University of Amsterdam

QuSoft
Research Center for Quantum Software

Quantum Information

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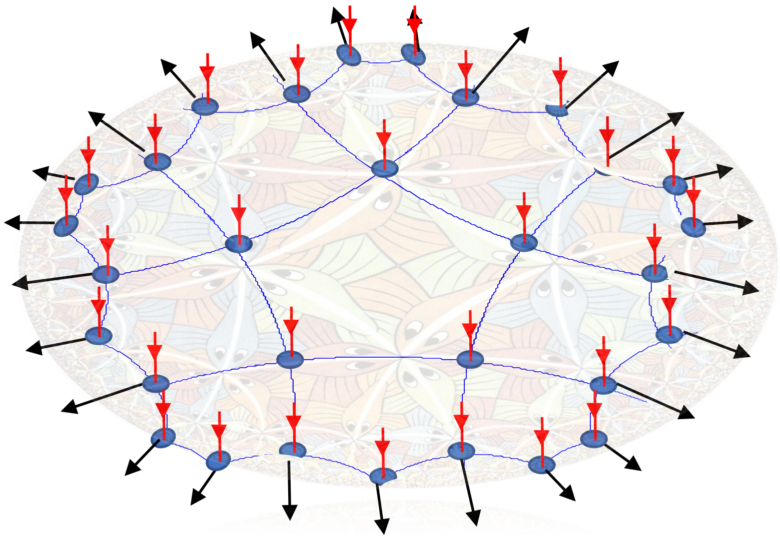


Solvay School 2021, Amsterdam

Seek to leverage laws of QM for information processing...

communication

cryptography

networks

algorithms

quantum bits

computation

complexity

Quantum Information

entropy

entanglement

error correction

tensor networks

quantum simulation

...but also toolbox and language for studying q. many-body systems.

Physics vs Information: Thermodynamics

Irreversibility (2nd law) vs coarse graining

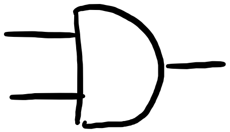
Boltzmann, Gibbs, ...

Thermodynamics of computation: Cost of erasing a bit?

W >= kT ln(2)

Landauer

Most logic gates are irreversible. Is there a fundamental cost to computing? No!



Bennett (1973):

Efficient reversible computing is possible!

Physics vs Information: Computation

Simulating quantum physics difficult for classical computers.

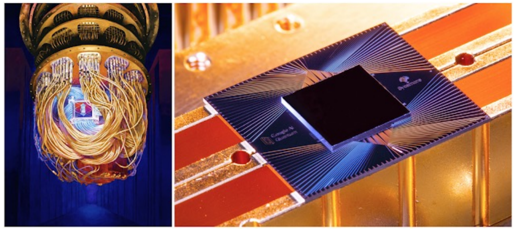
Hilbert space is exponentially large

Why don't we build a quantum computer?

Feynman, Deutsch, …

Shor's algorithm (1984): quantum computers may offer vast speedups for classical problems

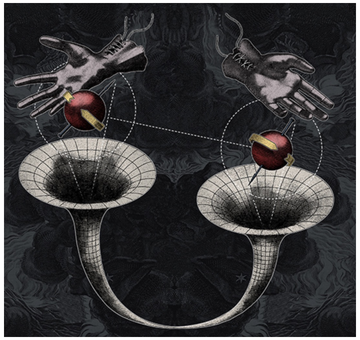
N = pq in time poly(log N)



Google "quantum supremacy" experiment (2019)

Today, quantum simulation still one of most promising applications.

Physics vs Information: Language and Toolbox



Quantum information is different: No cloning, uncertainty principle, Bell violations, entanglement, decoherence, ...

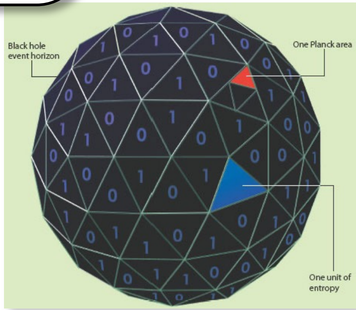
QIT offers language and toolbox to study and exploit these phenomena. Examples:

Uncertainty principle ➔ quantum cryptography

Bell violations ➔ device-independent control

Entanglement ➔ many-body physics

In recent years, exciting research at interface of quantum information with QFT and gravity.



Plan

Goal: Discuss language, toolbox, key concepts of quantum information. Survey applications to holography.

Today: States, Channels, Entropy, Entanglement

~~Entanglement in Mixed States~~

Entanglement in QFT & Holography

Tue: Toy Models of Holography, Tensor Network Models, Decoupling, Black Holes, Error Correction

Exercises and open problems throughout

Interrupt me!

If too slow (or too fast), please let me know. ☺

If not detailed enough, please ask. ☺

1. States, Channels, Entropy

Literature: Lectures Notes "Quantum Information Theory"
(https://staff.fnwi.uva.nl/m.walter/qit20/)

Quantum states

Density operators on Hilbert space:

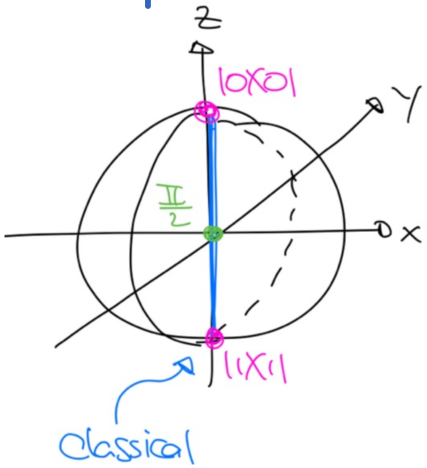
$$\rho = \sum_x \overset{\text{eigenvalues}}{p_x} \, |\underset{\text{eigenvectors}}{\psi_x}\rangle\langle\psi_x|$$

Pure states: $\rho = |\psi\rangle\langle\psi|$

Mixed states model ensembles $\{p_i, \rho_i\}$:

$$\rho = \sum_i p_i \, \rho_i$$

States of qubit: Bloch ball



Entropy

$\rho = \Sigma_x \, p_x \, |\Psi_x\rangle\langle\Psi_x|$

Von Neumann entropy: $\boxed{S(\rho) = -\mathrm{tr}\, \rho \log \rho}$ $= -\Sigma_x \, p_x \log p_x$

only depends on nonzero eigenvalues: $S(\rho) = S(U\rho U^\dagger)$

$$\boxed{0 \leq S(\rho) \leq \log(d)}$$

pure $\qquad$ $\rho = I/d$

Modular Hamiltonian:

$$\boxed{K_\rho = -\log \rho}$$

state-dependent, often nonlocal

"First law of entanglement"

$$\boxed{S(\rho + \delta\rho) = S(\rho) + \mathrm{tr}[\delta\rho \, K_\rho] + \ldots}$$

Proof? Homework!

Renyi entropies and replica trick

Von Neumann entropy often difficult to compute ➔ Renyi entropies:

$$\boxed{S_n(\rho) = \frac{1}{1-n} \log \mathrm{tr}[\rho^n]} \quad = (1-n)^{-1} \log \Sigma_x\, p_x{}^n$$

$S_0(\rho) = \log$ #nonzero eigenvalues
$S_1(\rho) = S(\rho)$
$S_2(\rho) = -\log \mathrm{tr}[\rho^2]$

equal if ρ flat spectrum

$$\boxed{\log(d) \geq S_0(\rho) \geq S(\rho) \geq S_2(\rho) \geq \ldots \geq 0}$$

Easy to calculate for integer n>1:

$\boxed{\mathrm{tr}[\rho^2] = \mathrm{tr}[\rho^{\otimes 2} F]}$ where $\boxed{F\,|xy\rangle = |yx\rangle}$ swap trick

$\boxed{\mathrm{tr}[\rho^n] = \mathrm{tr}[\rho^{\otimes n} C_n]}$ where $\boxed{C_n\,|x_1 x_2 \ldots\rangle = |x_2 x_3 \ldots x_1\rangle}$

Proof? Just expand it.

Joint systems

Reduced states of global states ρ_{AB} are given by partial trace:

$$\boxed{\rho_A = \mathrm{tr}_B(\rho_{AB})}$$

$$\boxed{\langle a|\rho_A|a'\rangle = \sum_b \langle ab|\rho_{AB}|a'b\rangle}$$

$$\rightarrow \langle O_A\rangle_{\rho_A} = \langle O_A\rangle_{\rho_{AB}}$$

Maximally entangled state (Bell/EPR pair):

$$\boxed{|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$$

$$\Longrightarrow \rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$\Longrightarrow \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{I}{2}$$ maximally mixed

Thus, pure states often have mixed reduced states. Conversely:

$$\boxed{\text{Any state } \rho_A \text{ has a purification } \rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|.}$$

Correlations

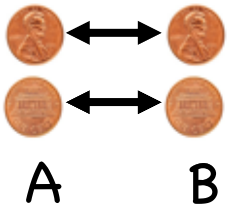
We say that a state is correlated if not a product:

$$\boxed{\rho_{AB} \neq \rho_A \otimes \rho_B}$$

$$\boxed{\langle O_A O'_B \rangle \neq \langle O_A \rangle \langle O'_B \rangle}$$

for some pair of observables

Correlations can have quantum or classical origin:



Maximally entangled state:

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Max. classically correlated:

$$\gamma_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

In both cases, $\rho_A = \rho_B = I/2$, but $\rho_{AB} \neq I/4$.

How to quantify correlations?

Mutual information

Mutual information: $\boxed{I(A{:}B) = S(A) + S(B) - S(AB)}$ ≥ 0

$= 0$ iff product

$I(A{:}B) = 2\log(d)$ iff maximally entangled

$I(A{:}B) = \log(d)$ if max. classical correlated

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_x |xx\rangle$$

$$\gamma_{AB} = \frac{1}{d} \sum_x |xx\rangle\langle xx|$$

Pinsker's inequality bounds correlation functions:

$$\boxed{\left|\langle O_A O'_B\rangle - \langle O_A\rangle\langle O'_B\rangle\right| \leq \|O_A\|\,\|O'_B\|\sqrt{2\ln(2)\ I(A:B)}}$$

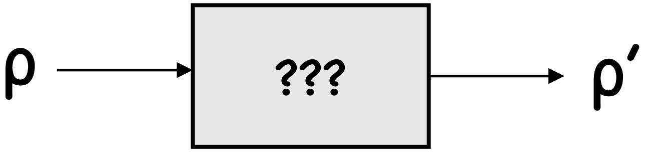
Strong subadditivity (SSA): $\boxed{I(A{:}BC) \geq I(A{:}B)}$

never more correlated with subsystem

Fundamental, intuitive, difficult to prove.

Quantum channels

What are the most general transformation of quantum states?



$$\rho \longrightarrow \boxed{???} \longrightarrow \rho'$$

$\rho \rightarrow U\rho U^\dagger$

$\rho \rightarrow \rho \otimes \sigma$

$\rho_{AB} \rightarrow \rho_A$

Quantum channel: Any combination of unitary evolution, partial traces, adding auxiliary systems.

Mathematically: Completely positive trace-preserving maps.

Data processing inequality:

$$\boxed{I(A:B) \geq I(A':B')}$$

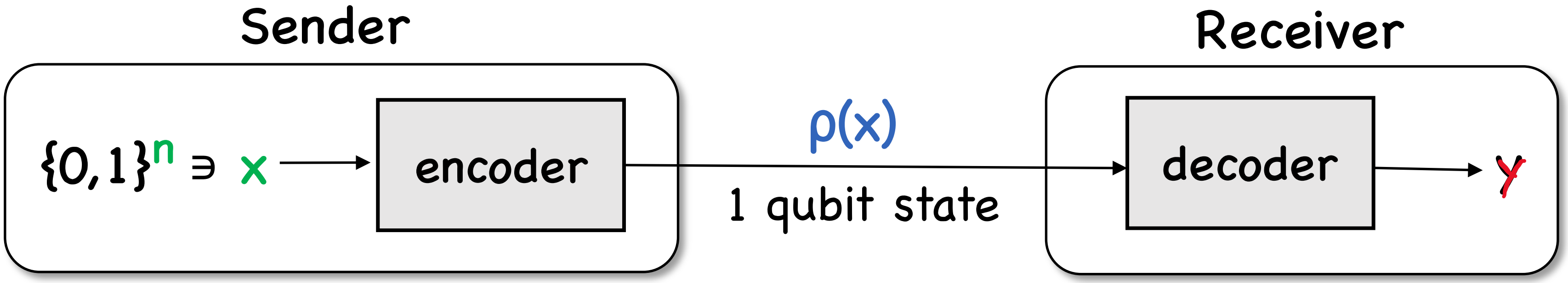
...if $\rho_{A'B'}$ obtained from ρ_{AB} by quantum channels $A \to A'$, $B \to B'$.

Homework: Prove this using SSA.

Application: Holevo bound

continuous
state space!?

How many bits can we communicate by sending 1 qubit?



Challenge: Do not know optimal states nor optimal decoder!

$$\boxed{\rho_{XB} = 2^{-n} \sum_x |x\rangle\langle x| \otimes \rho(x)} \Rightarrow \boxed{\rho_{XY} = 2^{-n} \sum_x |xx\rangle\langle xx|}$$

…if can decode perfectly. Using the data processing inequality:

$$\boxed{n = I(X:Y) \leq I(X:B) = S(B) - \sum_x p_x S(\rho(x)) \leq \log 2 = 1}$$

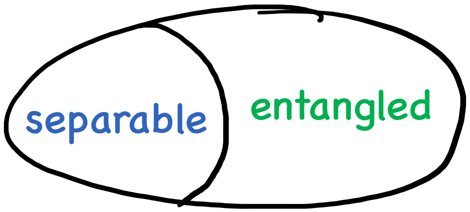
Homework: Verify this.

1 bit/qubit ➔ no quantum advantage!

2. Entanglement

Literature: Lectures on “Symmetry and Quantum Information”
(https://staff.fnwi.uva.nl/m.walter/qit18/)

Entanglement



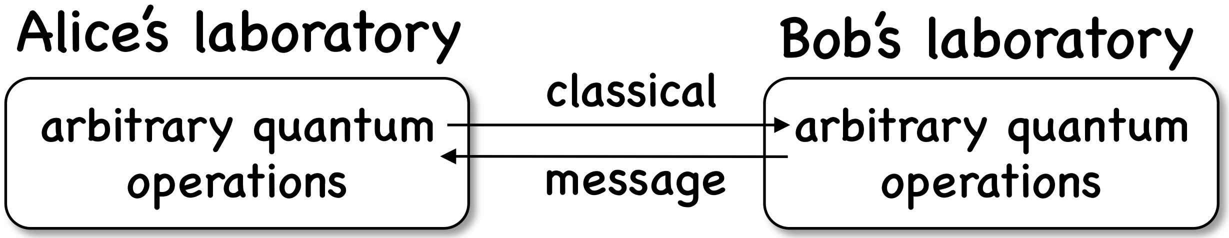
We say that a state is separable if mixture of product states:

$$\boxed{\rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}}$$

Motivation: classical correlations ≠ entanglement

Otherwise, the state is called entangled.

Separable states are precisely those that can be created by Local Operations and Classical Communication (LOCC).



Homework: Show this.

That is, to create entanglement need to send quantum systems.

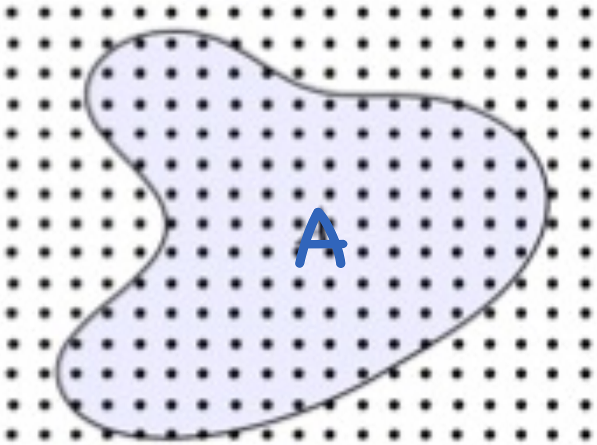
Entanglement in pure states

For pure states, the situation simplifies.

$|\Psi_{AB}\rangle$ is entangled if not a product:

$$|\Psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\phi_B\rangle$$

That is, all correlations in pure states boil down to entanglement.



[Headrick]

$B=A^c$

Schmidt decomposition

= SVD

Schmidt rank

orthogonal

$$|\Psi_{AB}\rangle = \sum_{i=1}^{r} s_i \, |e_i\rangle \otimes |f_i\rangle$$

Schmidt coefficients, >0

orthogonal

$$\rho_A = \sum_{i=1}^{r} s_i^2 \, |e_i\rangle\langle e_i|$$

$$\rho_B = \sum_{i=1}^{r} s_i^2 \, |f_i\rangle\langle f_i|$$

➔ Reduced states have same eigenvalues, entropies, … and characterize entanglement:

$|\Psi_{AB}\rangle$ product ⇔ $r = 1$ ⇔ ρ_A pure ⇔ ρ_B pure

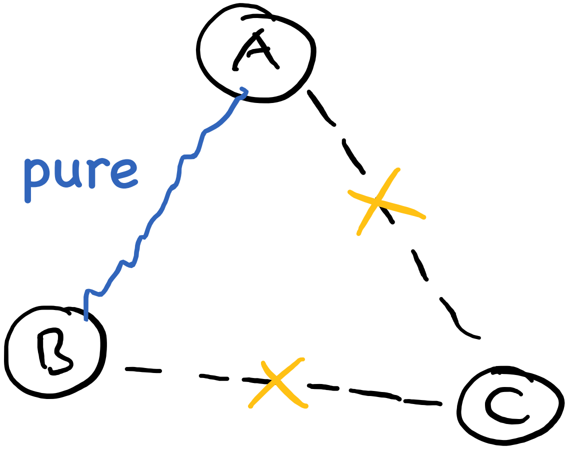
➔ Any two purifications of ρ_A are related by isometry on B

Extensions and Monogamy

Even if ρ_{AB} mixed: $\boxed{\rho_A \text{ pure} \rightarrow \rho_{AB} = \rho_A \otimes \rho_B}$

Take purification $|\Psi_{ABC}\rangle$ of ρ_{AB}. Since ρ_A pure, $|\Psi_{ABC}\rangle = |\Psi_A\rangle \otimes |\Psi_{BC}\rangle$.

