An introduction to discrete phase space and Schur-Weyl duality for the Clifford group

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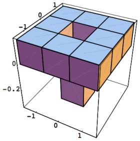
joint work with David Gross (Cologne) and Sepehr Nezami (Stanford)

Plan for today

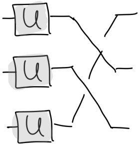
1. Introduction to discrete phase space

   Pauli & Clifford group, stabilizer states, motivation



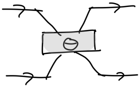
2. Schur-Weyl duality for the Clifford group

   higher moments, property testing, de Finetti, ...



Quantum optics motivation $[Q, P] = i$

Linear quantum optics described by Gaussian unitaries U_G (beam splitters, squeezing...), generate Gaussian states $|\psi\rangle = U_G |0\rangle$ (coherent, squeezed states...)

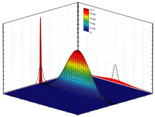


Phase space for n optical modes: $\mathbb{R}^{2n} \ni \boldsymbol{v} = (\boldsymbol{q}, \boldsymbol{p})$

- displacement operators $D_{\boldsymbol{v}} = e^{\mathrm{i}(\boldsymbol{pQ} - \boldsymbol{qP})}$
- commutation relations: $D_{\boldsymbol{v}} D_{\boldsymbol{w}} = e^{\mathrm{i}[\boldsymbol{v},\boldsymbol{w}]} D_{\boldsymbol{w}} D_{\boldsymbol{v}} \propto D_{\boldsymbol{v}+\boldsymbol{w}}$
- Gaussian unitaries act by symplectic transformations: $U_G D_{\boldsymbol{v}} U_G^\dagger \propto D_{\Gamma \boldsymbol{v}}$

Phase space distributions:

- *characteristic fn.* $\chi_\rho(\boldsymbol{v}) = \mathrm{tr}[\rho D_{\boldsymbol{v}}]$ and *Wigner function*
- Gaussian for Gaussian states $\rightsquigarrow$ mean & covariance



Highly useful – let's find a similar formalism in finite dimensions!

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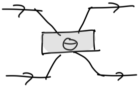

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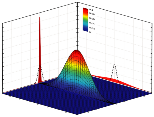

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Pauli operators and discrete phase space $(\mathbb{C}^2)^{\otimes n}$

Discrete phase space for n qubits: $\mathbb{F}_2^{2n} \ni \boldsymbol{v} = (\boldsymbol{q}, \boldsymbol{p})$.

Pauli operators:

$$P_{\boldsymbol{v}} = P_{v_1} \otimes \ldots \otimes P_{v_n} \text{ where } P_{00} = I,\ P_{01} = X,\ P_{10} = Z,\ P_{11} = Y$$

- commutation relations: $P_{\boldsymbol{v}} P_{\boldsymbol{w}} = (-1)^{[\boldsymbol{v}, \boldsymbol{w}]} P_{\boldsymbol{w}} P_{\boldsymbol{v}} \propto P_{\boldsymbol{v}+\boldsymbol{w} \pmod 2}$
- generate *Pauli group*
- orthogonal operator basis: can expand $\rho = \sum_{\boldsymbol{v}} \chi_\rho(\boldsymbol{v}) P_{\boldsymbol{v}}$, where $\chi_\rho(\boldsymbol{v}) = 2^{-n} \operatorname{tr}[\rho P_{\boldsymbol{v}}]$ *characteristic function*

Qudits: phase space $\mathbb{F}_d^{2n}$ corresponding to 'shift' and 'clock' operators:

$$X|q\rangle = |q + 1 \pmod d\rangle$$
$$Z|q\rangle = e^{2\pi i q/d} |q\rangle$$

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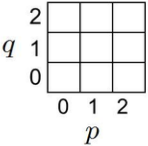


