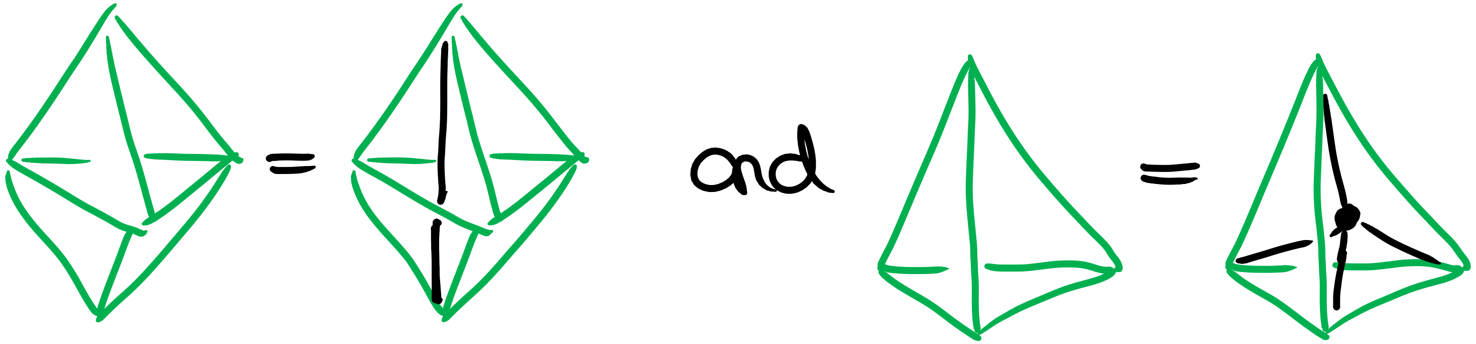
G^{ijm}_{kln} tensor for tetrahedra

Going Up: 2D States from 3D Triangulations

Tensor network for each triangulated N^3:

$$Z(N) \in \mathcal{H}(\partial N) = \bigotimes_{\substack{\text{edge } e \\ \in \partial N}} \mathbb{C}^I \otimes \bigotimes_{\substack{\text{triangle } t \\ \in \partial N}} \mathbb{C}^H$$

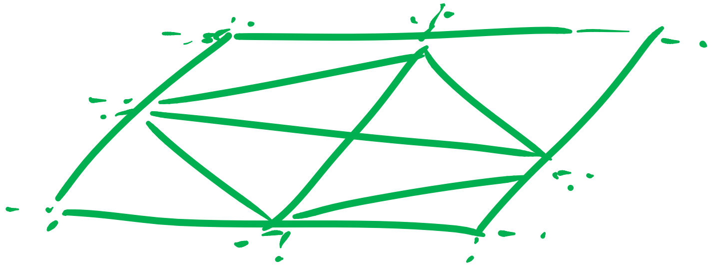
...independent of the bulk triangulation iff:



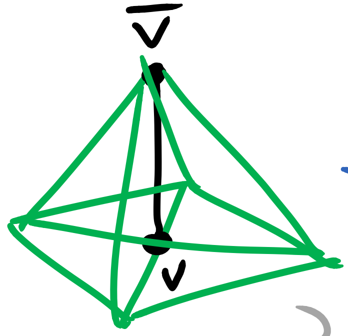
2D: Construction of Lattice Model

[Levin-Wen], [Koenig et al]

Given G^{ijm}_{kln} and triangulated M^2...



Hilbert space: $\mathcal{H}(M) = \bigotimes_{\substack{\text{edges } e \\ \in M}} \mathbb{C}^I \otimes \bigotimes_{\substack{\text{tris } t \\ \in M}} \mathbb{C}^H$

Hamiltonian: $H = -\sum_{\text{vertex } v} H_v$ where $H_v = Z\Big($  $\Big)$

local operator!

The H_v are commuting projectors! (As before...)