This implies that pure state entanglement is monogamous:



AB pure → AB uncorrelated with C

Monogamy: AB and AC cannot both be pure entangled.

In contrast, classical correlations can be arbitrarily shared.

Entanglement entropy

Schmidt decomposition suggests to quantify entanglement by the entropy of reduced states ➔ Entanglement entropy:

$$0 \leq \boxed{S_E = S(A) = S(B)} \begin{array}{l} \leq \log d_A \\ \leq \log d_B \end{array}$$

product state

maximally entangled

Interpretation: Optimal conversion rate with Bell pairs:

$$\boxed{|\Psi_{AB}\rangle^{\otimes n} \xleftrightarrow{\text{LOCC}} (|00\rangle + |11\rangle)^{\otimes S_E n}}$$

n→∞ copies
error → 0

➔ entanglement transformations "reversible"

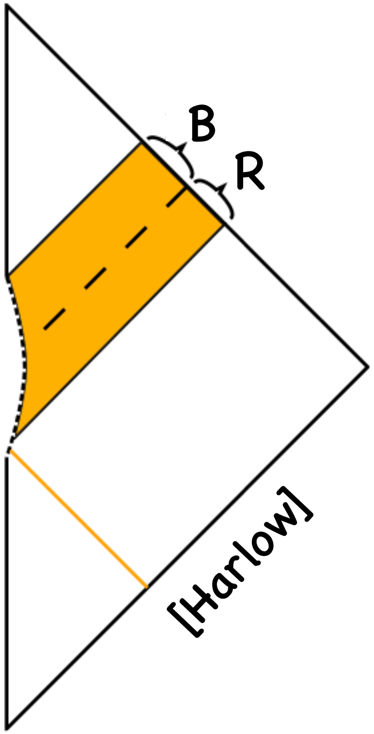
➔ Bell pairs = unit of entanglement

} for pure states

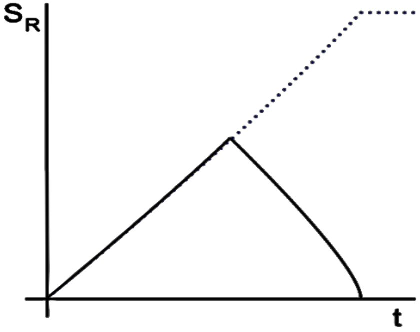
Application: Page curve

Suppose a black hole is created from infalling matter and we watch it evaporate.

R = Hawking radiation emitted up to some time
B = black hole = later Hawking radiation

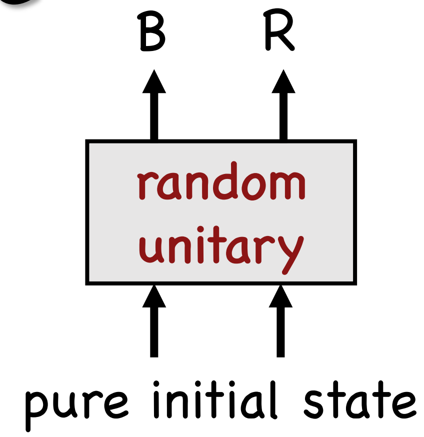


A semiclassical calculation suggests entropy of radiation increases until the end. But in a unitary theory, radiation will be pure once BH has evaporated...



Intuitively, early radiation is entangled with black hole, while late radiation is entangled with early radiation.

Application: Page curve



pure initial state

Simplest toy model: Assume that evaporation described by random unitary evolution.

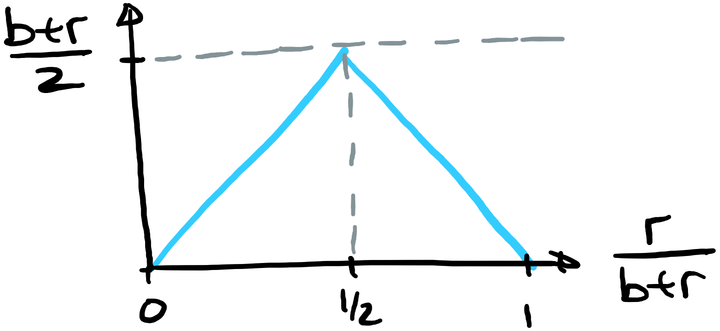
$|\Psi_{BR}\rangle$ = random pure state

$b = \log d_B$
$r = \log d_R$

Page theorem: For typical states,

$$S_E = \min(b,r) - O(1)$$

almost maximal!



It would be more physical to consider a random state in a fixed total energy subspace or a random Hamiltonian evolution.

Derivation of the Page formula

Idea: Lower-bound average Renyi-2 entropy $S_2(R)$ using swap trick.

Key formula: $\boxed{\overline{\psi^{\otimes 2}} = \frac{I+F}{d(d+1)}}$ for random ψ=|ψ><ψ|

Apply this to |ψ> = $|\psi_{BR}>$:

$$\overline{\psi_{BR}^{\otimes 2}} = \frac{I_{BB} \otimes I_{RR} + F_{BB} \otimes F_{RR}}{d_B d_R (d_B d_R + 1)}$$

$$\Rightarrow \quad \overline{\operatorname{tr} \psi_R^2} = \operatorname{tr} \overline{\psi_{BR}^{\otimes 2}} F_{RR} \leq \frac{\operatorname{tr}\left(I_{BB} \otimes F_{RR} + F_{BB} \otimes I_{RR}\right)}{d_B^2 d_R^2} = \frac{1}{d_R} + \frac{1}{d_B}$$

swap trick

$$\Rightarrow \quad \overline{S_2(R)} \geq -\log \overline{\operatorname{tr} \psi_R^2} \geq -\log\left(\frac{1}{d_R} + \frac{1}{d_B}\right) \geq \min(b, r) - 1$$

Jensen inequality

Homework: Verify this.

Entanglement as a resource

What is entanglement good for? Four examples where entanglement enables otherwise impossible capabilities:

1) Superdense coding: communicate 2 bits by sending 1 qubit

   Holevo bound shows that impossible w/o entanglement

2) Teleportation: communicate 1 qubit by sending 2 bits

3) Violating Bell inequalities: produce non-classical correlations

4) Quantum cryptography: distill a shared secret key

It is also necessary for any quantum computational speedup.

Superdense coding

[q→q] + [ebit] ≥ 2[c→c]

If Alice and Bob share EPR pair, they can use it to communicate 2 bits by sending 1 qubit!

“beating” the Holevo bound!

$$|\Phi_{AB}^{(00)}\rangle = (|00\rangle + |11\rangle)/\sqrt{2} = (I \otimes I)|\Phi_{AB}^{+}\rangle$$

$$|\Phi_{AB}^{(01)}\rangle = (|00\rangle - |11\rangle)/\sqrt{2} = (Z \otimes I)|\Phi_{AB}^{+}\rangle$$

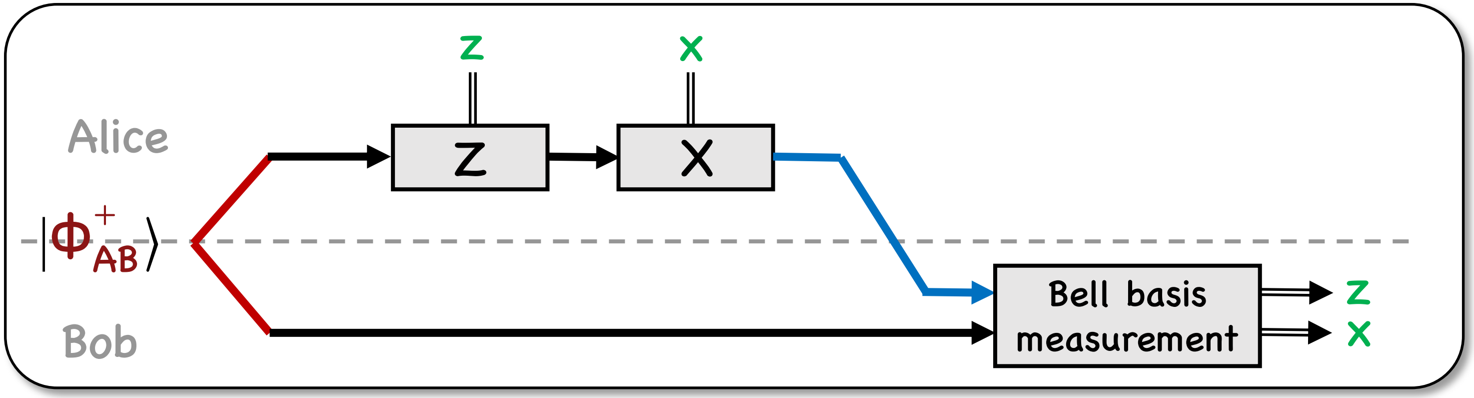
$$|\Phi_{AB}^{(10)}\rangle = (|10\rangle + |01\rangle)/\sqrt{2} = (X \otimes I)|\Phi_{AB}^{+}\rangle$$

$$|\Phi_{AB}^{(11)}\rangle = (|10\rangle - |01\rangle)/\sqrt{2} = (XZ \otimes I)|\Phi_{AB}^{+}\rangle$$

“Bell basis”

4 orthogonal states

created by local operation

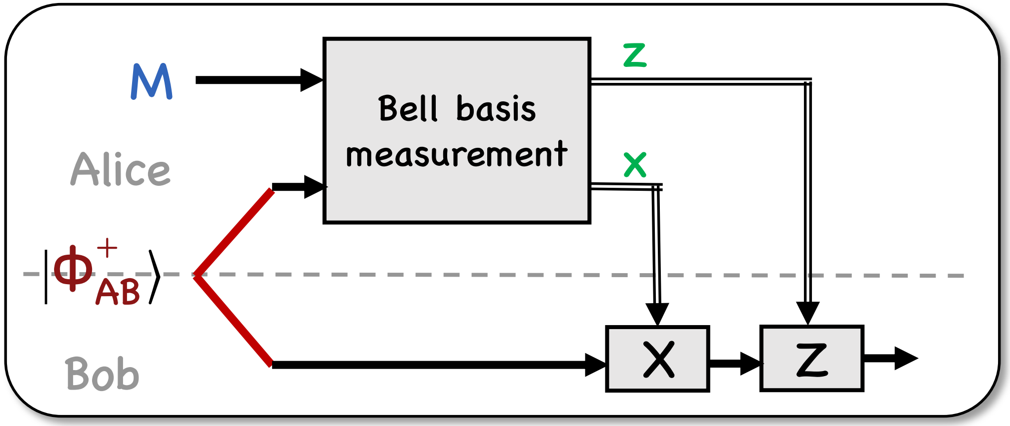


Teleportation

2[c→c] + [ebit] ≥ [q→q]

If Alice and Bob share EPR pair, they can use it to communicate 1 qubit by sending 2 bits!

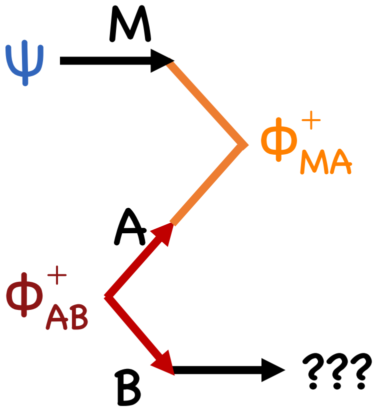
#qubit states = ∞!



x, z completely random

➔ Alice learns nothing!

Why does it work? If outcome x=z=0, post-measurement state:



$$= \left(\left(\langle\Phi^+_{MA}| \otimes I_B\right)\left(|\Psi_M\rangle \otimes |\Phi^+_{AB}\rangle\right)\right)$$

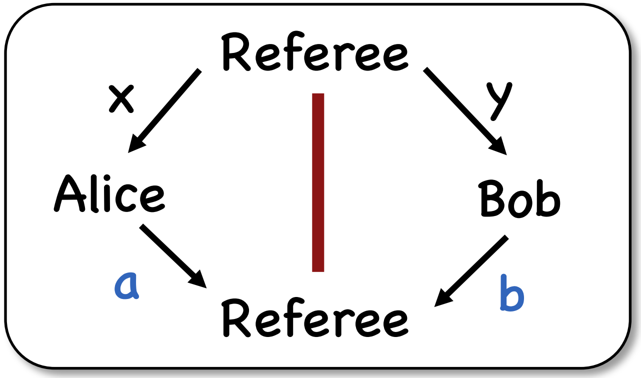
$$= \frac{1}{2}\sum_{j,k}\left(\left(\langle j|_M \otimes \langle j|_A \otimes I_B\right)\left(|\Psi_M\rangle \otimes |k\rangle_A \otimes |k\rangle_B\right)\right.$$

$$= \frac{1}{2} I_{M\to B}|\Psi_M\rangle = \frac{1}{2}|\Psi_B\rangle$$

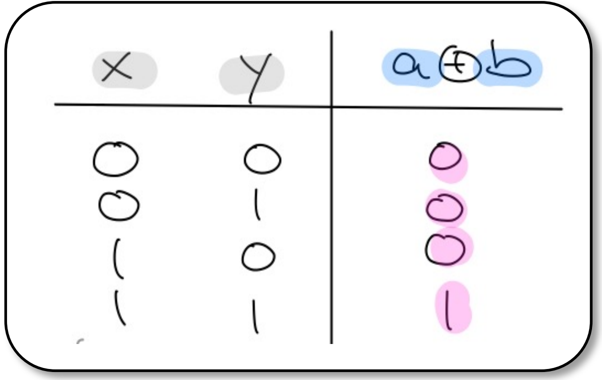
Nonlocal correlations

Clauser-Horne-
Shimony-Holt

Alice and Bob play CHSH game:



winning condition:



x	y	a⊕b
0	0	0
0	1	0
1	0	0
1	1	1

Local classical strategy: $a=a(x)$, $b=b(y)$

$$a(0)\oplus b(0) \oplus a(0)\oplus b(1) \oplus a(1)\oplus b(0) \oplus a(1)\oplus b(1) \equiv 0$$

➔ will get at least one answer wrong: $\boxed{p_{\mathrm{win}} \leq \frac{3}{4}}$

shared randomness does not help

This is a Bell inequality – a bound on classical correlations!

Nonlocality and quantum cryptography

If Alice and Bob share EPR pair, they can do better and achieve

$$\boxed{p_{win,q} = 0.85..}$$

Tsirelson: optimal

strategy “unique” (rigidity)

➔ can certify entanglement from correlations alone!

Application: In quantum key distribution, Alice and Bob want to create a key secret from everyone else.

1) Play nonlocal game to ensure that state $|\Phi^+_{AB}\rangle$ by rigidity

2) Then $|\Psi_{ABE}\rangle = |\Phi^+_{AB}\rangle \otimes |\psi_E\rangle$ by monogamy

3) Now measure EPR pair to get random secret bit.

Very rough sketch!

3. Entanglement in Mixed States

Literature: Lecture notes “Symmetry and Quantum Information”,
https://staff.fnwi.uva.nl/m.walter/qit18/

Entanglement in mixed states

Recall that a state is separable if mixture of product states:

$$\rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$$

not canonical, typically non-orthogonal ↯

Bad news: NP-hard to check if ρ_{AB} separable

➔ no entanglement measure is faithful <u>and</u> easy to compute

A practical problem – meaningful calculations are difficult.

Similarly, multipartite entanglement. ρ_{AB} vs purification $|\Psi_{ABC}\rangle$

Bound entanglement

Can create any entangled state by LOCC given enough Bell pairs.

teleportation

Bad news: Transformation usually irreversible.

$$|\Psi_{AB}\rangle^{\otimes n} \xleftrightarrow{\text{LOCC}} (|00\rangle + |11\rangle)^{\otimes m}$$

conversion rates not equal ↯

There even exist "bound entangled" states such that no Bell pairs can be obtained from any number of copies!

➔ Zoo of entanglement measures: entanglement cost E_C, distillable entanglement E_D, ...

with different interpretations

Yet there are some practically useful criteria...

PPT criterion

Idea: Necessary for separability ⇔ sufficient for entanglement

Partial transpose (PT): $\boxed{\langle ab|\rho_{AB}^{\Gamma}|a'b'\rangle = \langle ab'|\rho_{AB}|a'b\rangle}$

"partial time reverse"

If ρ_{AB} separable then ρ_{AB}^{Γ} is again a density operator.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \Rightarrow \quad \rho_{AB}^{\Gamma} = \sum_i p_i \rho_A^{(i)} \otimes (\rho_B^{(i)})^T$$

PPT criterion: $\boxed{\rho_{AB}^{\Gamma} \text{ negative eigenvalues} \rightarrow \rho_{AB} \text{ entangled}}$

e.g. $|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & & & 0 \\ \hline 0 & & & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\Gamma} \frac{1}{2}\left(\begin{array}{cc|cc} 1 & & & \\ & 0 & 1 & \\ \hline & 1 & 0 & \\ & & & 1 \end{array}\right)$

Negativity

Partial transpose has tr=1. Thus, has negative eigenvalues ⇔ sum of absolute eigenvalues is > 1.