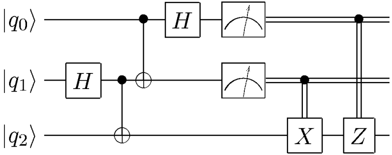
Clifford unitaries

$(\mathbb{C}^d)^{\otimes n}$

Clifford group: Unitaries U_C such that P Pauli $\Rightarrow U_C P U_C^\dagger \propto$ Pauli.
For qubits, generated by

$$\text{CNOT}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

E.g.,



Clifford unitaries = classical dynamics on discrete phase space

- for any symplectic matrix Γ, exists Clifford U_Γ with $U_\Gamma P_{\mathbf{x}} U_\Gamma^\dagger \propto P_{\Gamma \mathbf{x}}$
- conversely, any Clifford unitary is of form $U_C \propto U_\Gamma P_{\mathbf{v}}$

Clifford circuits can be simulated efficiently on a classical computer (Gottesman-Knill)

Stabilizer states

Stabilizer states: States of the form $|S\rangle = U_C |0\rangle^{\otimes n}$.

- computational basis states, maximally entangled states, GHZ states...
- QEC, MBQC, topological order, ...

Equivalently, stabilized by maximal commutative subgroup G of Paulis:

$$|S\rangle\langle S| = d^{-n} \sum_{P \in G} P$$

E.g., $|00\rangle + |11\rangle$ defined by $G = \langle XX, ZZ\rangle$.

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In terms of discrete phase space: $G = \{e^{2\pi i f(\mathbf{v})/d} P_{\mathbf{v}} \mid \mathbf{v} \in V\}$, where

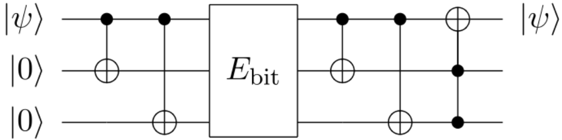
- $V \subseteq \mathbb{F}_d^{2n}$ isotropic: $[\mathbf{v}, \mathbf{w}] = 0$ for all $\mathbf{v}, \mathbf{w} \in V$
- maximal dimension: $\dim V = n$

E.g., $|00\rangle + |11\rangle$ defined by $G = \langle XX, ZZ\rangle$, $V = \langle (0011), (1100)\rangle$.

Stabilizer codes and quantum error correction

Obtain stabilizer *codes* if V isotropic but not of maximal dimension. E.g.,

- $1 \to 3$ *bit flip code* is defined by ZZI, IZZ



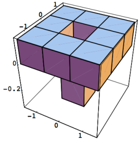
- $1 \to 5$ *qubit code* has stabilizers $XZZXI$, $IXZZX$, $XIXZZ$, $ZXIXZ$

Useful properties:

- 'stabilizers = syndrome'
- encoding and error correction circuits are Clifford
- q. error correction condition only depends on V (for Pauli errors)!

Wigner function and classical simulation

For odd d, every quantum state has a discrete Wigner function:



$$W_\rho(\boldsymbol{v}) = \hat{\chi}_\rho(\boldsymbol{v}) = d^{-2n} \sum_{\boldsymbol{w}} e^{-2\pi i [\boldsymbol{v},\boldsymbol{w}]/d} \operatorname{tr}[\rho P_{\boldsymbol{v}}]$$

- quasi-probability distribution on phase space $\mathbb{F}_d^{2n}$
- Clifford-covariant
- *discrete Hudson theorem* (Gross): for pure states, $W_\psi \geq 0$ iff stabilizer

Non-negative Wigner function $\Rightarrow$ efficient classical simulation

- Wigner negativity $\operatorname{sn}(\psi) = \sum_{\boldsymbol{v}: W_\rho(\boldsymbol{v})<0} |W_\rho(\boldsymbol{v})|$
- resource theory of stabilizer computation, contextuality, ...
- see work by Veitch *et al*, Pashayan *et al*, Raussendorf *et al*, Howard-Campbell, ...

Derandomization and designs

Randomized constructions often rely on *Haar measure*. Simple to analyze, often near-optimal – but inefficient!

