When is a quantum state a stabilizer state?

Joint work w/ David Gross, Sepehr Nezami

① Introduction

Pauli operators: $I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $Y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\underset{00}{I}$, $\underset{01}{X}$, $\underset{10}{\overbrace{Y}^{-iZX}}$, $\underset{10}{Z}$

n qubits: $\{I, X, Y, Z\}^{\otimes n} \rightsquigarrow 4^n$ many

Stabilizer state: unique eigenvector of subgroup of Paulis

$$\left.\begin{aligned} P^{(1)} |S\rangle &= \pm |S\rangle \\ &\vdots \\ P^{(n)} |S\rangle &= \pm |S\rangle \end{aligned}\right\} \text{ n Paulis are necessary \& sufficient}$$

* Ex: $|\Phi^+\rangle = \sqrt{\frac{1}{2}}\left(|00\rangle + |11\rangle\right)$ is characterized by

$$X_1 X_2 |\Phi^+\rangle = Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle$$

* Stab. group $S = \langle P^{(1)}, \ldots, P^{(n)} \rangle$, $\#S = 2^n$ (commute)

* Can be highly entangled, but efficient to represent

↳ Gottesman-Knill

↳ cf. Gaussian states (covariance = Paulis, mean = phases)

* Applications: QEC, contextuality, meas-based QC, topo order (toric code), toy models of holography, ... estimation/benchmarking

WHY? t-design for $t \leq 3$

$$\frac{1}{Z} \sum_{S} |S\rangle\langle S|^{\otimes t} = \int d\psi \, |\psi\rangle\langle\psi|^{\otimes t}$$

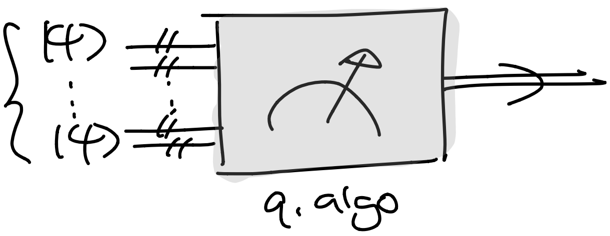
How large t? Worse, but...

RESULT: Complete theory for general t. → QIP

Not what I want to talk about... instead: New q. algo!

RESULT (informal): Q. algo that, given $|\psi\rangle^{\otimes t}$, detects if $|\psi\rangle$ is stab. state or NOT.

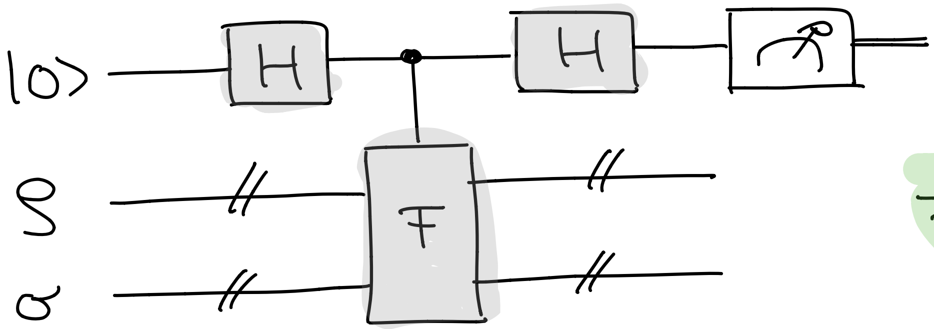
Q. property testing: $|\psi\rangle$ many-body state

t copies  yes/no

What properties can we test efficiently? (using few copies even if many qubits)

Can learn complete state from $\tilde{O}(2^n/\varepsilon^2)$ copies!