Negativity: $\boxed{N(\rho) = (\Sigma_i |\lambda_i| - 1)/2}$

Logarithmic negativity: $\boxed{E_N(\rho) = \log \Sigma_i |\lambda_i|}$

= 0 for separable states (but not only)

How to calculate?

1) Compute "Renyi negativities" $\mathrm{tr}\,(\rho_{AB}{}^{\Gamma})^{2n}$ and let n → ½

2) Use replica trick: $\mathrm{tr}\,(\rho_{AB}{}^{\Gamma})^{2n} = \mathrm{tr}\,(\rho_{AB}{}^{\Gamma})^{\otimes 2n}\,(C_{2n} \otimes C_{2n}{}^{-1})$

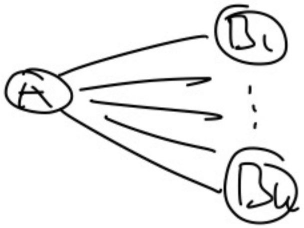
➔ Feasible in field theory and holography!

Calabrese-Cardy-Tonni, Kusuki-Kudler-Flam-Ryu, …, Dong-Qi-W

Extendibility criterion

Say ρ_{AB} has k-extension if there is state σ on $AB_1 \ldots B_k$ with

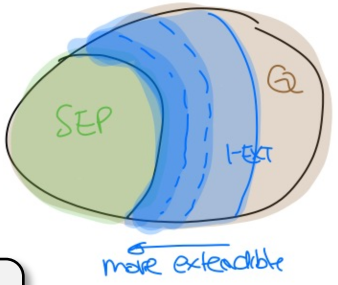
$$\boxed{\rho_{AB} = \sigma_{AB_1} = \cdots = \sigma_{AB_k}}$$



If ρ_{AB} separable then has k-extension for all k.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \Rightarrow \quad \sigma_{AB_1 \cdots B_k} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \ldots \otimes \rho_B^{(i)}$$

Conversely, if k-extension then O(1/k) to separable.

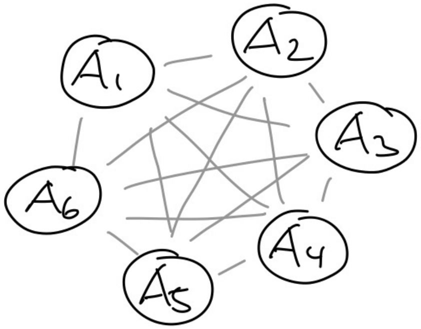


Criterion: $\boxed{\rho_{AB} \text{ separable} \Leftrightarrow \text{has k-extension for all k}}$

➔ Entanglement is monogamous also for mixed state!

Bonus: De Finetti theorem

Suppose that $A_1 \ldots A_n$ is permutation-symmetric. Then reduced states are close to mixtures of product states:



De Finetti Theorem:

$$\boxed{\rho_{A_1 \ldots A_k} \approx \int d\sigma \, p(\sigma) \, \sigma^{\otimes k}}$$

if $k \ll n$

e.g. $|00\ldots0\rangle + |11\ldots1\rangle$ and any $k < n$

- ➔ another version of monogamy
- ➔ justifies for why in mean field theory it suffices to consider product states

Bonus: Squashed entanglement

While mutual information is not a good entanglement measure, we can construct one using the conditional mutual information:

$$\boxed{I(A:B|C) = I(A:BC) - I(A:C)} = S(AC) + S(BC) - S(ABC) - S(C) \geq 0$$

Squashed entanglement:

$$\boxed{E_{sq}(A:B) = \frac{1}{2} \min_{\rho_{ABC}} I(A:B|C)}$$

Intuition: entanglement = correlations that cannot be shared

Properties:

1) $0 \leq E_{sq} \leq \frac{1}{2} I(A:B) \leq \log \min(d_A, d_B)$
2) For pure states: $E_{sq} = \frac{1}{2} I(A:B) = S_E$
3) Separable $\Leftrightarrow$ $E_{sq} = 0$
4) Monogamy: $E_{sq}(A:B) + E_{sq}(A:C) \leq E_{sq}(A:BC)$

Homework:
Show all but
← in 3.

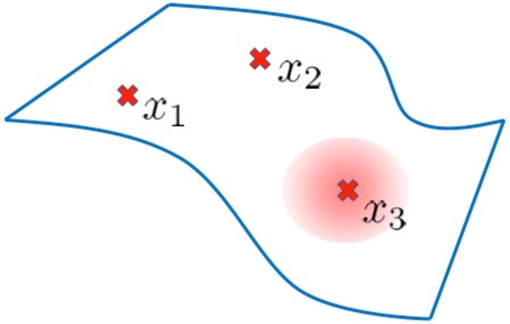
4. Entanglement in Field Theory

Literature: Harlow Jerusalem lectures (https://arxiv.org/abs/1409.1231),
Headrick lectures (https://arxiv.org/abs/1907.08126)

Quantum information & field theory

Do quantum information tools apply to quantum field theory?

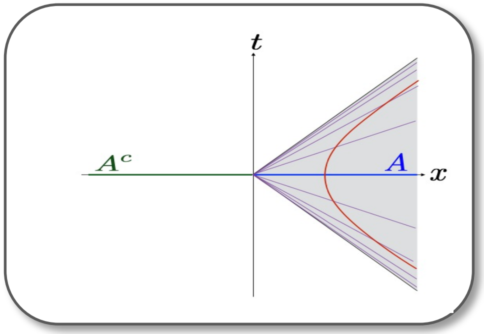
Challenge: Basic notions such as subsystems, entanglement, entropy, … more subtle!

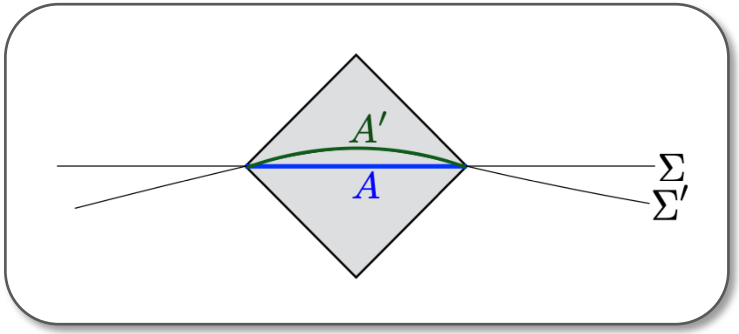


Theoretical insights: c-theorem from strong subadditivity, Bekenstein bound from relative entropy, renormalization vs QEC…

Another motivation: Quantum computers can simulate quantum mechanics. Can we simulate QFTs or even quantum gravity…?

Subsystems in relativistic QFT





[Headrick]

Causal domain of A:

D(A) = {p : every maximal causal curve through p intersects A}

Σ is Cauchy slice if acausal and D(Σ) = everything.

Time slice axiom:

Σ	⇔	global state	⇔	Hilbert space H
$A \subseteq \Sigma$	⇔	reduced state in D(A)	⇔	"$H = H_A \otimes H_B$"

D(A) = D(A′) ➔ ρ_A and $\rho_{A'}$ should be unitarily related

Correlations in QFT

Consider e.g. free scalar field with mass m in Minkowski space:

$$\boxed{H = \int d^3x\, \pi(x)^2 + (\nabla\phi(x))^2 + m^2\, \phi(x)^2}$$

$[\pi(x), \phi(y)] = i\delta^3(x-y)$

Correlation functions: $\boxed{\langle\phi(x)\rangle = 0}$

Amusing to compare
with Bell pair:

$\langle X\rangle = \ldots = \langle Z\rangle=0$
$\langle XX\rangle = \ldots = \langle ZZ\rangle=1$

UV divergence

$$\boxed{\langle\phi(x)\phi(y)\rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \exp(-|x-y|/\xi) & \text{if } |x-y| \gg \xi \end{cases}}$$

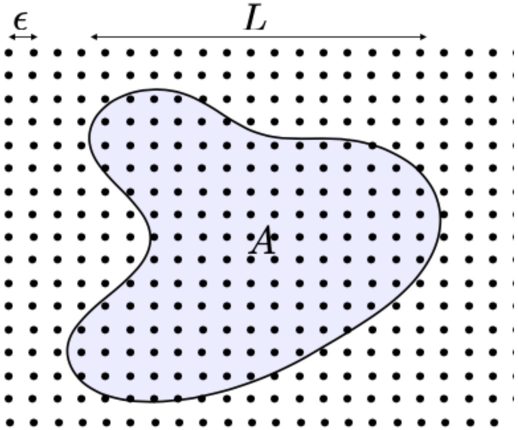
$\xi \sim 1/m$ correlation length

General form (short-distance power law, long-distance decay) believed to hold in any relativistic QFT. If m=0, decay can be power law.

Entanglement in QFT

Correlation functions:

$$\langle \phi(x)\phi(y)\rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \approx 0 & \text{if } |x-y| \gg \xi \end{cases}$$



[Headrick]

Thus, might expect that entanglement entropy satisfies an area law:

$$S(A) \propto |\partial A| \,/\, \varepsilon^{d-2}$$

UV cutoff

More generally, might expect that all divergences arise from local integrals over entangling surface ∂A.

That is, assuming $\xi < \infty$. E.g. for CFTs in d=1+1, power law decay leads to $\log(|A|/\varepsilon)$ divergence, as we will discuss momentarily.

Entanglement in QFT

$H \neq H_A \otimes H_B$

Observables in A, B commute, but Hilbert space does not factorize.

cf. divergence across entangling surface

➔ Reduced states not described by density operators
➔ Entanglement entropies not obviously well-defined

What can be said rigorously?

→ algebraic QFT literature, Witten's review

Reeh-Schlieder:

"$\{O_A \, |\Omega_{AB}\rangle\}$ dense"

Confusing? No, O_A will not be unitary!

Homework: Show that in finite dim any $|\Psi_{AB}\rangle$ can be written as $O_A \, |\Phi_{AB}^+\rangle$.

Relative entropies & various entanglement measures can be rigorously defined and computed/bounded

e.g., still makes sense to distill EPR pairs!

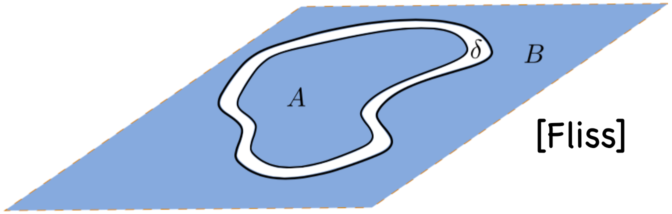
Bisognano-Wichmann: "modular Hamiltonian" of Rindler wedge

Entanglement Entropy in QFT

We will proceed cavalierly since we must anyways regulate entanglement entropy to obtain finite answer.

General strategy: UV regulate and compute universal quantities

coefficient of $\log(|A|/\varepsilon)$



[Fliss]

relative entropy

$D(\rho\|\sigma) = \mathrm{tr}\,\rho\,(\log\rho - \log\sigma)$

mutual information I(A:B)

If A, B don't touch: "$H_{AB} = H_A \otimes H_B$"
➔ rigorously defined in QFT!

Intuition: divergences cancel

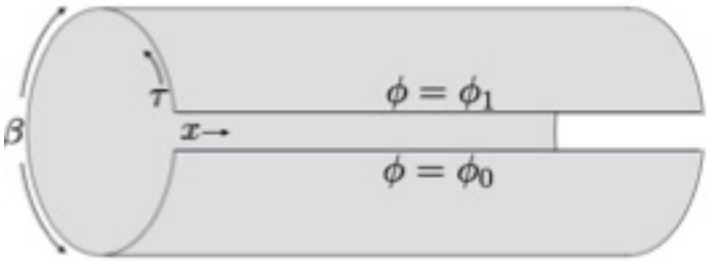
Euclidean path integrals

Let us consider states that are prepared by Euclidean path integrals.
E.g., unnormalized thermal state:

$$\boxed{\rho = e^{-\beta H}}$$

$\langle \phi_0 | e^{-\beta H} | \phi_1 \rangle =$

[Headrick]



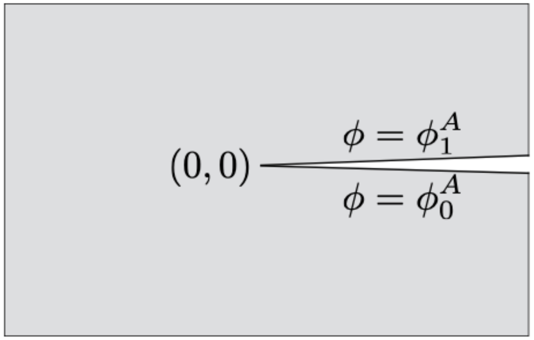
path integral on [0,β] x Σ

For $\beta \to \infty$, obtain vacuum state.

➔ Reduced state of $A \subseteq \Sigma$:

$$\boxed{\rho_A = \mathrm{tr}_B\, e^{-\beta H}}$$

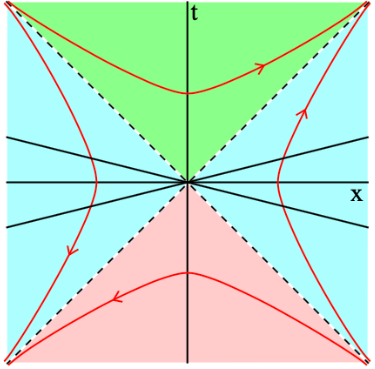
[Headrick]



path integral on plane with half-slit

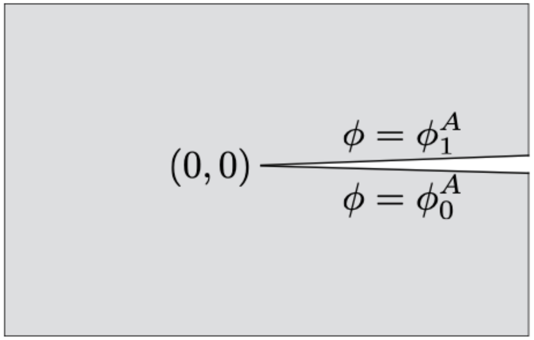
Rindler decomposition

[Harlow]



Minkowski space-time

[Headrick]



Euclidean path integral

Rindler wedges correspond to A = $[0,\infty)$ and B = $(-\infty,0]$.

Lorentz boost generator K acts by **rotations** in Euclidean signature ➔ $\boxed{\rho_A = e^{-2\pi K}}$ "thermal"

Similarly, Schmidt decomposition: $\boxed{|\Omega_{AB}\rangle = \Sigma_i \, e^{-\pi\omega_i} \, |i'\rangle|i\rangle}$ Homework!

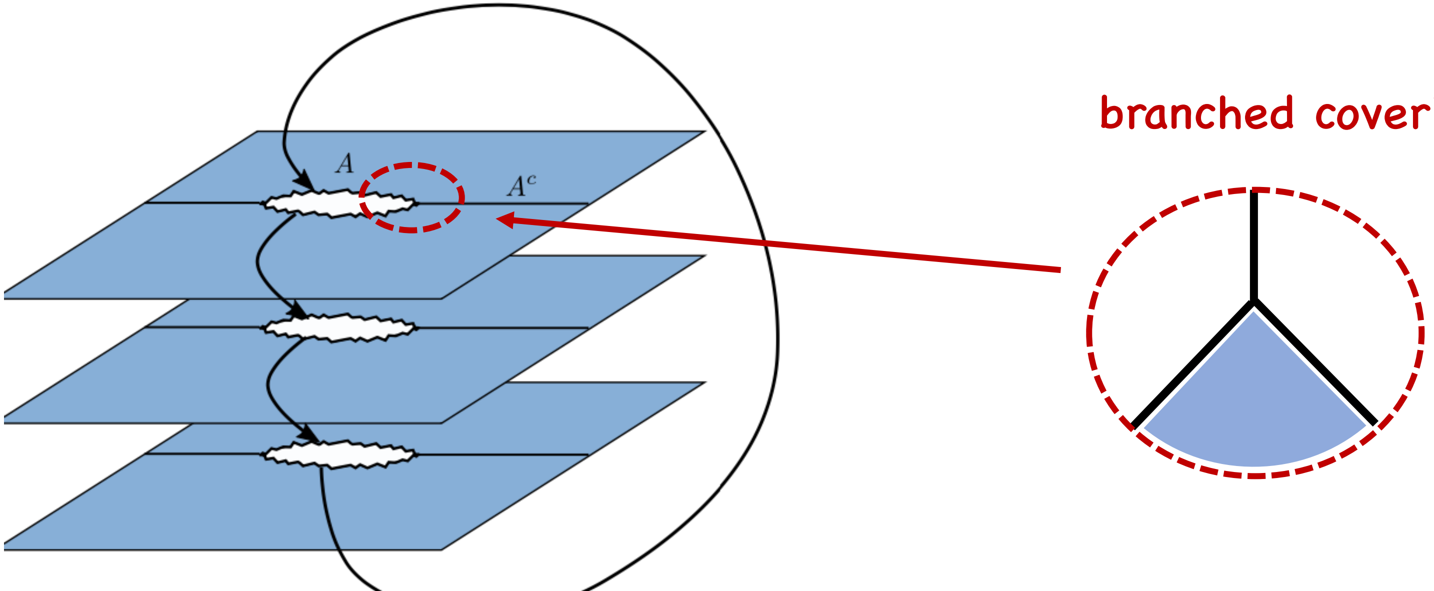
Amusing: If $|\Omega_{AB}\rangle$ were product ➔ "firewall" between A:B.

