Schur-Weyl Duality for the Clifford Group: Property testing, *de Finetti representations*, and a robust Hudson theorem

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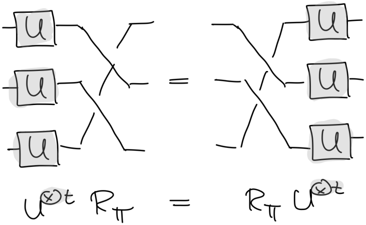
Schur-Weyl duality

$(\mathbb{C}^D)^{\otimes t}$

$$U^{\otimes t} |x_1, \ldots, x_t\rangle = U |x_1\rangle \otimes \ldots \otimes U |x_t\rangle$$
$$R_\pi |x_1, \ldots, x_t\rangle = |x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(t)}\rangle$$

Schur-Weyl duality: These actions generate each other's commutant.



Two *symmetries* that are ubiquitous in quantum information theory:

- i.i.d. quantum information: $[\rho^{\otimes t}, R_\pi] = 0$
- eigenvalues, entropies, ...: $\rho \equiv U \rho U^\dagger$
- randomized constructions: $E_{\text{Haar}}[|\psi\rangle\langle\psi|^{\otimes t}]$

See several other talks at this QIP...

Clifford unitaries and stabilizer states $\mathbb{C}^D = (\mathbb{C}^d)^{\otimes n}$

Clifford group: Unitaries U_C such that P Pauli $\Rightarrow U_C P U_C^\dagger$ Pauli.
For qubits, generated by CNOT, H, P.

Stabilizer states: States of the form $|S\rangle = U_C |0\rangle^{\otimes n}$.

These are important classes of unitaries & states:

- QEC, MBQC, topological order, randomized benchmarking, ...
- can be highly entangled, but efficient to represent and compute with
- 2-design; 3-design for qubits $\Rightarrow$ efficient random constructions

Motivates a Schur-Weyl duality for the Clifford group!

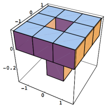
Our results

"Schur-Weyl duality" for the Clifford group: We characterize precisely which operators commute with $U_C^{\otimes t}$ for all Clifford unitaries U_C.

Fewer unitaries $\rightsquigarrow$ larger commutant (more than permutations).

Applications:

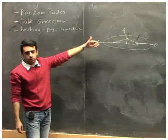
- Property testing $\quad |S\rangle^{\otimes t} \longleftrightarrow |\psi\rangle^{\otimes t}$
- De Finetti theorems with increased symmetry $\quad \Psi_s \approx \sum_S p_S \, |S\rangle\langle S|^{\otimes s}$
- Higher moments of stabilizer states $\quad E_S[|S\rangle\langle S|^{\otimes t}]$
- t-designs from Clifford orbits
- Robust Hudson theorem



Towards Schur-Weyl duality for the Clifford group

Plan:

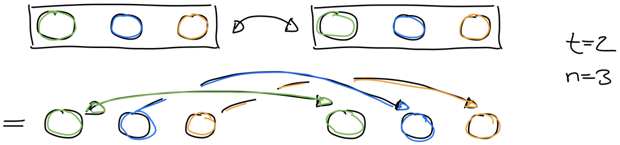
1. Write down permutation action in clever way.
2. Generalize.
3. Prove it!



Towards Schur-Weyl duality for the Clifford group

1. Write down permutation action in clever way:

Permutation of t copies of $(\mathbb{C}^d)^{\otimes n}$:



$$R_\pi = r_\pi^{\otimes n}, \quad r_\pi = \sum_{\boldsymbol{x}} |\pi \boldsymbol{x}\rangle \langle \boldsymbol{x}|$$

Here, we think of π as $t \times t$-permutation matrix, and $|\boldsymbol{x}\rangle = |x_1, \ldots, x_t\rangle$ is computational basis of $(\mathbb{C}^d)^{\otimes t}$.

Towards Schur-Weyl duality for the Clifford group

2. Generalize:

$$R_O = r_O^{\otimes n}, \quad r_O = \sum_{\boldsymbol{x}} |O\boldsymbol{x}\rangle \langle \boldsymbol{x}|$$

Allow all orthogonal and stochastic $t \times t$-matrices O with entries in $\mathbb{F}_d$.