Ex: SWAP TEST



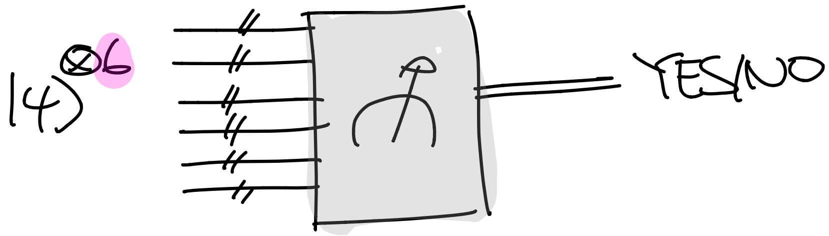
P(outcome 0)
$$= \frac{1}{2}\left(1 + \mathrm{tr}[\rho\sigma]\right)$$

↳ $t=2$ copies suffice to test EQUALITY, PURITY, ⊗, ...

Arguably only "meaningful" properties of many-body states.

But not so many properties known!

RESULT: Q. algo



S.th.

$|\psi\rangle$ Stab. state $\longrightarrow$ $\Pr(\mathrm{YES}) = 1$

$\max_S |\langle\psi|S\rangle|^2 \leq 1-\varepsilon^2 \longrightarrow \Pr(\mathrm{NO}) \geq \boxed{\frac{1}{4}\varepsilon^2}$

INDEPENT OF #QUBITS

$O(1/\varepsilon^2)$ repetitions

* resolved open question in q. complexity theory

[best prior algo: $O(n)$ copies, to learn state!
Aaronson–Gottesman, Zhao et al, Montanaro]

* new tools

* Simple + efficient: constant-depth, 2-qubit coherence only

② Intuition: Given $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$:

* Easy to check if eigenvector of some Pauli P: measure on two copies & compare result

* Need to find candidate Paulis !

Write state in Pauli basis:

$$|\psi\rangle\langle\psi| = 2^{-n/2} \sum_P \underbrace{c_\psi(P)}_{\text{characteristic function}} \cdot P$$

Since pure: $\sum_P \boxed{c_\psi^2(P)} = 1$

probability distribution $\in [0, 2^{-n}]$

* If stabilizer: $|S\rangle\langle S| = 2^{-n} \sum_{P \in S} \pm \cdot P$

Stabilizer group

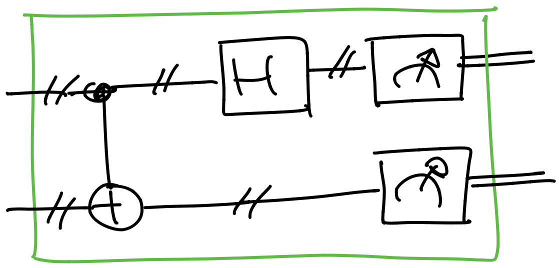
$$\hookrightarrow \boxed{c_\psi^2(P) = \begin{cases} 2^{-n} & P \in S \\ 0 & P \notin S \end{cases}}$$

maximally sparse!

Get candidate Paulis if can sample from c_ψ^2 ...

* BELL SAMPLING: measure $|\psi\rangle^{\otimes 2}$ in "Bell basis"