Entanglement entropy and replica trick

Using the replica trick, it is easy to compute Renyi entropies:

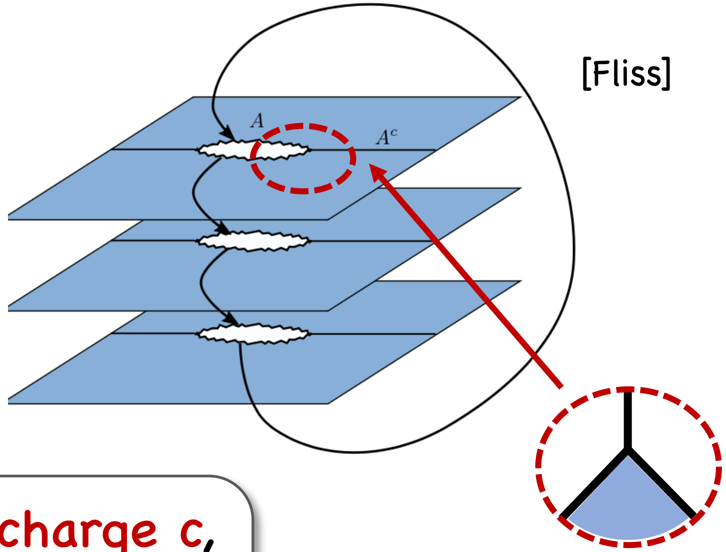
$$\boxed{S_n(\rho) = \frac{1}{1-n} \log \frac{\mathrm{tr}[\rho^n]}{\mathrm{tr}[\rho]^n} = \frac{1}{1-n} (\log Z_n - n \log Z_1)}$$

where $Z_n = \mathrm{tr}[\rho^n] = \mathrm{tr}[\rho^{\otimes n} C_n]$ is calculated by the following path integral:



[Fliss]

Entanglement entropy for single interval



[Fliss]

Can be explicitly computed for spherical regions in conformal field theory.

Cardy-Calabrese: In 1+1d CFT with central charge c,

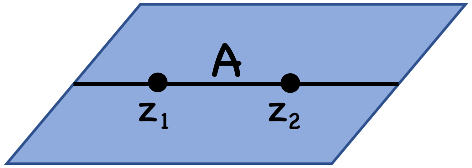
$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{L}{\varepsilon} \qquad S = \frac{c}{3}\log\frac{L}{\varepsilon}$$

Homework:
Prove this.

M_n is topologically sphere, compute Z_n from Weyl anomaly.

Alternatively, via 2-point function of twist operators in orbifold CFT:

$$Z_n = \langle \sigma_+(z_1)\sigma_-(z_2)\rangle_{\mathrm{CFT}^n/\mathbb{Z}_n}$$



Application: c-theorem

Casini-Huerta

Can use entanglement entropy to construct RG monotone and re-prove c-theorem.

$$\boxed{c_{UV} \geq c_{IR}}$$

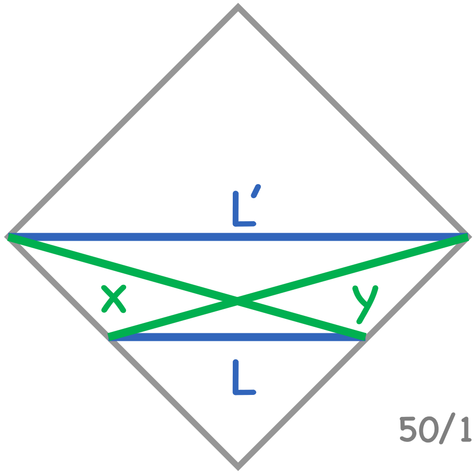
Suppose we deform "UV CFT" by relevant operator. Then:

$$\boxed{S(L \ll \xi) = \frac{c_{UV}}{3} \log \frac{L}{\varepsilon}} \qquad \boxed{S(L \gg \xi) = \frac{c_{IR}}{3} \log \frac{L}{\varepsilon'}}$$

Claim: $c(L) = 3\, L\, dS/dL$ interpolates c_{UV}, c_{IR} and decreases with L.

Key idea: Use strong subadditivity $S(AB) + S(BC) \geq S(ABC) + S(B)$.
Here:

$$\underbrace{\boxed{S(x) + S(y)}}_{= 2S(\sqrt{LL'})} \geq S(L') + S(L)$$

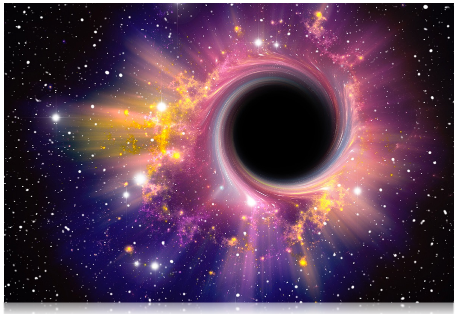


Choose $L' = L + \delta$ ➔ $d^2\ldots/d\delta^2 \propto -dc/dL \geq 0$

5. Entanglement in Holography

Literature: Headrick lectures (https://arxiv.org/abs/1907.08126)

Black holes and quantum information



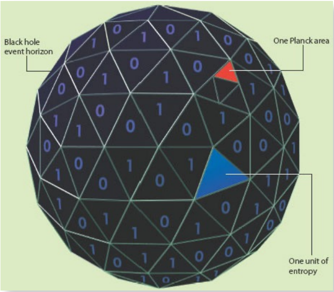
Black holes have a thermodynamic temperature and entropy. This entropy is proportional to the area of the event horizon:

$$S_{BH} = \frac{\text{Area}}{4G}$$

Bekenstein
Hawking

Surprising! Further puzzles arise when we try to quantize: Hawking radiation, information paradox(es), ...

A theory of quantum gravity ought to give microscopic explanations.



[Quanta]

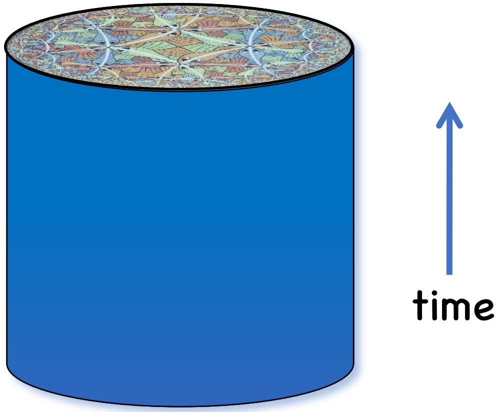
Holographic principle and practice

Holographic principle: Can all information in a region of space be represented as “hologram” living on boundary?

Susskind
‘t Hooft

AdS/CFT duality: Realization in Anti-de Sitter space

Maldacena



boundary: d-dim conformal field theory (CFT)

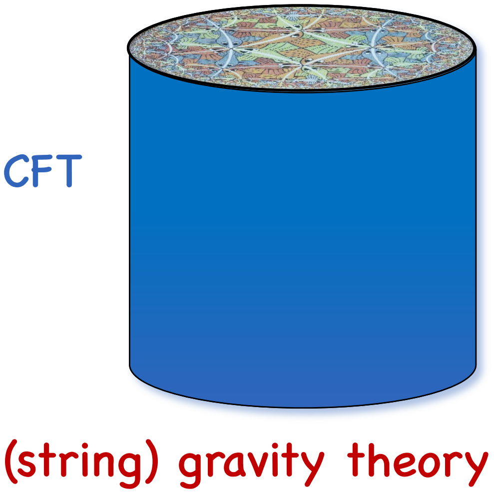
bulk: (d+1)-dim (string) gravity theory

Controlled setup to study quantum gravity; including black holes, wormholes, …

What can we learn by applying the QI toolkit?

AdS/CFT Dictionary

$$\frac{c}{3} = \frac{\ell}{2G_N}$$



Symmetries ✓

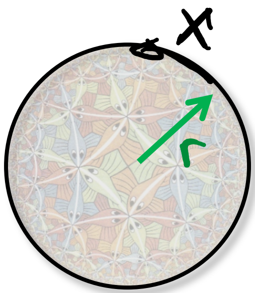
Partition functions: $\boxed{Z_{CFT} = Z_{string}}$

"Extrapolate dictionary":

$$\boxed{O(X) = \lim_{r\to\infty} r^{\Delta}\phi(r, X)}$$

➔ can compute CFT correlation functions:

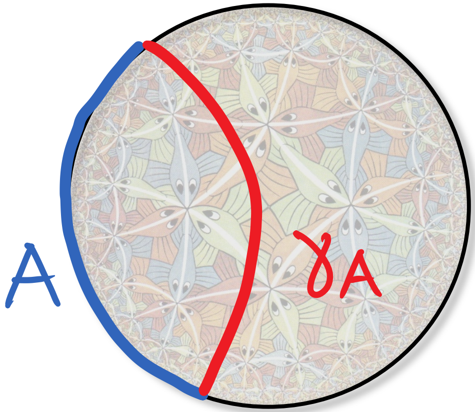
$$\boxed{\int D\phi\, e^{iS_{eff}}\, O_1 \cdots O_n = \langle O_1 \cdots O_n \rangle_{CFT}}$$



What is the bulk dual of entanglement entropy?

Ryu-Takayanagi formula

Ryu-Takayanagi (RT): For static space-times, boundary entropies are computed by area of bulk minimal surface homologous to A:



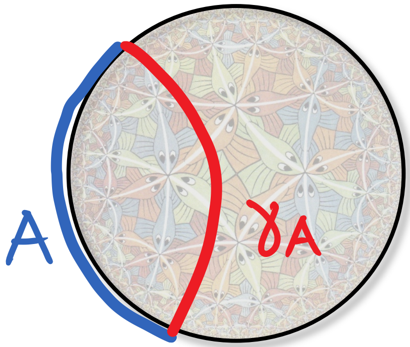
time slice

$$S(A) = \min \frac{|\gamma_A|}{4G} + \ldots$$

Entanglement ⇔ Geometry

Example: AdS$_3$

CFT vacuum state $|\Omega\rangle$ is dual to AdS$_3$ bulk:



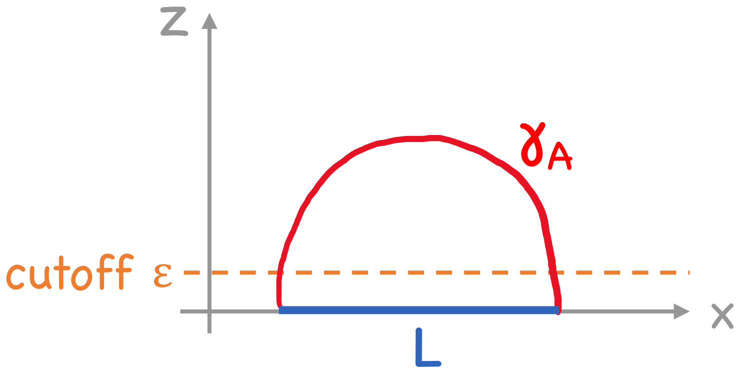
time slice

Pure state: $\boxed{S(\Sigma) = 0, \quad S(A) = S(A^c)}$ ✓

For an interval of length L, recover Cardy formula:

Poincaré coordinates

$$ds^2 = \ell^2/z^2 \, (dx^2+dz^2-dt^2)$$



minimal geodesics = coordinate semicircles

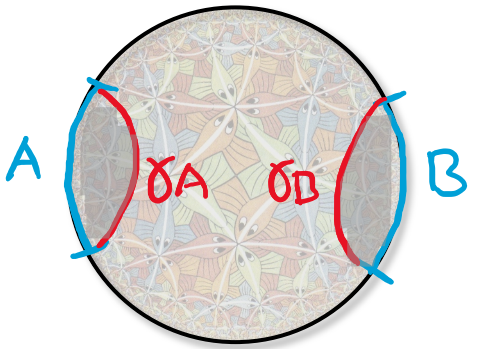
$$\boxed{|\gamma_A| = 2\ell \log(L/\varepsilon)}$$

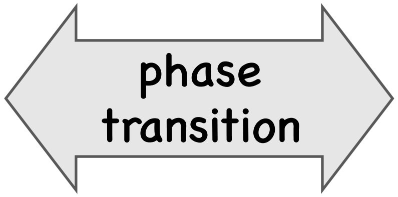
$$\rightarrow \boxed{S(L) = c/3 \log(L/\varepsilon)} \checkmark$$

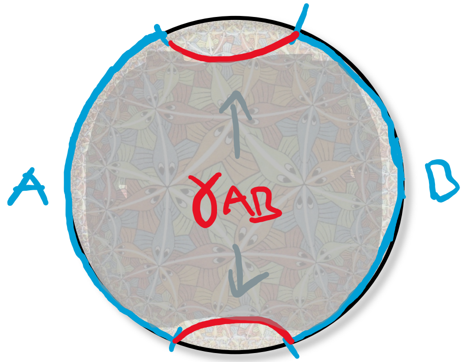
Homework: Verify this.

Example: Multiple subsystems

Two boundary subsystems:





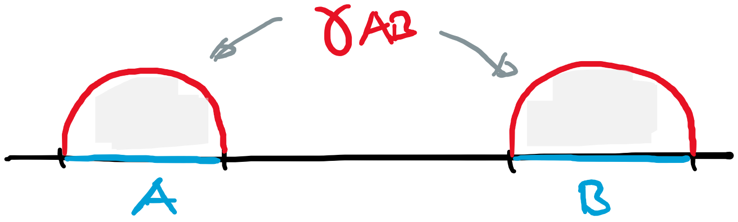


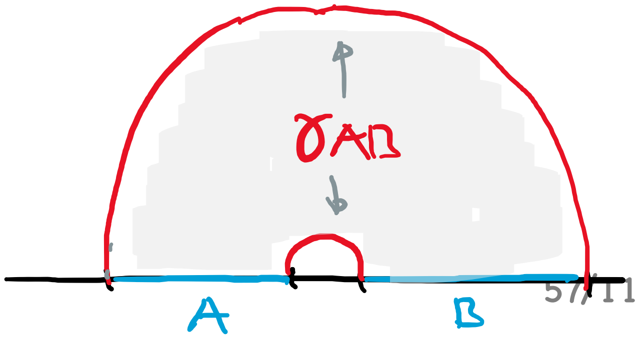
$$S(AB) = S(A) + S(B)$$
$$I(A:B) = 0$$

"uncorrelated phase"

$$S(AB) < S(A) + S(B)$$
$$I(A:B) > 0$$

"correlated phase"

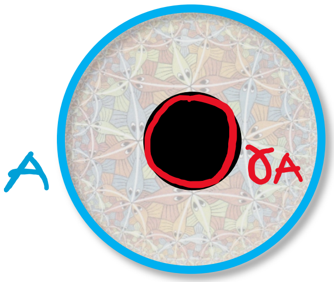




Example: BTZ black hole

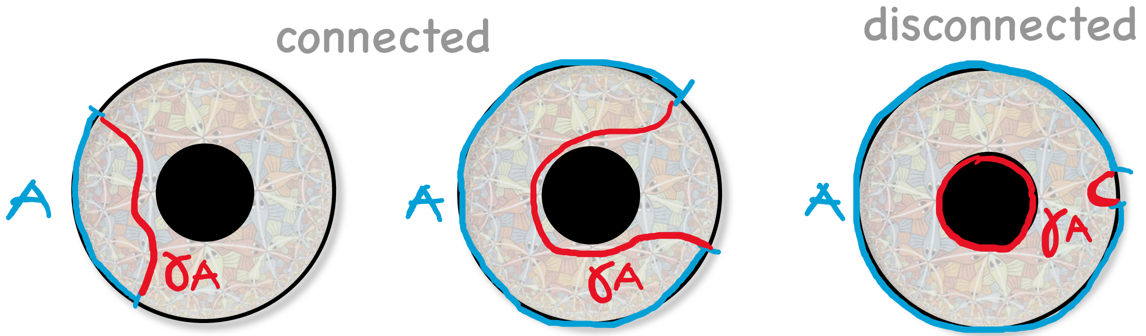
$T = 2\pi r_s$

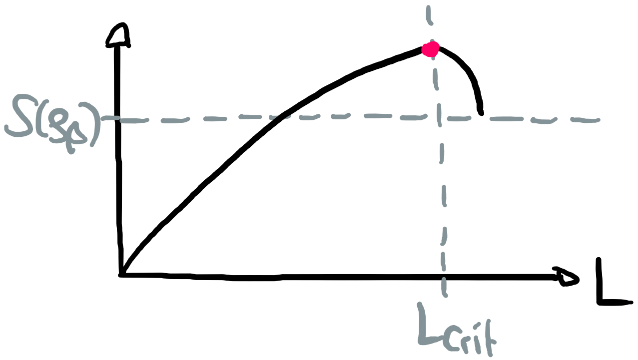
BTZ_3 black hole solution is dual to CFT_2 thermal state ρ_β:



Mixed state: $\boxed{S(\rho_\beta) = \frac{\text{horizon area}}{4G_N}}$ > 0 ✓

Phase transition in entanglement entropy:



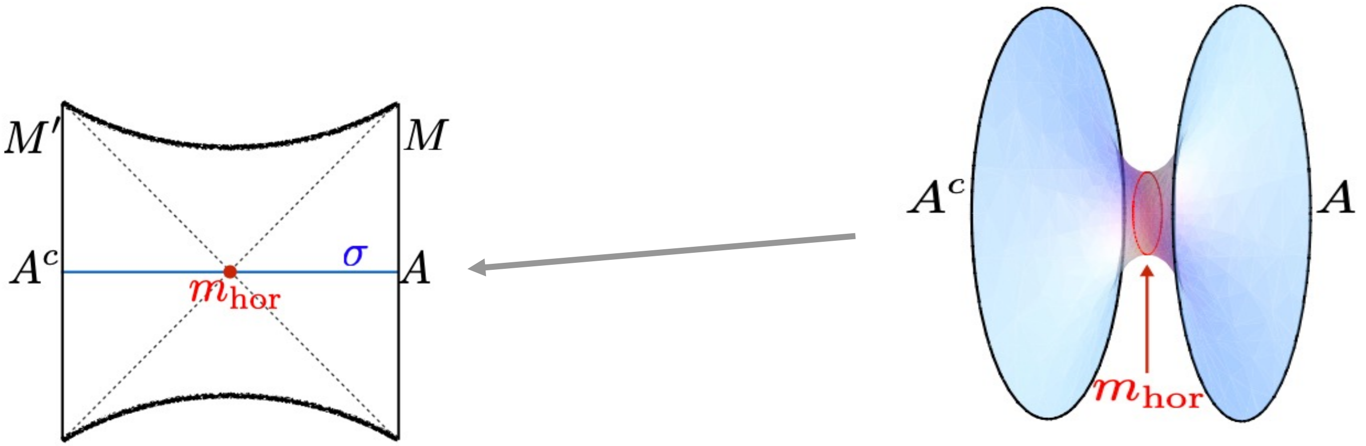


Entanglement shadow: minimal geodesics don't reach all the way to r_+.