For qubits, an example is the 6×6 anti-identity:

$$\overline{\mathrm{id}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$R_{\overline{\mathrm{id}}} |\boldsymbol{x}_1, \ldots, \boldsymbol{x}_6\rangle = |\boldsymbol{x}_2 + \ldots + \boldsymbol{x}_6, \ldots, \boldsymbol{x}_1 + \ldots + \boldsymbol{x}_5\rangle$$

The operator $R_{\overline{\mathrm{id}}}$ commutes with $U_C^{\otimes 6}$ for every n-qubit Clifford unitary.

Towards Schur-Weyl duality for the Clifford group

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Towards Schur-Weyl duality for the Clifford group

2. Generalize further:

$$R_T = r_T^{\otimes n}, \quad r_T = \sum_{(\boldsymbol{y},\boldsymbol{x})\in T} |\boldsymbol{y}\rangle\langle\boldsymbol{x}|$$

Allow all subspaces $T \subseteq \mathbb{F}_d^{2t}$ that are self-dual, i.e. $\boldsymbol{y}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot\boldsymbol{x}^\dagger$ and of dimension t, and contain $\mathbf{1} = (1,\ldots,1)$.

†For qubits, require modulo 4 ('doubly even' code).

Schur-Weyl duality for the Clifford group $(\mathbb{C}^d)^{\otimes n}$

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Result

For $n \geq t - 1$, the operators R_T are $\prod_{k=0}^{t-2}(d^k + 1)$ many linearly independent operators that span the commutant of $\{U_C^{\otimes t}\}$.

Sketch of proof: Phase space formalism. Compute cardinalities and compare. $\square$

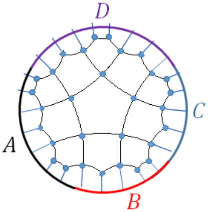
Independent of n! Rich algebraic structure (see paper).

Application 1: Higher moments of stabilizer states

Result (t-th moment)

$E[|S\rangle\langle S|^{\otimes t}] \simeq \sum_T R_T$

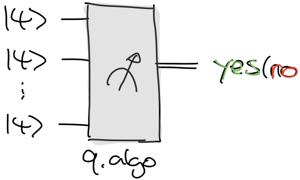
- When stabilizer states form t-design, reduces to $\sum_\pi R_\pi$ (Haar average)
- Summarizes all previous results on statistical properties
- ...but also holds for larger t!



We can also write t-th moment as weighted sum of certain CSS codes.

Application 2: Stabilizer testing

Given t copies of an unknown state in $(\mathbb{C}^d)^{\otimes n}$, decide if it is a stabilizer state or ε-far from it.



Idea: Use the anti-identity. Measure POVM element $\frac{1+R_{\overline{\mathrm{id}}}}{2}$ on $t = 6$ copies.

Result

Let ψ be a pure state of n qubits. If ψ is a stabilizer state then this accepts always. But if $\max_S |\langle\psi|S\rangle|^2 \leq 1 - \varepsilon^2$, acceptance probability $\leq 1 - \varepsilon^2/4$.

- Power of test independent of n. Answers q. by Montanaro & de Wolf.
- Similar result for qudits & for testing if blackbox unitary is Clifford.

Why does it work? How to implement?

Stabilizer testing using Bell difference sampling

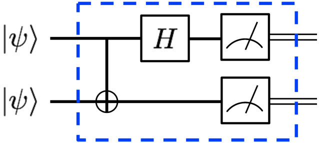
Any state ψ can be expanded in Pauli basis[†]:

$$\psi = \sum_{\boldsymbol{v}} c_\psi P_{\boldsymbol{v}}$$

- If pure, then $p_\psi(\boldsymbol{v}) = 2^n |c_\psi(\boldsymbol{v})|^2$ is a probability distribution.
- If stabilizer state, then support of p_ψ is stabilizer group (up to sign).

Key idea: Sample & verify!

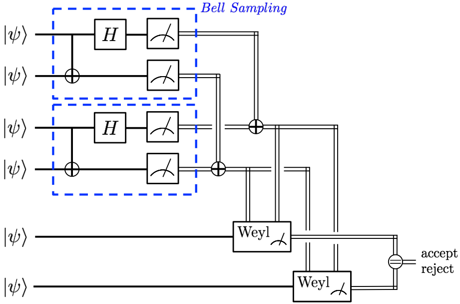
How to sample? If ψ is real, can simply measure in Bell basis $(P_{\boldsymbol{v}} \otimes I) |\Phi^+\rangle$ (Bell sampling; Montanaro, Zhao *et al*).