$$|P\rangle := (P \otimes I)|\Phi^+\rangle$$



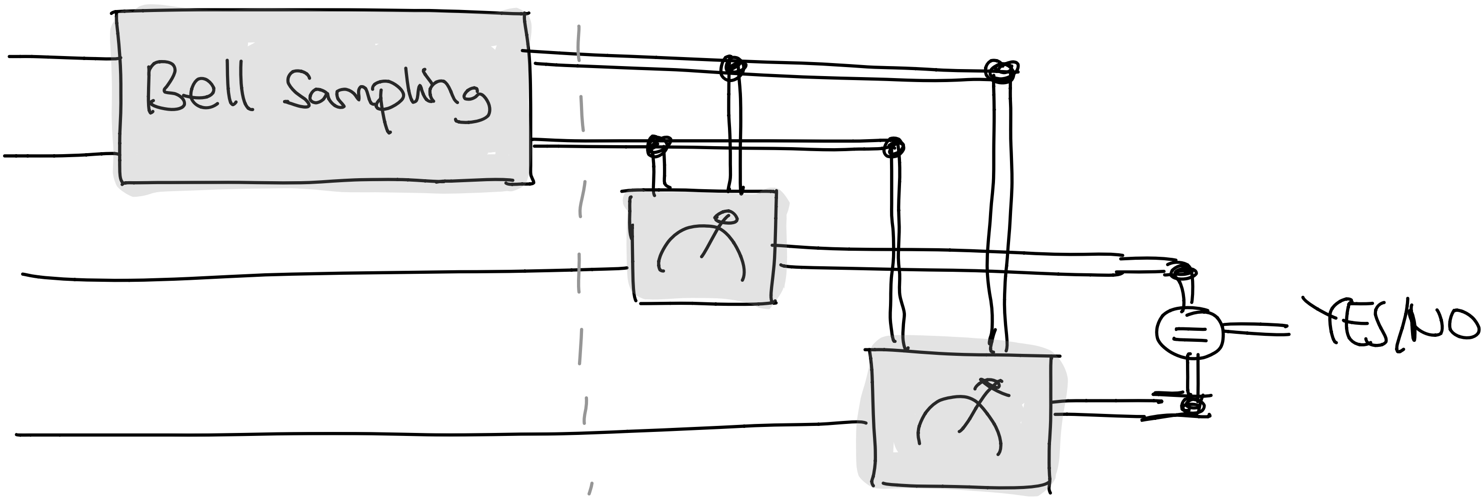
cf. teleportation

$\hookrightarrow \Pr(\text{outcome } P) = |\langle \psi^{\otimes 2} | P \rangle|^2$

$= 2^{-n} |\langle \psi | P | \bar{\psi} \rangle|^2 = c_\psi^2(P)$

if wavefunction is real

③ Algorithm (real case): need 4 copies



measure Pauli twice, accept if same outcome

* fully transversal

* $O(n)$ gates, $O(1)$ depth

Analysis: If stabilizer: Bell sampling selects uniform $P \in S$

$\hookrightarrow P|S\rangle = \pm|S\rangle \to$ get same outcome, always accept.

Converse:

$$\Pr(\text{NO}) = \sum_P c_\psi^2(P) \, \mathrm{tr}\left[\frac{I - P \otimes P}{2} |\psi\rangle\langle\psi|^{\otimes 2}\right]$$

$$= \sum_P c_\psi^2(P) \frac{1}{2}\left[1 - 2^n c_\psi^2(P)\right]$$

↳ if accept WHP: $\boxed{c_\psi^2(P) \approx 2^{-n}}$ for typical P

Approximate version of stabilizer situation... !!!

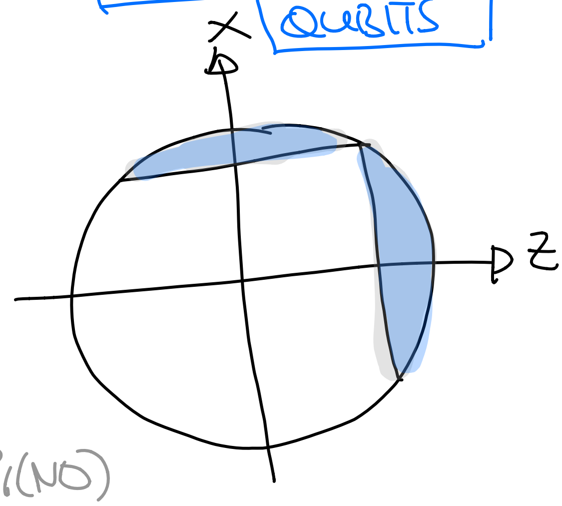
* <u>Let's guess a nearby stabilizer!</u> Define

$$S_0 := \{ P : c_\psi^2(P) \geq 2^{-n} \tfrac{3}{4} \}$$

MARKOV

$$\sum_{P \in S_0} c_\psi^2(P) \geq 1 - 8 \cdot \Pr(\text{NO})$$

UNCERTAINTY RELATION FOR QUBITS



* Paulis in S_0 commute!