Example: Thermofield double

$$\boxed{|\mathrm{TFD}_\beta\rangle = 1/Z \, \Sigma_E \, e^{-\beta E/2} \, |E'\rangle|E\rangle}$$

Thermofield double state is purification of thermal state to two CFTs.
Bulk dual: Two-sided black hole in static asymptotic AdS space-time.



[Headrick]

contains Einstein-Rosen (ER) bridge connecting asymptotic AdS regions

"ER = EPR" Susskind-Maldacena

cf. Rindler wedge analysis

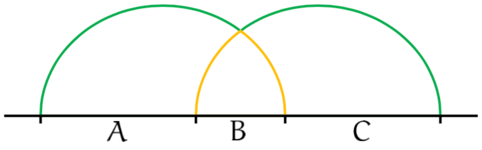
Why should Ryu-Takayanagi hold?

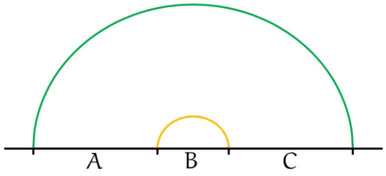
Intuitive generalization of Bekenstein-Hawking formula.

Matches CFT calculations. ✓

Proved under plausible assumptions. ✓ Lewkowycz-Maldacena

Satisfies many nontrivial consistency checks. For example, easy to verify strong subadditivity:

$S(AB) + S(BC) =$ 

$\geq$ 

$= S(B) + S(ABC)$

✓

However, we can prove "too much"...

Holographic entropy laws

Ryu-Takayanagi formula satisfies non-standard entropy inequalities.
These are constraints for CFTs to have a gravity dual!

"Monogamy" inequality: $I(A{:}B) + I(A{:}C) \leq I(A{:}BC)$

Hayden-
Headrick-
Maloney

Does not hold general states – not even for all probability distributions.
Correlations are not monogamous!

➔ excludes plausible states such as

$$\sum_n e^{-\beta E_n/2} |n\rangle |n\rangle |n\rangle |n\rangle \quad \neq$$

➔ can be used to witness multipartite correlations

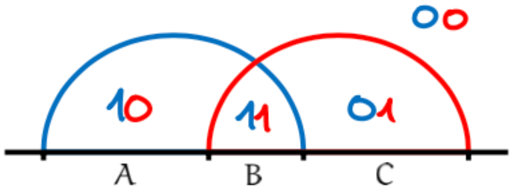
"=" for bipartite correlated states $\rho_{AB} \otimes \rho_{AC} \otimes \rho_{BC}$

Bao-…-Ooguri-W

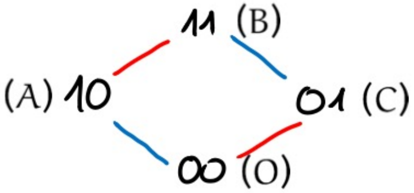
How to prove holographic entropy inequalities?

$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$

General method that abstracts inclusion/exclusion reasoning:

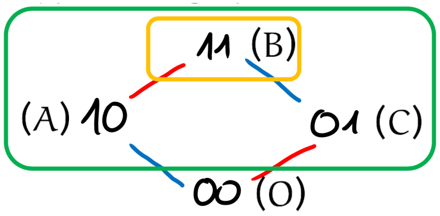


"Homology regions" for LHS minimal surfaces partition bulk into 2^{LHS} regions. ➔ Hypercube:



vertices = bulk regions

edges = surfaces between regions



➔ use subsets of hypercube to define homology regions for RHS surfaces

not necessarily minimal

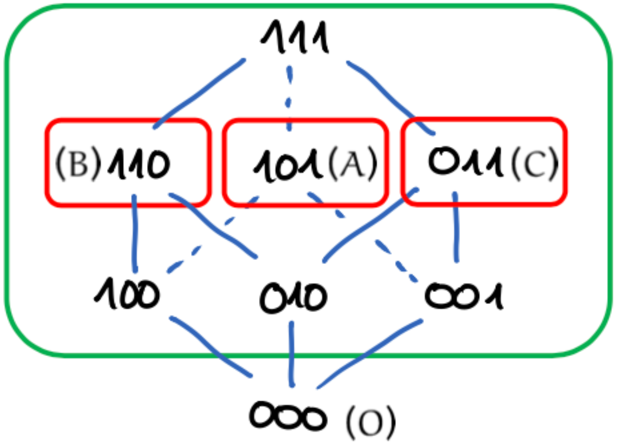
Homework: Work out details.

If each edge cut at most once: Entropy inequality is valid!

Hypercube proof of monogamy inequality

To illustrate the method, let us prove the "monogamy inequality", which expands to:

$$S(AB) + S(BC) + S(AC) \geqslant S(A) + S(B) + S(C) + S(ABC)$$



Infinitely many holographic entropy inequalities can so be proved. How to organize systematically?

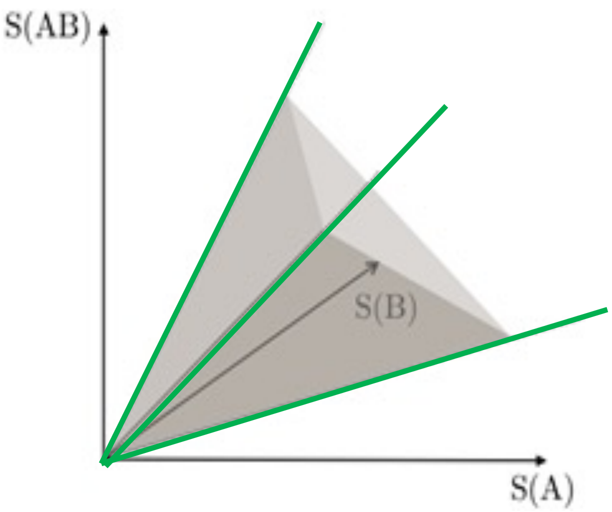
Holographic entropy cones

Bao-…-
Ooguri-W

For fixed number of subsystems,
consider all possible entropy vectors:

$$C_n = \{(S_{RT}(A_1), \ldots, S_{RT}(A_1A_2\ldots A_n)\}$$

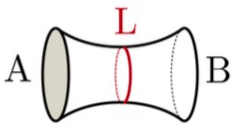
arbitrary geometries allowed!



This is a polyhedral convex cone – the holographic entropy cone.

faces: entropy inequalities such as S(A) + S(B) ≥ S(AB)

rays: entropy vectors that cannot be written as mixture of others. represented by "extremal geometries".



can these be identified with microscopic building blocks??

Constraints from entropy inequalities

Can also go the other way and exploit known entropy inequalities to derive gravitational constraints. E.g., using relative entropy:

$$\boxed{S(\rho\|\sigma) = \mathrm{tr}\ \rho \log \rho - \mathrm{tr}\ \rho \log \sigma \geq 0}$$

Perturb around vacuum state:

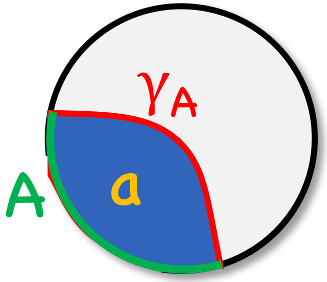
1st order: linearized Einstein equations — Faulkner et al

2nd order: positive energy inequalities — Lin et al, Lashkari et al

e.g. $$\boxed{\int T_{00} \sqrt{g} \geq 0}$$

Much more to be said about holographic entropies (monotonicity of relative entropy, Freedman-Headrick bit threads, ...)

Generalizations



Entropy of bulk fields in region enclosed by RT surface contribute O(1) corrections to entropy:

$$\boxed{S(A) = \frac{|\gamma_A|}{4G} + S(a)}$$

Faulkner-Lewkowycz-Maldacena

better: minimize joint expression ("generalized entropy") Engelhardt-Wall, ...

RT holds in static situations (more generally, in time-reflection symmetric situations). In general, consider extremal area codimension-2 spacelike bulk surfaces.

Hubeny-Rangamani-Takayanagi (HRT)

Equivalently, Wall's maximin procedure: $S(A) = \max_\Sigma \min_{\gamma A} \frac{|\gamma_A|}{4G}$

6. Toy Models of Holography

Literature: Harlow TASI lectures (https://arxiv.org/abs/1802.01040)

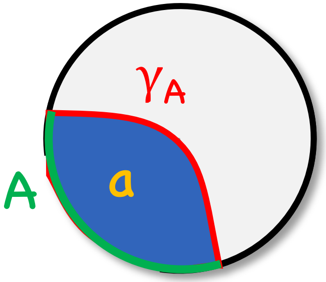
Holography is mysterious…

1) "Extrapolate" dictionary: $r^{\Delta}\phi(X,r) \xrightarrow{r\to\infty} O(X)$

A puzzle: $\boxed{[\phi(y), O(X)] = 0}$!?

2) Ryu-Takayanagi with bulk corrections:

$$\boxed{S(A) = \min \frac{|\gamma_A|}{4G} + S(a)}$$

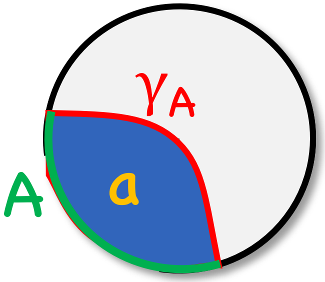


3) Bulk reconstruction problem: Every bulk operator should be dual to some boundary operator.

$$\phi(x) \overset{!?}{=} \int O(X)K(X|x)dX$$

Why do we care? Extrapolate dictionary insufficient if want to study processes behind horizons, understand bulk locality.

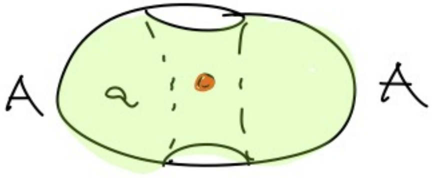
Subregion duality

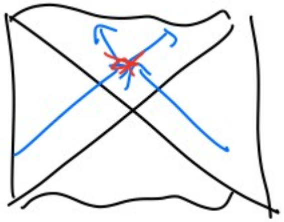


Subregion duality: Can write any bulk operator in a as boundary operator in A!

Proved using QI tools. ✓ Dong-Harlow-Wall, Cotler-…-W

Not known how to do explicitly in most tantalizing situations:



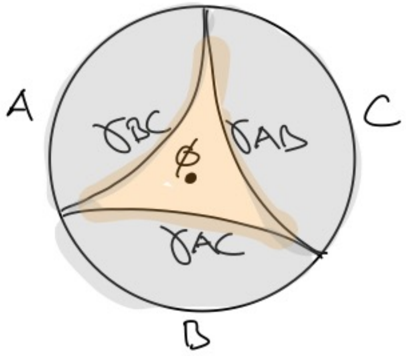


Only when A = everything or $\phi(x)$ in (smaller) causal wedge of A.

➔ Hamilton-Kabat-Lifschytz-Lowe, Banks et al, Heemskerk et al, …, Harlow TASI

Holography is mysterious

Subregion duality leads to another puzzle:



$$\phi = O_{AB} = O_{AC} = O_{BC}$$

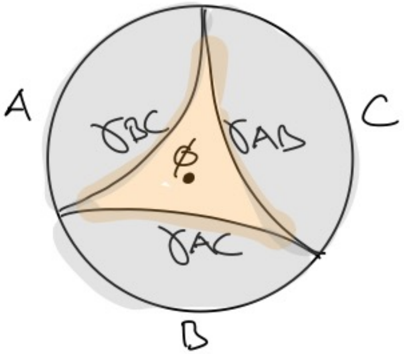
no common support ↯

$AB \cap AC \cap BC = \emptyset$

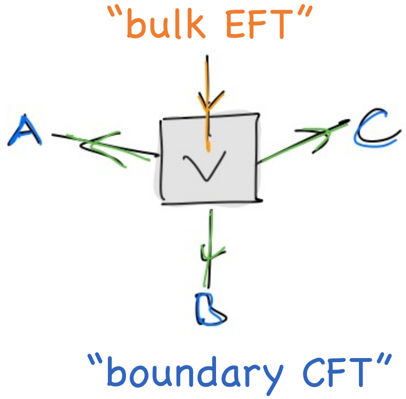
Resolution: Only "few" states correspond to any particular semiclassical bulk description.

"$[\phi(y),O(X)]=0$" or "$O=\phi$" only hold (make sense!) on small subspaces of CFT Hilbert space, known as "code subspaces"

Plan: Discuss toy models that reproduce 1)-3) and resolve puzzles by simple QI mechanisms.



Three-Qutrit code



$$\mathbb{C}^3 \rightarrow \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

$$\boxed{V|i\rangle = |\tilde{i}\rangle = \frac{1}{\sqrt{3}}\sum_{j=0}^{2}|j, j+i, j-i\rangle}$$

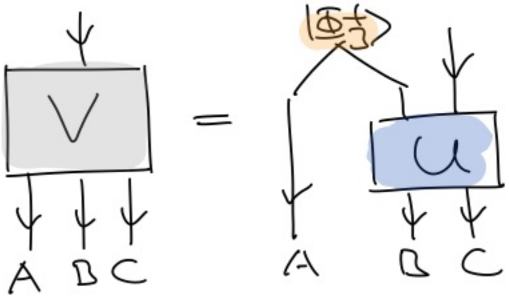
encodes 3-dim in 27-dim space

states ρ are encoded by $\tilde{\rho}_{ABC} = V\rho V^{\dagger}$
operators ϕ are encoded by $\tilde{\phi}_{ABC} = V\phi V^{\dagger}$

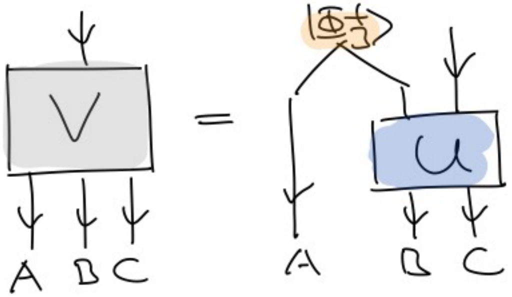
$$\langle\phi\cdots\rangle_{\rho} = \langle\tilde{\phi}\cdots\rangle_{\tilde{\rho}}$$

Key fact: $\boxed{V|i\rangle = (I_A \otimes U_{BC})(|\Phi^{+}_{AB}\rangle \otimes |i_C\rangle)}$

where $U_{BC}|j,i\rangle = |j+i,j-i\rangle$



Three-Qutrit code



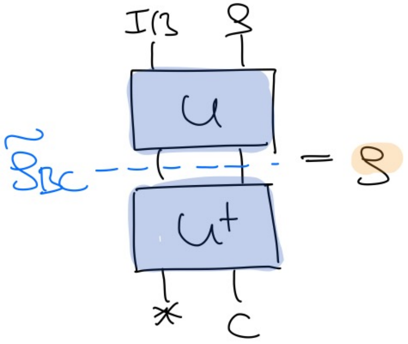
This has remarkable consequences:

Ryu-Takayanagi:

$$S(A) = \log(3) \quad = S(B) = S(C)$$

$$S(AB) = \log(3) + S(\rho) \quad = S(AC) = S(BC)$$

Subregion duality: can decode ρ from BC alone! likewise from AB, AC



"erasure code": can correct for loss of single qutrit!

Heisenberg picture:

$$O_{BC} = U_{BC}(I \otimes \phi)U_{BC}^\dagger$$

➔ $O_{BC}V = V\phi$, $O_{BC}^\dagger V = V\phi$

"$O_{BC} = \phi$"

➔ resolves second puzzle!

Three-Qutrit code

Similarly, if ϕ is any bulk and O_A any boundary operator on A:

$$\langle\tilde{i}|[O_A, \tilde{\phi}_{ABC}]|\tilde{j}\rangle = \langle\tilde{i}|\underbrace{[O_A, O_{BC}]}_{=0}|\tilde{j}\rangle = 0 \qquad \text{“}[\phi(x), O(Y)] = 0\text{”}$$

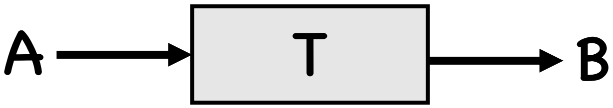
➔ resolves first puzzle!

Quantum error correction plays important role in recent research in holography (emergence of bulk locality, black hole information paradox, ...)