A unitary t-design $\{U_j\}$ has same t-th moments as Haar measure on $U(D)$:

$$E_j[(U_j \otimes U_j^\dagger)^{\otimes t}] = E_{\text{Haar}}[(U \otimes U^\dagger)^{\otimes t}]$$

A state t-design $\{\psi_j\}$ has same t-th moments as 'Haar measure' on pure states:

$$E_j[|\psi_j\rangle\langle\psi_j|^{\otimes t}] = E_{\text{Haar}}[|\psi\rangle\langle\psi|^{\otimes t}]$$

- Clifford unitaries and stabilizer states are 2-design; 3-design for qubits (Küng-Gross, Zhu, Webb)
- many applications: randomized benchmarking, phase retrieval, low-rank matrix recovery, ...

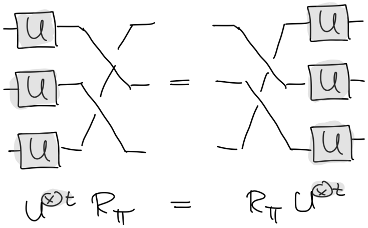
Schur-Weyl duality for the Clifford group

Schur-Weyl duality

$(\mathbb{C}^D)^{\otimes t}$

Two *symmetries* that are ubiquitous in quantum information theory:

$$U^{\otimes t} |x_1, \ldots, x_t\rangle = U |x_1\rangle \otimes \ldots \otimes U |x_t\rangle$$
$$R_\pi |x_1, \ldots, x_t\rangle = |x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(t)}\rangle$$



- i.i.d. quantum information: $[\rho^{\otimes t}, R_\pi] = 0$
- eigenvalues, entropies, ...: $\rho \equiv U \rho U^\dagger$
- randomized constructions: $E_{\text{Haar}}[|\psi\rangle\langle\psi|^{\otimes t}]$

Would like a version for Clifford unitaries U_C!

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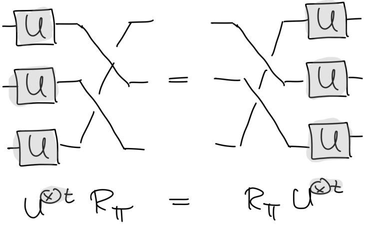

Schur-Weyl duality: These actions generate each other's commutant.

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- eigenvalues, entropies, ...: $\rho \equiv U \rho U^\dagger$
- randomized constructions: $E_{\text{Haar}}[|\psi\rangle\langle\psi|^{\otimes t}] \propto \sum_{\pi \in S_t} R_\pi$

Would like a version for Clifford unitaries U_C!

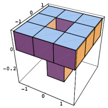
Our results

"Schur-Weyl duality" for the Clifford group: We characterize precisely which operators commute with $U_C^{\otimes t}$ for all Clifford unitaries U_C.

Fewer unitaries $\rightsquigarrow$ larger commutant (more than permutations).

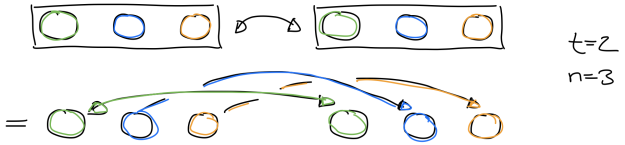
Applications:

- Property testing $\quad |S\rangle^{\otimes t} \longleftrightarrow |\psi\rangle^{\otimes t}$
- De Finetti theorems with increased symmetry $\quad \Psi_s \approx \sum_S p_S \, |S\rangle\langle S|^{\otimes s}$
- Higher moments of stabilizer states $\quad E_S[|S\rangle\langle S|^{\otimes t}]$
- t-designs from Clifford orbits
- Robust Hudson theorem



Permuting blocks

Permutation of t copies of $(\mathbb{C}^d)^{\otimes n}$:



$$R_\pi = r_\pi^{\otimes n}, \quad r_\pi = \sum_{\boldsymbol{x}} |\pi \boldsymbol{x}\rangle \langle \boldsymbol{x}|$$

Here, we think of π as $t \times t$-permutation matrix, and $|\boldsymbol{x}\rangle = |x_1, \ldots, x_t\rangle$ is computational basis of $(\mathbb{C}^d)^{\otimes t}$.