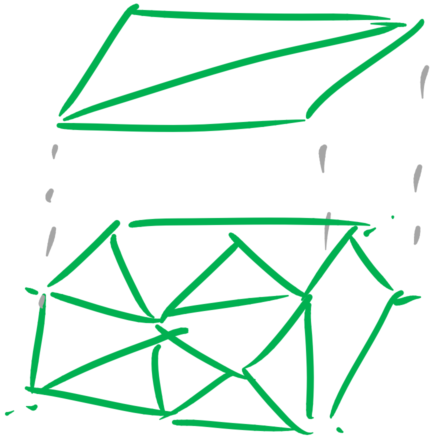
2D: Ground Space & Renormalization

GS projector: $P_0 = \prod_v H_v = Z(M \times [0,1])$ $\rightsquigarrow$ tensor network

GS degeneracy: $\mathrm{tr}\, P_0 = Z(M \times S^1)$ $\leftarrow$ topo invariant

Renormalization:

coase lattice

$Z($  $)$ $\leftarrow$ partial iso GS $\mapsto$ GS

fine lattice

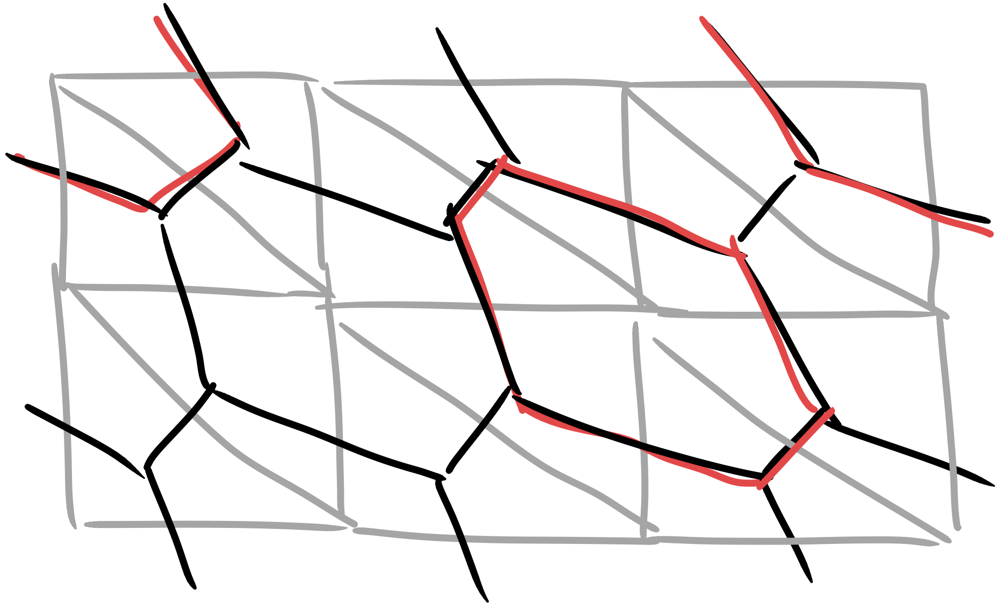
Example: Toric Code

[Kitaev]

$I = \mathbb{Z}_2$, fusion rule: $\boxed{i+j+k \equiv 0}$, $G = 1$ if allowed

dual complex

GS are Σ of closed strings, locally indistinguishable



GSD $= 2^{2g}$

Topological Qubits

Logical operators = strings. Excitations well-understood, also on TN level.

Going Up: Algebra vs Topology

Associativity: $(a_i \cdot a_j) \cdot a_k = a_i \cdot (a_j \cdot a_k)$

$\Downarrow$ Categorification

Associator: $(i \otimes j) \otimes k \xrightarrow[\cong]{\alpha_{ijk}} i \otimes (j \otimes k)$

$\hookrightarrow$ Monoidal tensor categories

2D: Topological Tensors from Fusion Categories [Barrett-Westbury]

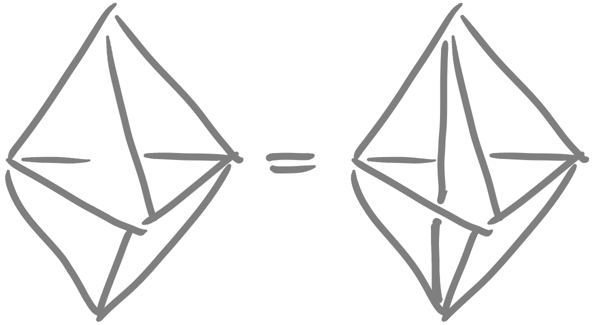
$\mathcal{C}$ spherical fusion category $\leftarrow \mathrm{Rep}(\mathcal{G})$