Verlinde², Almheiri-Dong-Harlow, ...

7. Error Correction, Decoupling, and Black Holes

Recall: Quantum channels



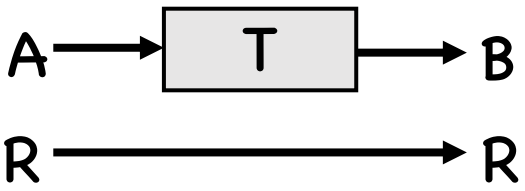
Quantum channel: Any combination of unitary evolution, partial traces, adding auxiliary systems.

$\rho \to U\rho U^\dagger$
$\rho \to \rho \otimes \sigma$
$\rho_{AB} \to \rho_A$

Equivalently, any map that sends states ρ_{AR} ➔ states ρ_{BR}.

$$\boxed{\rho_{BR} = (T \otimes \mathrm{id})(\rho_{AR})}$$

completely positive & trace-preserving (CPTP)



Examples:

Basis measurement:

$$\boxed{M(\rho) = \Sigma_x \langle x|\rho|x\rangle\, |x\rangle\langle x|}$$

Depolarizing noise:

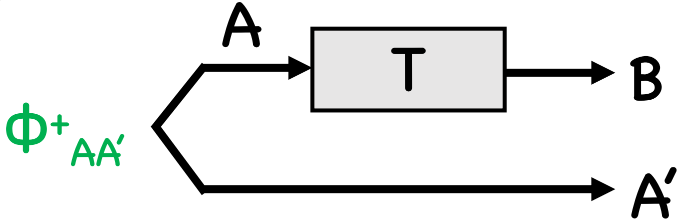
$$\boxed{D_p(\rho) = p\rho + (1-p)I/d}$$

Homework: Check this.

Tools for quantum channels

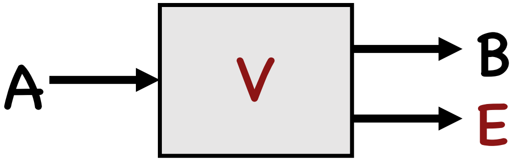
Choi state: characterizes channel completely!

$$\boxed{\Omega_{A'B} = (\mathrm{id} \otimes T)(\Phi^+_{A'A})}$$



Stinespring extension: Isometry V such that:

$$\boxed{T(\rho) = \mathrm{tr}_E(V\rho V^\dagger)}$$



➔ complementary channel: $\boxed{T^c(\rho) = \mathrm{tr}_B(V\rho V^\dagger)}$ what leaks to environment!

Together: Solve channel problems by (pure) state reasoning!

Example: Basis Measurement

$$\boxed{M(\rho) = \Sigma_x \langle x|\rho|x\rangle \, |x\rangle\langle x|}$$

$$\boxed{\Omega_{A'B}} = (\mathrm{id} \otimes M)(|\Phi^+_{A'A}\rangle\langle\Phi^+_{A'A}|) = 1/d \; \Sigma_{x,y} \, (\mathrm{id} \otimes M)(|xx\rangle\langle yy|)$$

$$= 1/d \; \Sigma_{x,y} \, |x\rangle\langle y| \otimes M(|x\rangle\langle y|) = 1/d \; \Sigma_x \, |x\rangle\langle x| \otimes |x\rangle\langle x|$$

$$= \boxed{1/d \; \Sigma_x \, |xx\rangle\langle xx|}$$

$$\boxed{V|x\rangle = |xx\rangle} \qquad \mathrm{tr}_E(V|x\rangle\langle y|V^\dagger) = \mathrm{tr}_E(|xx\rangle\langle yy|) = \delta_{xy} \, |x\rangle\langle x| = M(|x\rangle\langle y|)$$

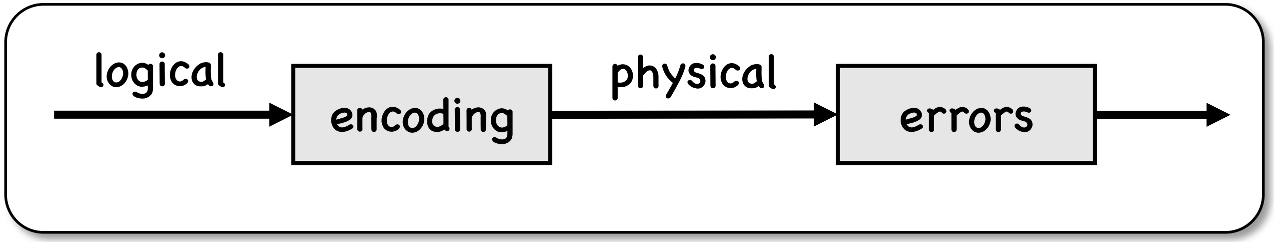
→ Complementary channel: $M^c = M$!

Homework: Compute Choi + Stinespring for other examples.

Quantum error correction

When building quantum computers, we want to protect against errors (imperfections, noise, decoherence, ...).

To achieve this, redundantly encode "logical" into "physical" qubits:



logical → encoding → physical → errors →

quantum channels!

For example, 3-qutrit code corrects again erasure of any 1 qutrit.

Holography:

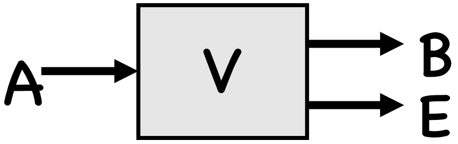
bulk d.o.f → dictionary → boundary → trace over part of bdry →

Questions:

1) When can we in principle correct?

2) How to correct in practice?

Decoupling criterion



The question: Given a channel $T_{A\to B}$, when can we reverse it?

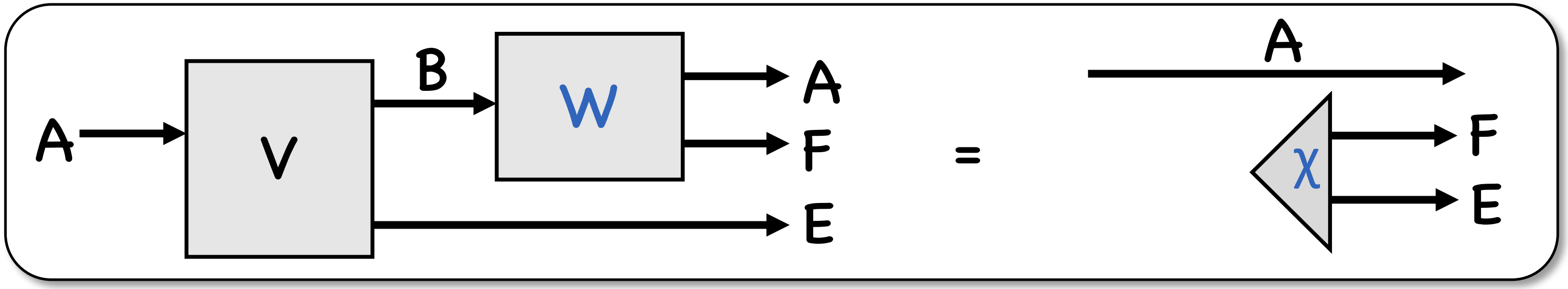
Decoupling criterion: Can reverse $T_{A\to B}$ if and only if the complementary channel $T^c_{A\to E}$ is constant.

$\Omega_{A'E} = \Omega_{A'} \otimes \chi_E$

$I(A':E) = 0$

- ➔ exactly what we found for 3-qutrit code
- ➔ very strong form of "no cloning" statement

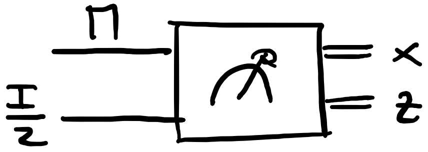
If reversible: There exists state $|\chi\rangle$ and isometry W such that:



or $|\Omega_{A'AEF}\rangle = |\Phi^+_{AA'}\rangle \otimes |\chi_{EF}\rangle$ Homework: Prove this.

Teleportation revisited

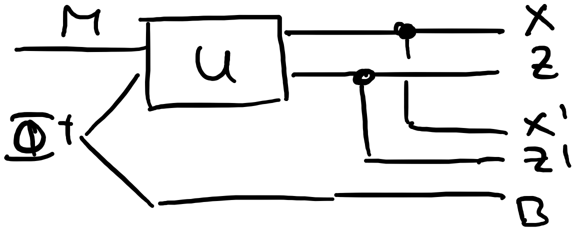
It is instructive to revisit teleportation from this perspective.
Consider channel which performs Bell measurement on $\rho_M \otimes I_A/2$:



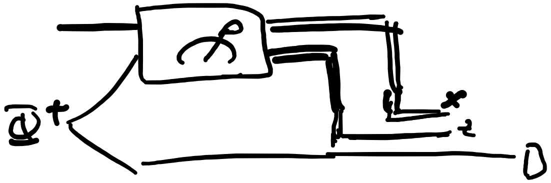
This is a constant channel since all outcomes are equally likely. By the decoupling criterion, can decode from complementary channel!

First, compute Stinespring extension:

$$U\ |\Phi^{(xz)}> \ = \ |xzxz>$$

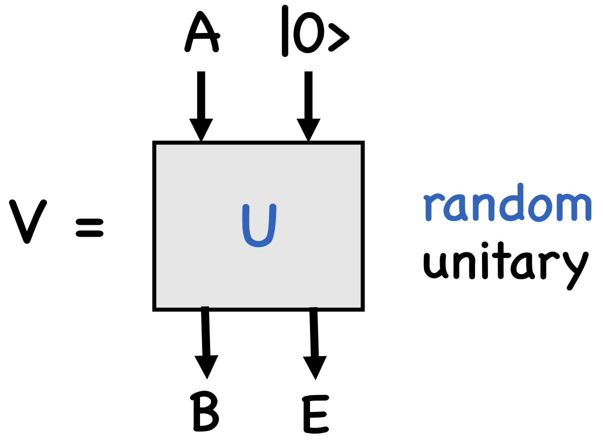


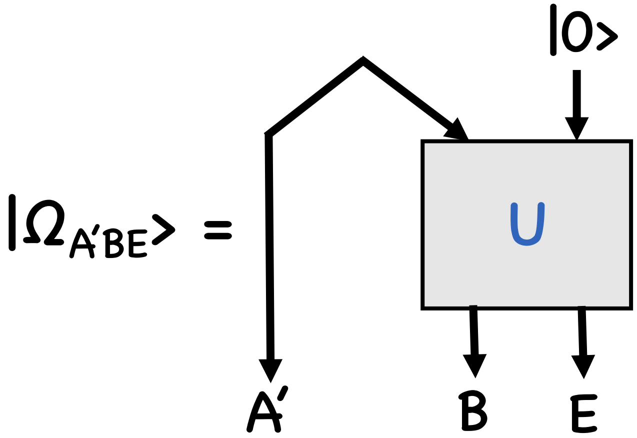
Thus, complementary channel looks like teleportation w/o correction:



Decoupling inequality

In information theory, random codes are often almost optimal.





When can we decode A from B?

Need $\Omega_{A'E} \approx \Omega_{A'} \otimes \Omega_E$!

The following result addresses these kind of problems:

Decoupling Inequality: Let ρ_{ABE} state, U_{BE} random. Then:

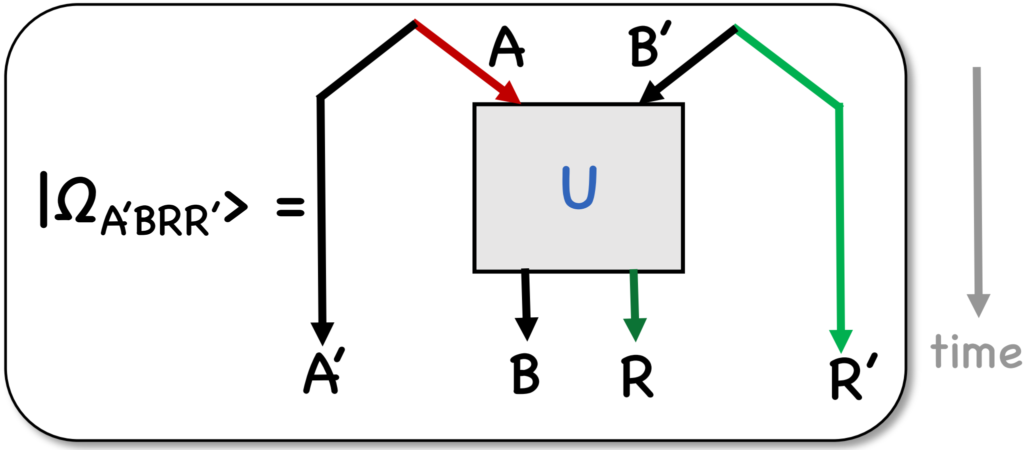
$$\int dU_{BE} \left\| \mathrm{tr}_B(U_{BE}\, \rho_{ABE}\, U_{BE}^\dagger) - \rho_A \otimes I_E/d_E \right\|_1^2 \leq \frac{d_{AE}}{d_B}\, 2^{-S_2(\rho)}$$

Hayden-Preskill protocol

We again model an evaporating black hole by random unitary. **After Page time**, assume black hole maximally entangled with old radiation.

Now suppose Alice throws her diary into black hole.

How much further radiation do we need to collect so that we can recover diary?



That is, when can we decode A from RR′?

Need $\Omega_{A'B} \approx \Omega_{A'} \otimes \Omega_B$!

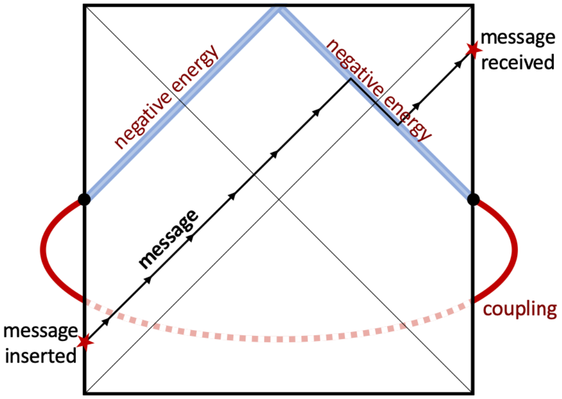
Answer: $d_A \ll d_R$

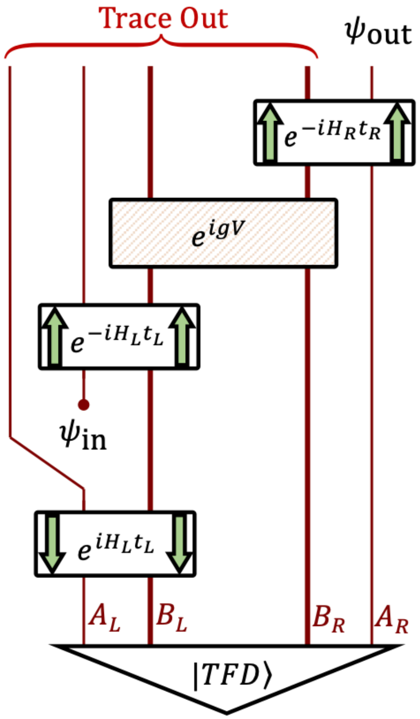
Little more than size of diary – independent of size of black hole. Black hole after Page time is like a mirror, information comes right out.

Homework: Show this using the decoupling theorem with B → R, E → B.

Holographic teleportation

Gao–Jafferis–Wall,
Maldacena–Stanford–Yang





Wormhole in thermofield double state can be made traversable by weak local classical coupling between the two CFTs.

This holographic teleportation protocol is remarkable: "self-decoding" even though CFT time evolution scrambling!

Recent work constructed toy models using e.g. random unitaries and proposed general QI mechanisms.

Brown–...–W

Effective interaction only depends on operator sizes.

Bonus: Relative entropy

$$\boxed{D(\rho\|\sigma) = \operatorname{tr}\rho\,(\log\rho - \log\sigma)}$$

≥ 0

$= 0$ iff $\rho = \sigma$

well-defined in QFT

➔ $S(\rho) = \log d - D(\rho\|I/d)$, $I(A{:}B) = D(\rho_{AB}\|\rho_A\otimes\rho_B)$, ...

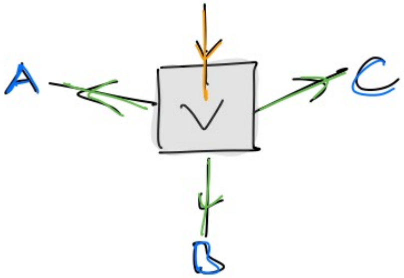
Pinkser inequality: $\boxed{D(\rho\|\sigma) \geq {}^1/_{2\ln 2}\,\|\rho - \sigma\|_1^2}$

Data processing inequality: $\boxed{D(\rho\|\sigma) \geq D(T(\rho)\|T(\sigma))}$ ➔ strong subadditivity

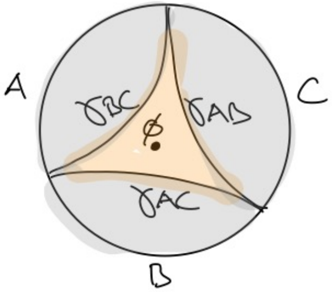
"=" ⇔ can reverse channel on pair of states ρ,σ

How so? Use Petz map: $\boxed{D(\rho) = \sigma^{1/2}\,T^\dagger(T(\sigma)^{-1/2}\,\rho\,T(\sigma)^{-1/2})\,\sigma^{1/2}}$

Back to our toy model



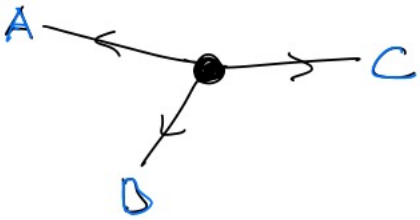
$$V|i\rangle = |\tilde{i}\rangle = \frac{1}{\sqrt{3}}\sum_{j=0}^{2}|j,j+i,j-i\rangle$$

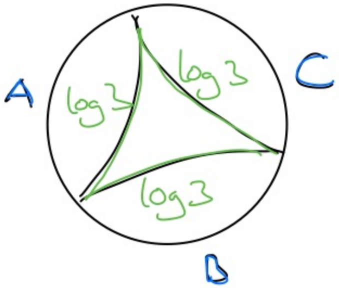


Toy model has no geometry – just a single site!