The commutant of $\{U_C^{\otimes t}\}$ is given by a straightforward generalization...

Schur-Weyl duality for the Clifford group $(\mathbb{C}^d)^{\otimes n}$

$$R_T = r_T^{\otimes n}, \quad r_T = \sum_{(\boldsymbol{y},\boldsymbol{x}) \in T} |\boldsymbol{y}\rangle \langle \boldsymbol{x}|$$

Allow all subspaces $T \subseteq \mathbb{F}_d^{2t}$ that are self-dual codes, i.e. $\boldsymbol{y} \cdot \boldsymbol{y}' \equiv \boldsymbol{x} \cdot \boldsymbol{x}'$ and of maximal dimension t. Moreover, require $|\boldsymbol{y}| \equiv |\boldsymbol{x}|$ (for qubits, modulo 4).

Theorem

For $n \geq t-1$, the operators R_T are $\prod_{k=0}^{t-2}(d^k+1)$ many linearly independent operators that span the commutant of $\{U_C^{\otimes t}\}$.

Independent of n (just like in ordinary Schur-Weyl duality)! Rich algebraic structure (see paper).

Examples of commutant

Want subspaces $T \subseteq \mathbb{F}_d^{2t}$ that are self-dual codes, i.e. $\boldsymbol{y} \cdot \boldsymbol{y}' \equiv \boldsymbol{x} \cdot \boldsymbol{x}'$ and of maximal dimension t. Moreover, require $|\boldsymbol{y}| \equiv |\boldsymbol{x}|$ (for qubits, modulo 4).

For qubits, an example is the following code for $t = 4$:

$$T = \operatorname{ran}\begin{pmatrix} 1\,1\,1\,1 & 0\,0\,0\,0 \\ 0\,0\,0\,0 & 1\,1\,1\,1 \\ 1\,1\,0\,0 & 1\,1\,0\,0 \\ 1\,0\,1\,0 & 1\,0\,1\,0 \end{pmatrix},$$
$$R_T = 2^{-n}\left(I^{\otimes 4} + X^{\otimes 4} + Y^{\otimes 4} + Z^{\otimes 4}\right)$$

The projector R_T commutes with $U_C^{\otimes 4}$ for every n-qubit Clifford unitary. Central to 4-th moments of multiqubit stabilizer states (Zhu *et al*, later).

Examples of commutant

Can also obtain subspaces as graphs $T = \{(O\boldsymbol{x}, \boldsymbol{x})\}$ of $t \times t$ orthogonal stochastic matrices. Then $R_O = r_O^{\otimes n}$, $r_O = \sum_{\boldsymbol{x}} |O\boldsymbol{x}\rangle \langle \boldsymbol{x}|$ is in commutant.

For qubits, an example is the 6×6 anti-identity:

$$\overline{\mathrm{id}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$R_{\overline{\mathrm{id}}} |\boldsymbol{x}_1, \ldots, \boldsymbol{x}_6\rangle = |\boldsymbol{x}_2 + \ldots + \boldsymbol{x}_6, \ldots, \boldsymbol{x}_1 + \ldots + \boldsymbol{x}_5\rangle$$

The unitary $R_{\overline{\mathrm{id}}}$ commutes with $U_C^{\otimes 6}$ for every n-qubit Clifford unitary.

The $\{R_O\}$ are symmetries of stabilizer tensor powers $\rightsquigarrow$ de Finetti (later).

Why should the theorem be true?

$$R_T = r_T^{\otimes n}, \quad r_T = \sum_{(\boldsymbol{y},\boldsymbol{x}) \in T} |\boldsymbol{y}\rangle \langle \boldsymbol{x}|$$

When is R_T in the commutant? Need that $T \subseteq \mathbb{F}_2^{2t}$ is...