[†] $P_{\boldsymbol{v}} = P_{v_1} \otimes \ldots \otimes P_{v_n}$ where $P_{00} = I$, $P_{01} = X$, $P_{10} = Z$, $P_{11} = Y$

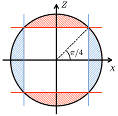
Stabilizer testing using Bell difference sampling

In general, need to take difference of two Bell measurement outcomes:



- Constant-depth circuit, only need coherent two-qubit operations.
- Circuit is equivalent to measuring the anti-identity!

Proof of converse uses uncertainty relation.



Application 3: Stabilizer de Finetti theorems

Any tensor power $|\psi\rangle^{\otimes t}$ has S_t-symmetry. De Finetti theorems provide 'partial' converse: If $|\Psi\rangle$ has S_t-symmetry, $\Psi_s \approx \int d\mu(\psi)\psi^{\otimes s}$ for $s \ll t$.

Stabilizer tensor powers have increased symmetry:

$$R_O |S\rangle^{\otimes t} = |S\rangle^{\otimes t} \qquad \text{for all orthogonal and stochastic } O$$

Result

Assume that $|\Psi\rangle \in ((\mathbb{C}^d)^{\otimes n})^{\otimes t}$ has this symmetry, $d > 2$. Then:

$$\|\Psi_s - \sum_S p_S |S\rangle\langle S|^{\otimes s}\|_1 \lesssim d^{2n(n+2)} d^{-(t-s)/2}$$

- Approximation is exponentially good, yet *bona fide* stabilizer states.
- Similar to Gaussian de Finetti (Leverrier *et al*). Applications to QKD?

Can reduce symmetry requirements at expense of approximation.

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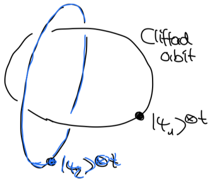
Application 4: t-designs from Clifford orbits

When $t > 2$ or 3 (qubits), stabilizer states fail to be t-design. Yet, hints in the literature that this failure is relatively *graceful* (Zhu *et al*, Nezami-W). E.g., Clifford orbit of non-stabilizer qutrit states can be 3-design!

We prove in general:

Result

For every t, there exists ensemble of $N = N(d, t)$ many fiducial states in $(\mathbb{C}^d)^{\otimes n}$ such that corresponding Clifford orbits form t-design.

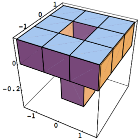


- Number of fiducials does not depend on n!
- Efficient construction?

Application 5: Robust Hudson theorem

For odd d, every quantum state has a discrete Wigner function:

$$W_\rho(\boldsymbol{v}) = d^{-2n} \sum_{\boldsymbol{w}} e^{-2\pi i [\boldsymbol{v},\boldsymbol{w}]/d} \operatorname{tr}[\rho P_{\boldsymbol{v}}]$$



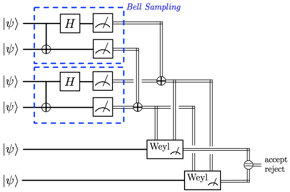
- Quasi-probability distribution on phase space $\mathbb{F}_d^{2n}$
- Discrete Hudson theorem: For pure states, $W_\psi \geq 0$ iff ψ stabilizer
- Wigner negativity $\operatorname{sn}(\psi) = \sum_{\boldsymbol{v}: W_\rho(\boldsymbol{v})<0} |W_\rho(\boldsymbol{v})|$: monotone in resource theory of stabilizer computation; witness for contextuality

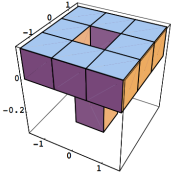
Result (Robust Hudson)

There exists a stabilizer state $|S\rangle$ such that $|\langle S|\psi\rangle|^2 \geq 1 - 9d^2 \operatorname{sn}(\psi)$.

Summary and outlook

arXiv:1712.08628





Schur-Weyl duality for the Clifford group:

- clean algebraic description in terms of self-dual codes
- resolve open question in quantum property testing
- new tools for stabilizer states: moments, de Finetti, Hudson, ...

Thank you for your attention!

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