To go beyond, let's focus on the RT formula:

$$|\tilde{0}\rangle = \frac{1}{\sqrt{3}}\sum_{j=0}^{2}|j,j,j\rangle$$





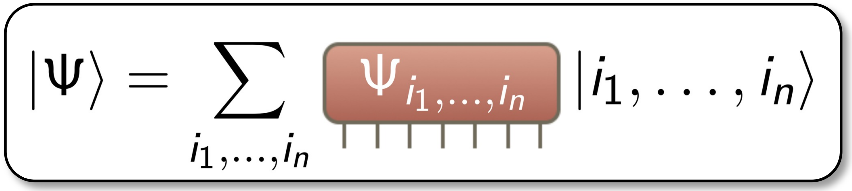
Can we glue together many such states (or codes)?

8. Tensor Network Toy Models

Literature: https://arxiv.org/abs/1503.06237, http://pirsa.org/20110023

Many-body quantum states

Many-body quantum states have exponentially large description



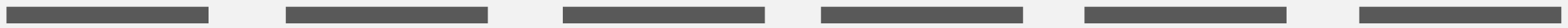
$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle$$

tensor with n indices

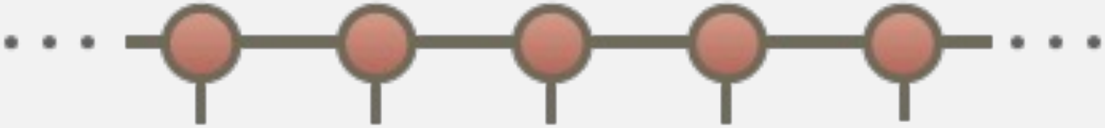
In practice: entanglement is local, correlations decay rapidly

→ can hope for more efficient description:

Key idea: start with entangled pairs...



...and apply local transformations:



e.g. ‘cat’ state |0...00> + |1...11> from |00> → |0>, |11> → |1>

Tensor networks as a tool

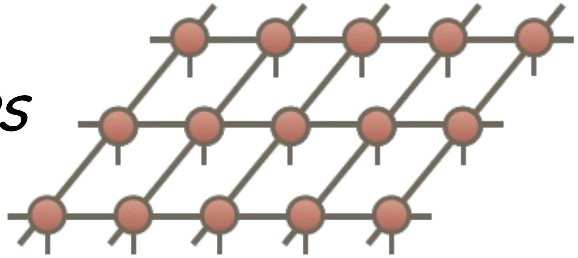
Tensor network: state defined by contracting network of (local) tensors

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \boxed{\Psi_{i_1,\ldots,i_n}} \, |i_1,\ldots,i_n\rangle$$

e.g. *MPS*

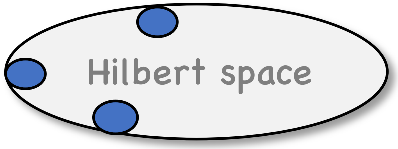
White, Fannes-Nachtergaele-Werner, Östlund-Rommer

PEPS



Verstraete-Cirac

Numerical tool: efficient variational classes



provably so for gapped theories in 1+1d (Hastings)

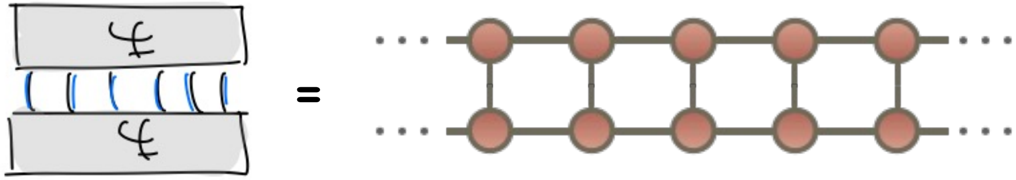
can even have interpretation as quantum circuits

Powerful theoretical formalism, provides "dual" descriptions of complex phenomena → quantum phases, topological order, …

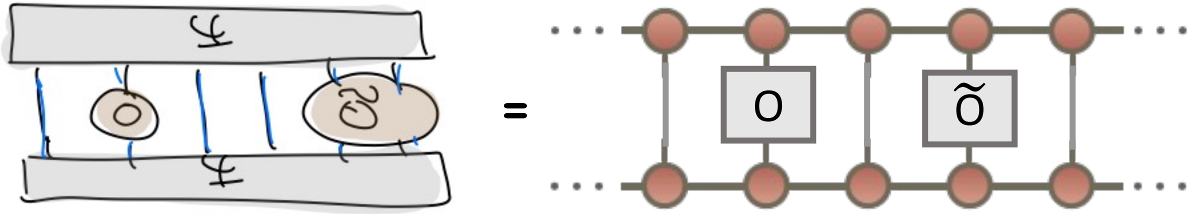
Computing with tensor networks

Very similar to path integral reasoning:

$\langle\psi|\psi\rangle =$



$\langle\psi|O_A\tilde{O}_B|\psi\rangle =$

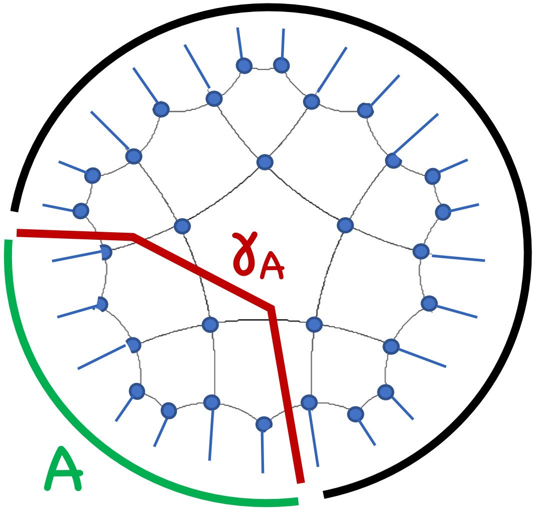


Can formally obtain tensor networks by trotterizing $e^{-\beta H}$.

What is the role of the network geometry?

Entropy in tensor networks

Entanglement entropy satisfies "Ryu-Takayanagi bound":



$$\boxed{S(A) \leq N\,|\gamma_A|}$$

N qubits/bond

γ_A = minimal cut

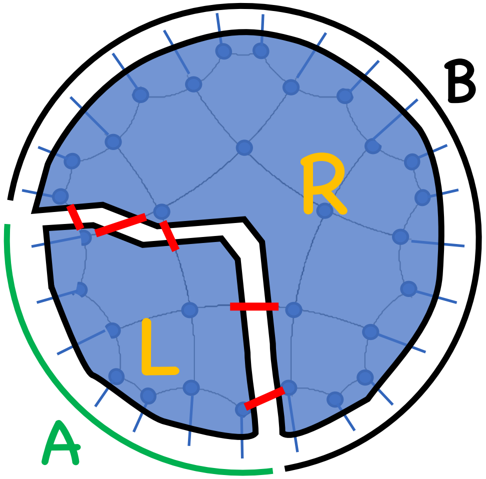
In general, the bound is not saturated...

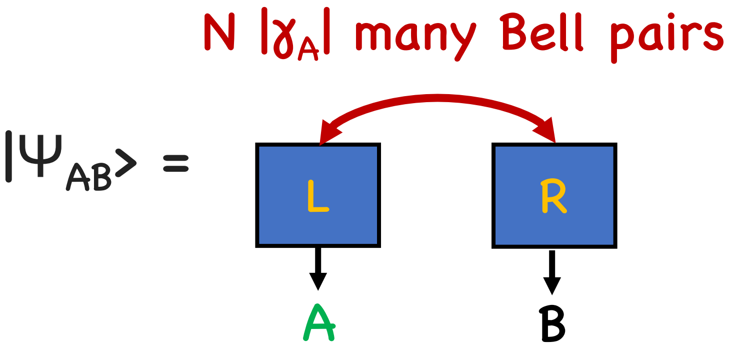
Tantalizing: Picture shows Vidal's MERA tensor network.

Used for critical theories, it looks like a time slice of AdS!

Swingle

Why does the bound hold?




N $|\gamma_A|$ many Bell pairs

$|\Psi_{AB}\rangle =$

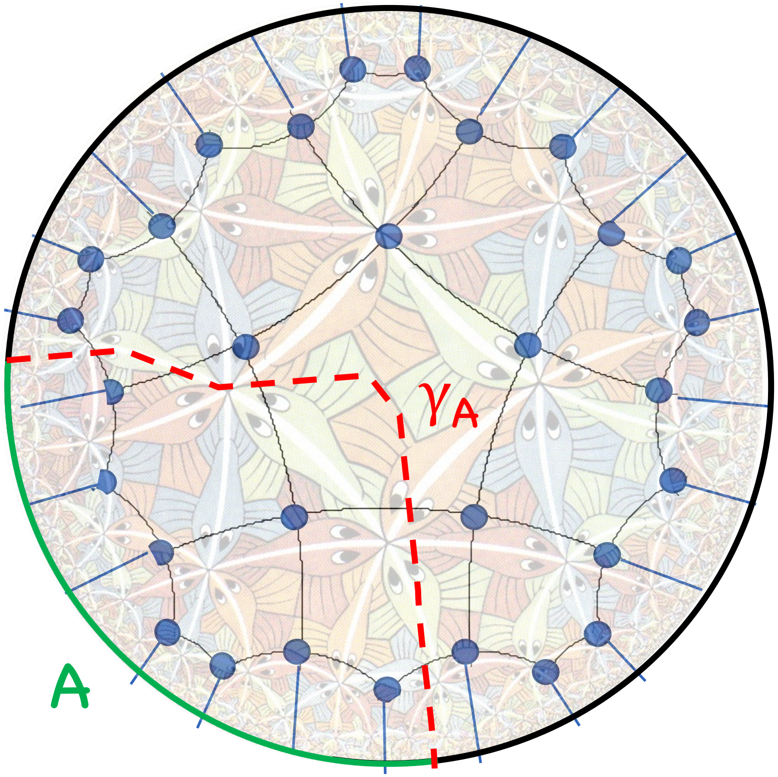
Thus, the Schmidt rank is at most $2^{N\,|\gamma_A|}$.

$$\rightarrow \boxed{S(A) \le S_0(A) \le N\,|\gamma_A|}$$

NB: Bound saturated if L, R are unitaries (or isometries)!

Holography from tensor networks

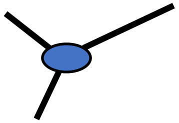
Want "exactly solvable" toy models of holographic duality:



Harlow et al, Hayden-…-W

Approach: Define boundary state via tensor network in bulk

simple bulk tensors, e.g. random and large N



➔ emergent Ryu-Takayanagi law!

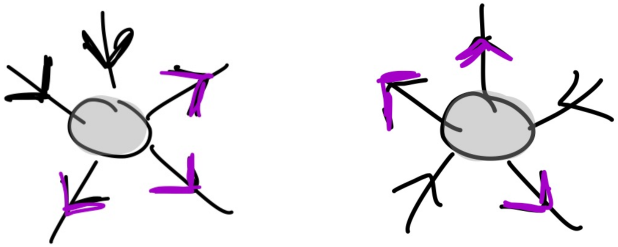
$$S(A) \simeq N\,|\gamma_A|$$

Mostly works in any geometry. By now, many variations known.

HaPPY model

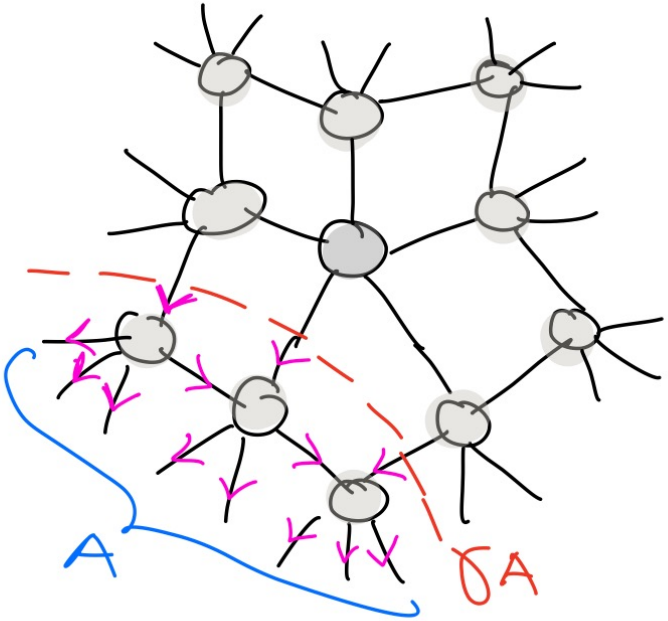
Harlow-Pastawski-
Preskill-Yoshida

Assume each local tensor is perfect = isometry in all possible directions.



#in ≤ #out

exist! e.g. 3-qutrit code, 5-qubit code, ...



Choose orientations such that $\gamma_A \to A, A^c$
Then: V, W isometries and RT formula holds

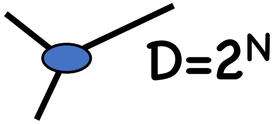
Always possible for graphs with "negative curvature" and A "single interval".

☺ Concrete and intuitive!

How to generalize?

Random tensor model

Hayden-Nezami-Qi-
Thomas-W-Yang



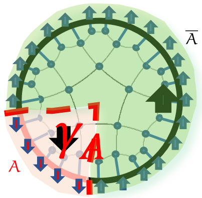
Choose random bulk tensors of large bond dimension.

➔ emergent RT formula

Three interpretations:

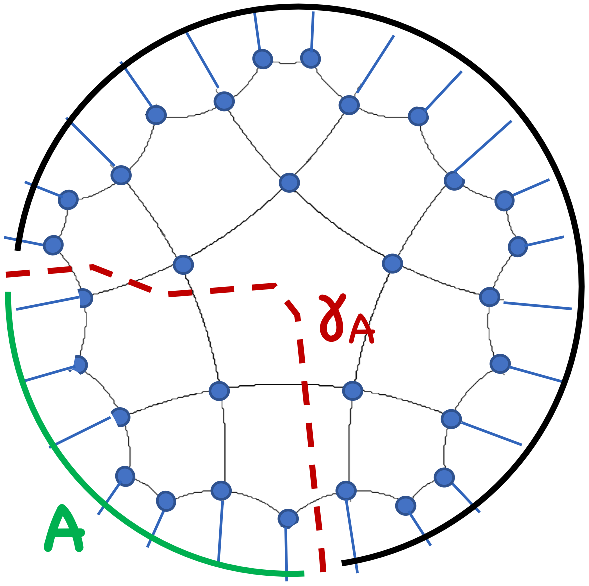
1. Random tensors ≈ perfect
2. Entanglement distillation protocol
3. Disorder average → ferromagnetic spin model

large N → low T



Derivation of Ryu–Takayanagi law

Setup:

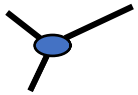


$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

max. entangled states

$$|xy\rangle = \sum_{\mu=1}^{D} |\mu,\mu\rangle$$

random tensors



Arbitrary lattice or graph. Tensors are chosen i.i.d. from Haar measure.

Recall: In any tensor network: $S(A) \leq N\,|\gamma_A|$.

Strategy: Lower bound $S_2(A)$ using replica trick.