- subspace: $\mathrm{CNOT}^{\otimes t} \, r_T^{\otimes 2} \, \mathrm{CNOT}^{\otimes t} = \sum_{(\boldsymbol{y},\boldsymbol{x}),(\boldsymbol{y}',\boldsymbol{x}') \in T} |\boldsymbol{y}\rangle\langle\boldsymbol{x}| \otimes |\boldsymbol{y}+\boldsymbol{y}'\rangle\langle\boldsymbol{x}+\boldsymbol{x}'| = r_T^{\otimes 2}$
- self-dual: $H^{\otimes t} \, r_T \, H^{\otimes t} = \sum_{(\boldsymbol{y}',\boldsymbol{x}') \in T^\perp} |\boldsymbol{y}'\rangle \langle \boldsymbol{x}'| = r_T$
- modulo 4: $P^{\otimes t} \, r_T \, P^{\dagger,\otimes t} = \sum_{(\boldsymbol{y},\boldsymbol{x}) \in T} i^{|\boldsymbol{y}|-|\boldsymbol{x}|} |\boldsymbol{y}\rangle \langle \boldsymbol{x}| = r_T$

Remainder of proof: Show that R_T's linearly independent. Compute dimension of commutant (#group orbits) & number of subspaces as above (Witt's lemma). □

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Application 1: Higher moments of stabilizer states

Result (t-th moment)

$E[|S\rangle\langle S|^{\otimes t}] \propto \sum_T R_T$

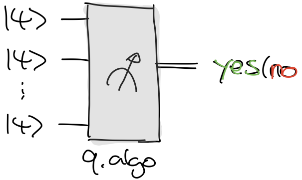
- When stabilizer states form t-design, reduces to $\sum_\pi R_\pi$ (Haar average)
- Summarizes all previous results on statistical properties
- ... but works for *any* t-th moment!

Many applications: Improved bounds for randomized benchmarking (Helsen et al, Bas' poster!), low-rank matrix recovery (Kueng et al); analytical studies of scrambling in Clifford circuits; toy models of holography (Nezami-W); ...

We can also write t-th moment as weighted sum of certain CSS codes.

Application 2: Stabilizer testing

Given t copies of an unknown state in $(\mathbb{C}^d)^{\otimes n}$, decide if it is a stabilizer state or ε-far from it.



Idea: Use the anti-identity. Measure POVM element $\frac{1+R_{\overline{\text{id}}}}{2}$ on $t = 6$ copies.

Result

Let ψ be a pure state of n qubits. If ψ is a stabilizer state then this accepts always. But if $\max_S |\langle\psi|S\rangle|^2 \leq 1 - \varepsilon^2$, acceptance probability $\leq 1 - \varepsilon^2/4$.

- Power of test independent of n. Answers q. by Montanaro & de Wolf.
- Similar result for qudits & for testing if blackbox unitary is Clifford.

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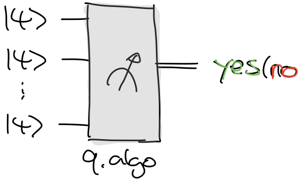

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Stabilizer testing using Bell difference sampling

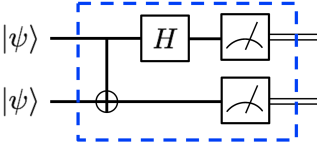
Any state ψ can be expanded in Pauli basis[†]:

$$\psi = \sum_{\boldsymbol{v}} \chi_\psi(\boldsymbol{v}) P_{\boldsymbol{v}}$$

- If pure, then $p_\psi(\boldsymbol{v}) = 2^n |\chi_\psi(\boldsymbol{v})|^2$ is a probability distribution.
- If stabilizer state, then support of p_ψ is stabilizer group (up to sign).

Key idea: Sample & verify!

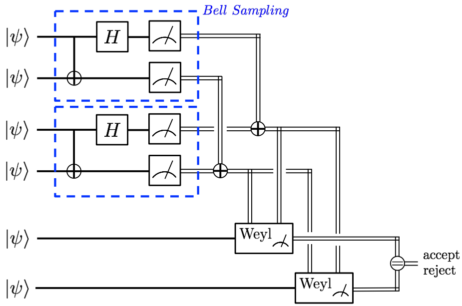
How to sample? If ψ is real, can simply measure in Bell basis $(P_{\boldsymbol{v}} \otimes I) |\Phi^+\rangle$ (Bell sampling; Montanaro, Zhao *et al*).