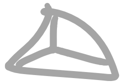
$i, j, \ldots \in \mathcal{I}$ simple objects, $\varphi_{ijk} \in \mathrm{Hom}(i, j \otimes k)$ basis

α_{ijk} associator $\longrightarrow$ structure constants G^{ijm}_{kln}

Pentagon equation:

$$\begin{tikzcd}
 & (ij)(kl) \arrow[dr, "\alpha"] & \\
((ij)k)l \arrow[ur, "\alpha"] \arrow[d, "\alpha"'] & & i(j(kl)) \\
(i(jk))l \arrow[r, "\alpha"'] & i((jk)l) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$



Semisimplicity $\rightsquigarrow$  = . Excitations $\hat{=}\ Z(\mathcal{C})$

Going Up: Topological Tensors in *n*D

Degrees of freedom on
n-skeleton of M^n:

$i, j, k, l, \ldots \in I$ on edges

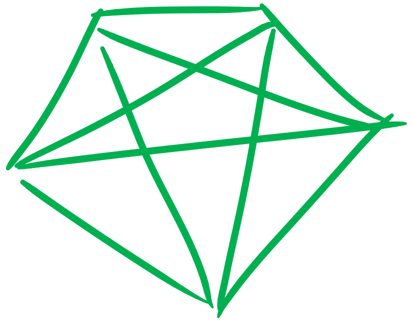
$f \in H(ijk)$ on triangles

$\Phi \in K(fg, hk)$ on tetrahedra

$\ldots$

Tensor for *n+1*-simplex:

$T_{ij\ldots fg\ldots \Phi\Psi\ldots}$



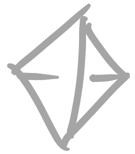
$\Rightarrow \quad \mathcal{H}(\partial N) \ni Z(N)$ for any N^{n+1}

Topological invariance from *n+1*-Pachner moves (boundary of *n+2*-simplex).

Associativity: $(a_i \cdot a_j)\cdot a_k = a_i \cdot (a_j \cdot a_k)$

$\Downarrow$

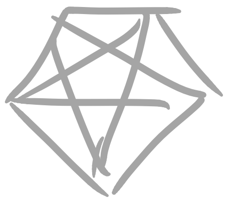
Associator: $(i \otimes j) \otimes k \xrightarrow{\alpha_{ijk}} i \otimes (j \otimes k)$



$\Downarrow$

Pentagonator:

$$\begin{tikzcd}
 & (ij)(kl) \arrow[dr, "\alpha"] & \\
((ij)k)l \arrow[ur, "\alpha"] \arrow[d, "\alpha"'] & \Downarrow \pi_{ijkl} & i(j(kl)) \\
(i(jk))l \arrow[r, "\alpha"'] & i((jk)l) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$



$\Downarrow$

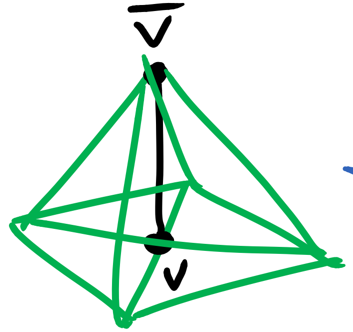
...

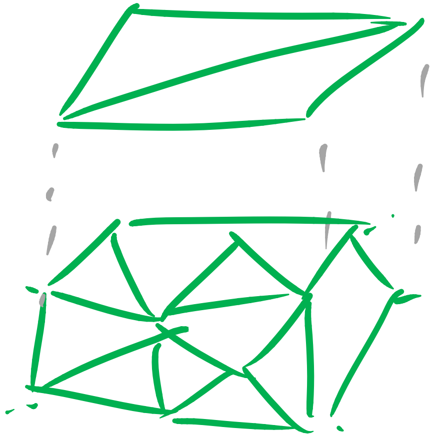
Higher Category Theory

Summary

Topological lattice models in *arbitrary* dimension

- Input: labels, “topological tensor” satisfying coherence equations
- Topological ground state degeneracy
- Tensor network, RG map

$H_r = Z($  $)$



2D: Recover all known topologically ordered models

3D: New models from tricats   Sahinoglu-W.

Many open questions: New phases from exotic ncats? Mathematical structure of the excitations? Classification?

Thank you for your attention