↳ Stab. <u>basis</u> $\exists |S\rangle$ s.th.

$$|\langle S|\psi\rangle|^2 \geq \sum_{P \in S_0} c_\psi^2(P) \geq 1 - 8 \cdot \Pr(\text{NO})$$

* Thus: If $\max_S |\langle S|\psi\rangle|^2 \leq 1 - \varepsilon$: $\Pr(\text{NO}) \geq \frac{\varepsilon}{8}$

④ <u>General case?</u>

$|\bar{S}\rangle = P_0 \cdot |S\rangle$ for any stab.

↳ Bell Sampling result "multiplied" by P_0

$\tilde{P} = PP_0$

$\tilde{Q} = QP_0$

$\to \tilde{P}\tilde{Q}^\dagger = PQ$

↳ Sample twice & "divide"

= BELL DIFFERENCE SAMPLING, 6 copies

5 Odds & Ends

Qudits

Wigner function for qudits: $W_\psi = \hat{C}_\psi$

- Gross (2006): $W_\psi \geq 0$ iff stabilizer
- Using our methods:

simple proof

$$\sum_{W_\psi(u)<0} |W_\psi(u)| \leq \varepsilon \Rightarrow f(\varepsilon)\text{-close to stabilizer}$$

↳ mana, resource theory of magic

How we discovered our measurement...

* wanted to understand how stabilizers fail to be a t-design (i.e. solve problem on average)

$$\frac{1}{N}\sum_{\phi \text{ stab}} |\phi\rangle\langle\phi|^{\otimes t} \quad \text{vs.} \quad \int_{\text{Haar}} d\psi\, |\psi\rangle\langle\psi|^{\otimes t}$$

Commutes w/ Clifford unitaries
↳ more such operators

FULL ALGEBRAIC UNDERSTANDING

commutes w/ all unitaries
↳ span of permutations

$$\frac{1}{t!}\sum_{\pi \in S_t} R_\pi$$

* $t=6$: "anti-permutations"

$$\tilde{\pi} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{pmatrix} - \pi$$

permut. matrix

- orthogonal matrix
- stochastic

$\Big\}$ mod 4

over $\mathbb{Z}$ just permut.

↳ unitary $U_{\tilde{\pi}} |x\rangle = |\tilde{\pi} x\rangle$ on $(\mathbb{C}^2)^{\otimes t}$

↳ $U_{\tilde{\pi}}^{\otimes n}$ commutes w/ Clifford group

* <u>Fact:</u> our POVM element is exactly

$$\boxed{\frac{I^{\otimes n} + U_{\tilde{\tau}}^{\otimes n}}{2}}$$

ANTI SWAP-TEST

where $\tau = (5\ 6) \in S_6$

Any other antipermutation also works.

$$\frac{1}{2t!} \sum_{\pi} \left(U_{\pi} + U_{\tilde{\pi}} \right)$$

is exactly projector onto "$\mathrm{Stab}(n)^{\otimes 6}$".

De Finetti

* $|\psi\rangle^{\otimes t} \Longrightarrow S_t\text{-inv.}$

  $|\Psi\rangle$ S_t-inv $\Longrightarrow \Psi_s \approx \int d\psi\, p(\psi)\, |\psi\rangle\langle\psi|^{\otimes s}$

  if $s \ll t$

* $|S\rangle^{\otimes t} \Longrightarrow \tilde{O}_t\text{-inv.}$ (ortho-stoch)

  $|\Psi\rangle$ $\tilde{O}_t$-inv. $\Longrightarrow \Psi_s \underset{\text{exp}}{\approx} \sum_S p(S)\, \underbrace{|S\rangle\langle S|^{\otimes s}}_{\text{Stabs}}$

  if $s \ll t$

Appl: QKD security proofs

Commutant:

$|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_t\rangle \qquad x \in \mathbb{F}_2^t$

$R_O |x\rangle = |Ox\rangle \qquad O \in \mathbb{F}_2^{t\times t}$