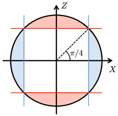
[†]recall $P_{\boldsymbol{v}} = P_{v_1} \otimes \ldots \otimes P_{v_n}$ where $P_{00} = I$, $P_{01} = X$, $P_{10} = Z$, $P_{11} = Y$

Stabilizer testing using Bell difference sampling

In general, need to take 'difference' of two Bell measurement outcomes:



- Fully transversal circuit, only need coherent two-qubit operations.
- Circuit is equivalent to measuring the anti-identity!



Proof of converse uses uncertainty relation.

How to test stabilizer rank?

Application 3: Stabilizer de Finetti theorems

Any tensor power $|\psi\rangle^{\otimes t}$ has S_t-symmetry. De Finetti theorems provide 'partial' converse: If $|\Psi\rangle$ has S_t-symmetry, $\Psi_s \approx \int d\mu(\psi)\psi^{\otimes s}$ for $s \ll t$.

Stabilizer tensor powers have increased symmetry:

$$R_O |S\rangle^{\otimes t} = |S\rangle^{\otimes t} \qquad \text{for all orthogonal and stochastic } O$$

Result

Assume that $|\Psi\rangle \in ((\mathbb{C}^d)^{\otimes n})^{\otimes t}$ has this symmetry. Then:

$$\|\Psi_s - \sum_S p_S |S\rangle\langle S|^{\otimes s}\|_1 \lesssim d^{2n(n+2)} d^{-(t-s)/2}$$

- Approximation is exponentially good, by *bona fide* stabilizer states.
- Similar to Gaussian de Finetti (Leverrier *et al*). Applications to QKD?

Can reduce symmetry requirements at expense of goodness.

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Application 4: t-designs from Clifford orbits

When $t > 2$ or 3 (qubits), stabilizer states fail to be t-design. Yet, hints in the literature that this failure is relatively *graceful* (Zhu *et al*, Nezami-W). E.g., Clifford orbit of non-stabilizer qutrit states can be 3-design!

We prove in general:

Result

For every t, there exists ensemble of $N = N(d, t)$ many fiducial states in $(\mathbb{C}^d)^{\otimes n}$ such that corresponding Clifford orbits form t-design.

- Number of fiducials does not depend on n!
- Efficient construction?

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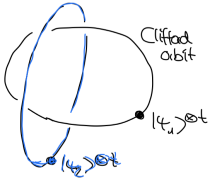
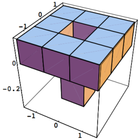


Application 5: Robust Hudson theorem

Recall: For odd d, every quantum state has a discrete Wigner function:

$$W_\rho(\boldsymbol{v}) = d^{-2n} \sum_{\boldsymbol{w}} e^{-2\pi i[\boldsymbol{v},\boldsymbol{w}]/d} \operatorname{tr}[\rho P_{\boldsymbol{v}}]$$



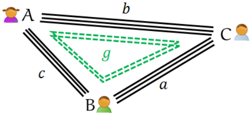
- Quasi-probability distribution on phase space $\mathbb{F}_d^{2n}$
- Discrete Hudson theorem: For pure states, $W_\psi \geq 0$ iff ψ stabilizer
- Wigner negativity $\operatorname{sn}(\psi) = \sum_{\boldsymbol{v}:W_\rho(\boldsymbol{v})<0} |W_\rho(\boldsymbol{v})|$: monotone in resource theory of stabilizer computation; witness for contextuality

Result (Robust Hudson)

There exists a stabilizer state $|S\rangle$ such that $|\langle S|\psi\rangle|^2 \geq 1 - 9d^2 \operatorname{sn}(\psi)$.

Application 6: Typical entanglement of stabilizer states

Tripartite stabilizer states decompose into EPR and GHZ entanglement:



How about typical stabilizer states? *Or even tensor networks?*

Result (Nezami-W)

In random stabilizer tensor network states: $g = O(1)$ *w.h.p.*

- can distill $\simeq \frac{1}{2} I(A:B)$ EPR pairs
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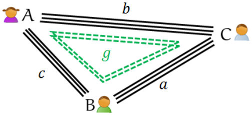

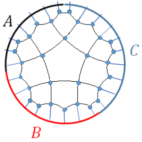

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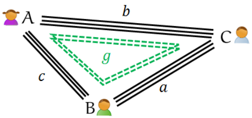

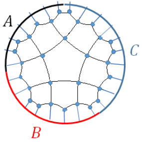

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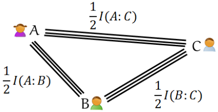
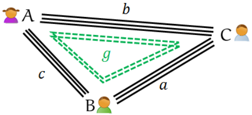


Bounding the amount of GHZ entanglement



$$I(A:B) = 2c + g$$

Diagnose via third moment of *partial transpose*:

$$g \log d = S(A) + S(B) + S(C) + \log \operatorname{tr}(\rho_{AB}^{T_B})^3$$

Compute via *replica trick*: For single stabilizer state

$$\operatorname{tr}(\rho_{AB}^{T_B})^3 = \operatorname{tr} |S\rangle\langle S|_{ABC}^{\otimes 3} \left(R_{\zeta,A} \otimes R_{\zeta^{-1},B} \otimes R_{\mathrm{id},C} \right)$$

where $\zeta = (1\ 2\ 3)$ three-cycle, hence

$$\mathbb{E}[\operatorname{tr}(\rho_{AB}^{T_B})^3] \propto \sum_T \left(\operatorname{tr} r_T r_\zeta\right)^{n_A} \left(\operatorname{tr} r_T r_{\zeta^{-1}}\right)^{n_B} \left(\operatorname{tr} r_T r_{\mathrm{id}}\right)^{n_C}$$

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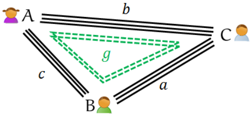

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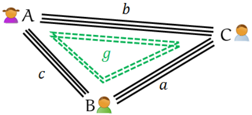

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$$\mathbb{E}[g] \leq 3n + \log \mathbb{E}[\mathrm{tr}(\rho_{AB}^{T_B})^3]$$

Since qubit stabilizers are three-design:

$$\mathbb{E}[\mathrm{tr}(\rho_{AB}^{T_B})^3] = \sum_{\pi \in S_3} 2^{-n\left(d(\zeta,\pi) + d(\zeta^{-1},\pi) + d(\mathrm{id},\pi)\right)}$$

where $d(\pi,\tau)$ = minimum number of swaps needed for $\pi \leftrightarrow \tau$. Thus:

$$\mathbb{E}[\mathrm{tr}(\rho_{AB}^{T_B})^3] \leq 3 \cdot \underbrace{2^{-3n}}_{\text{swaps}} + 3 \cdot \underbrace{2^{-4n}}_{\mathrm{id},\zeta,\zeta^{-1}} \quad \Rightarrow \quad \mathbb{E}[g] \lesssim \log 3 \quad \square$$

For $d > 2$, $\{T\} = \{even\} \cup \{odd\}$. Calculation completely analogous!

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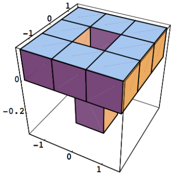
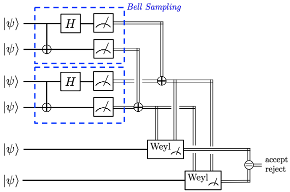
Summary and outlook

arXiv:1712.08628





Pauli & Clifford unitaries, stabilizer states in $(\mathbb{C}^d)^{\otimes n}$:

- best understood via discrete phase space $\mathbb{F}_d^{2n}$

Schur-Weyl duality for the Clifford group:

- clean algebraic description in terms of self-dual codes
- resolve open question in quantum property testing
- new tools for stabilizer states: moments, de Finetti, Hudson, ...

Thank you for your attention!