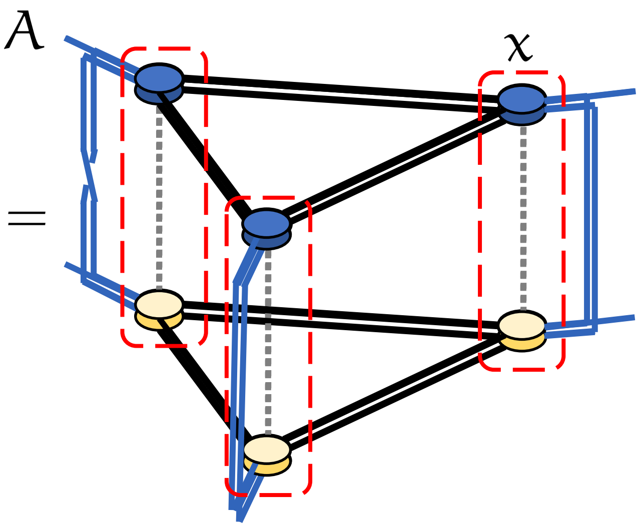
Replica trick for 2nd Rényi

$S_2(A) = -\log \mathrm{tr}[\rho_A{}^2]$

$$|\Psi\rangle = \Big(\bigotimes_{\langle x,y\rangle} \langle xy|\Big)\Big(\bigotimes_{x} |V_x\rangle\Big)$$

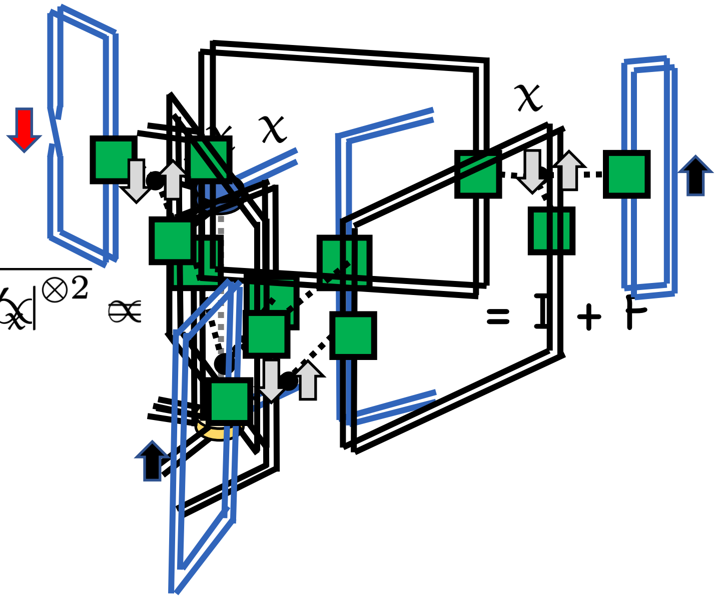
Replica trick: $\mathrm{tr}\,\rho_A^2 = \mathrm{tr}[(\rho \otimes \rho) F_A]$

$\mathrm{tr}\,\rho_A^2 =$



$\mathrm{tr}\,\rho_A^2 \propto$

$|V_x\rangle\langle V_x|^{\otimes 2} \propto$



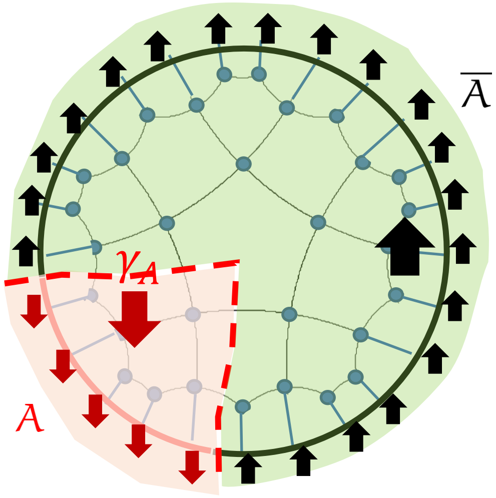
Pick I vs F at each vertex.

Each loop is trace: factor $D=2^N$

Ising variables & boundary conditions!

2^{nd} Rényi entropy

$S_2(A) = \log \text{tr}[\rho_A^n]$



domain wall!

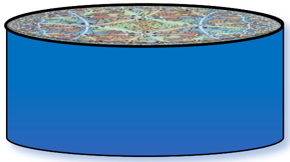
$\text{tr}[\rho_A^2] \approx$ partition function of ferromagnetic Ising model at $1/T = \log(D)$

Result: $\boxed{S_2(A) \approx -\log \text{tr}[\rho_A^2] \approx \log(D)\ |\gamma_A|}$

Ryu-Takayanagi formula!

What does it mean?

Random tensor networks (RTN) provide intuitive toy model. Reproduce Ryu-Takayanagi formula (+ much more). Analyzed using replica trick.

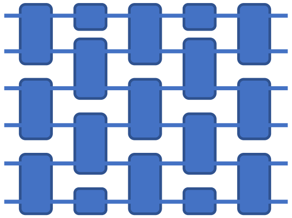


Relevance for holography? Ryu-Takayanagi formula is proved similarly. But: Einstein equations ➔ nontrivial spectrum!

Is all hope lost? No! Remarkably, RTN match precisely so-called fixed-area states in holography.

✓ Dong-Harlow-Marolf Penington et al

Moreover, general states can be expanded in terms of fixed-area states. Under certain "diagonal approximations", can lift results!



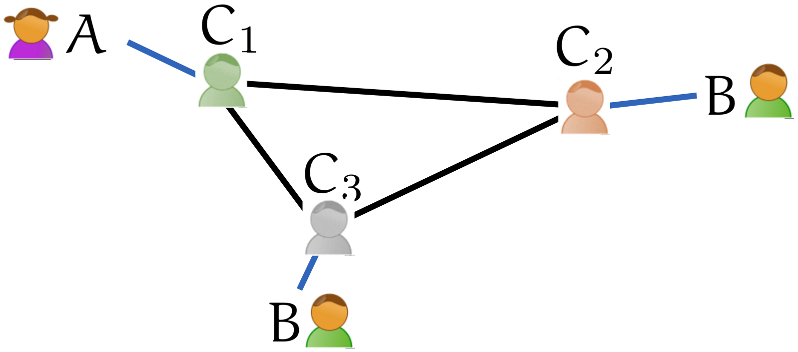
Similarly, random quantum circuit models have recently been studied, exhibit interesting phenomenology.

relevant to "quantum supremacy" proposals etc

Bonus: Entanglement of assistance

[Smolin-Verstraete-Winter]
[Hayden-Dutil]

Multiparty entanglement distillation: create entanglement between Alice & Bob with help of Charlies by measurements & classical communication.



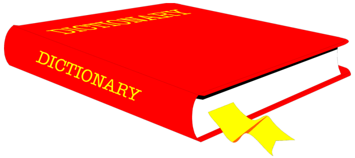
$$|\Psi\rangle = \underbrace{\Big(\bigotimes_{x} \langle V_x|\Big)}_{\text{measurement in random basis}} \overbrace{\Big(\bigotimes_{\langle x,y\rangle'} |xy\rangle\Big)}^{\text{initial collection of Bell pairs}}$$

optimal! merges state w.h.p.

$$\lim_{n\to\infty} \frac{1}{n} E_{assist}(A^n; B^n) = \min_{M \subset V \setminus AB} S(A \cup M) = S_{RT}(A)$$

General mechanism for producing Ryu-Takayanagi from area law state!

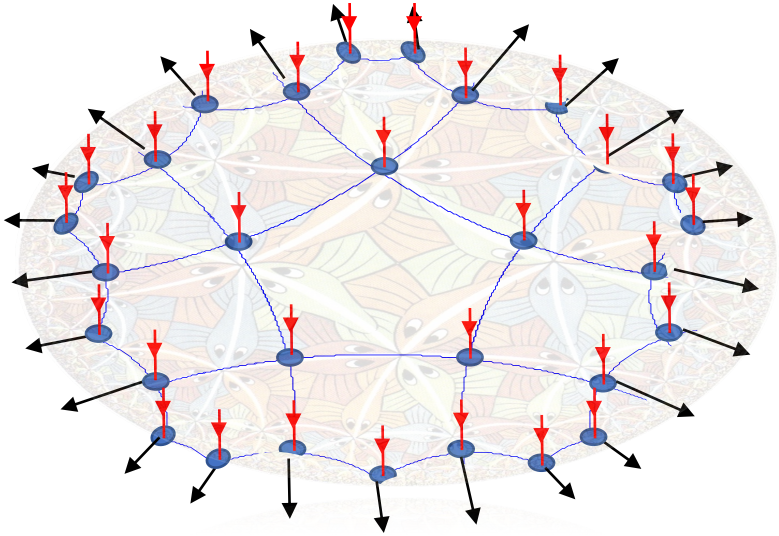
Holographic mappings



AdS/CFT is duality between two theories = “dictionary” that maps states & observables. How to incorporate into toy model?

Approach: Define bulk-boundary mapping via tensor network

= combination of both toy models

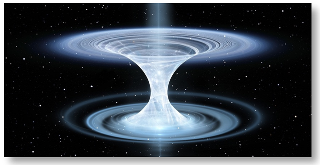


red legs: bulk degrees
black legs: boundary degrees

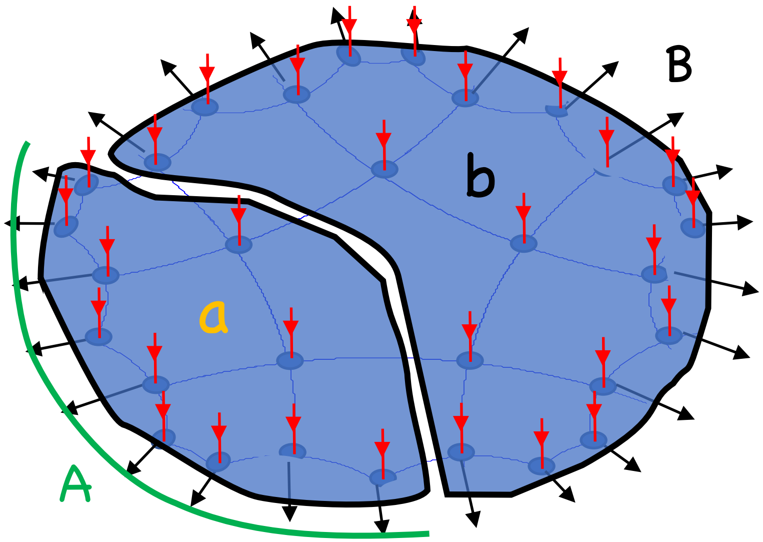
“logical” bulk states are encoded in “physical” boundary Hilbert space

Toy model of how bulk quantum fields get encoded in boundary CFT

Holographic codes



If bulks legs have small dimension d << D, obtain error correcting code that satisfies "subregion duality", a key QI feature of AdS/CFT:



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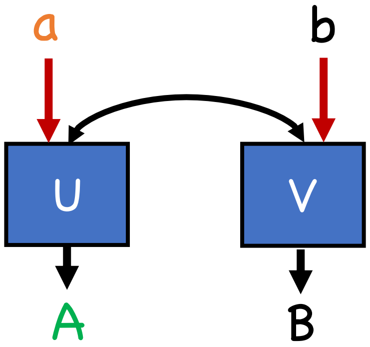


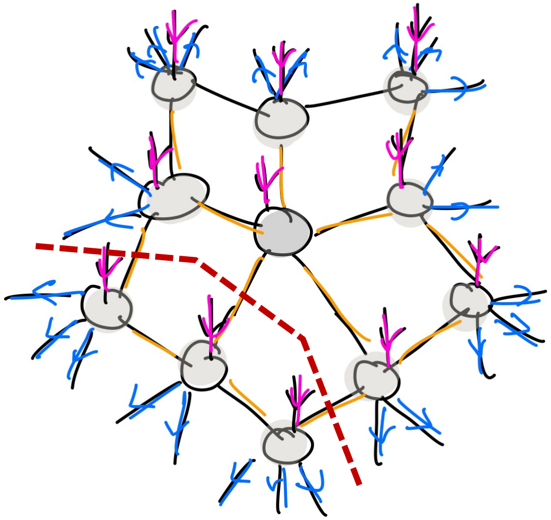
...for isometries U, V.

Bulk degrees of freedom in a (b) get encoded into A (B)! ✓

In particular, bulk corrections to entropy: $S(A) \approx N\,|\gamma_A| + S(a)$ ✓

Proof of subregion duality

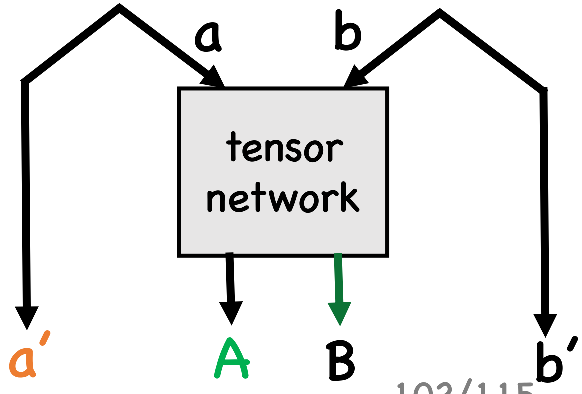
1) HaPPY argument: Choose orientations s.th. $a\gamma_A \rightarrow A$, $b\gamma_A \rightarrow B$.



Interpretation: Holographic codes are macroscopic erasure codes built from microscopic ones (perfect tensors).

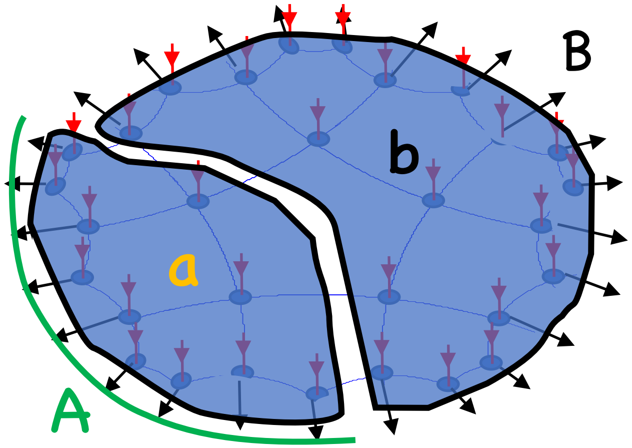
2) Decoupling argument: Only need to prove that $I(a':b'B) = 0$. Why? See later!

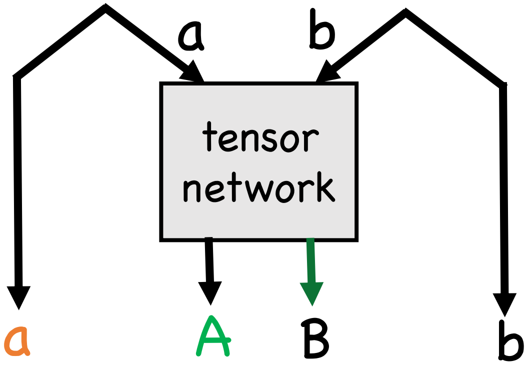
➔ Can prove geometrically since Choi state satisfies Ryu-Takayanagi!



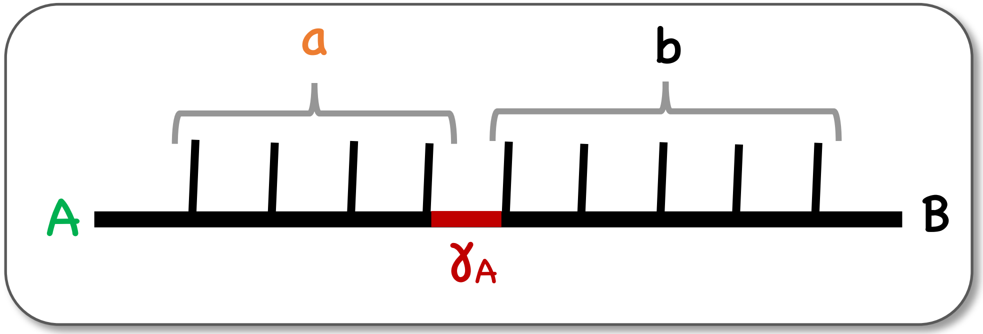
Subregion duality from decoupling

By decoupling, suffices to prove that $\boxed{I(a:bB) \approx 0}$ in Choi state:





Schematically:



$$S(a) = \log(d)\ |a|$$

$$S(bB) = \log(D)\ |\gamma_A|$$

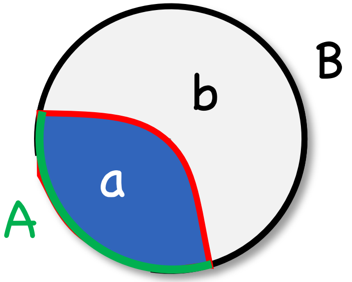
$$S(abB) = \log(D)\ |\gamma_A| + \log(d)\ |a|$$

Assume bulk legs have small dimension $d \ll D$.

Quantum minimal surfaces and islands 🏝️

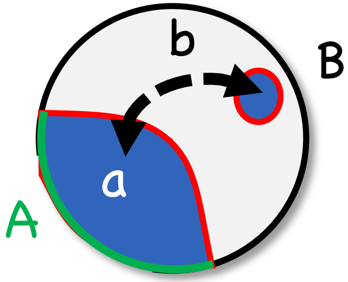
What if bulk entropy is not small?

$$\boxed{S_2(A) \simeq \min \{ N |\gamma_A| + S_2(a) \}}$$ ✓



“Quantum minimal surface”, minimizes “generalized entropy”.

Proof using replica trick (additional action from bulk state)!



E.g., if we add highly entangled state between distant bulk sites, obtain “island” disconnected from boundary.

> Holographic counterparts feature crucially in very recent developments on black hole information paradox that seek to give a bulk picture of black hole evaporation.

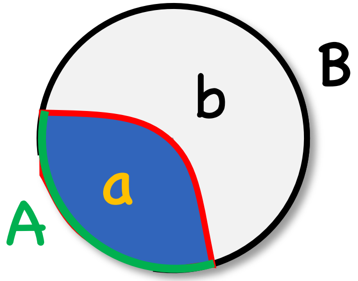
Penington
Almheira et al

Surprising that the simple RTN model reproduces these features!?

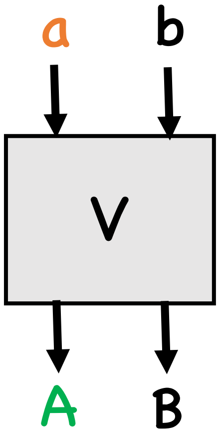
9. Subregion Duality and Subsystem Error Correction

Literature: https://arxiv.org/abs/1607.03901

Subregion duality



Let us talk more systematically about the quantum information structure of subregion duality. Consider an isometry:

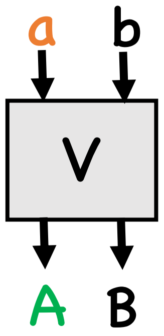


Notation: ρ_{ab} state ➔ $\rho_{AB} = V\rho_{ab}V^{\dagger}$

Subsystem error correction: When can we recover a from A?

More subtle than what we discussed last lecture. There we had no "b" system – now ρ_{ab} can be correlated or entangled!

Subsystem error correction



The following conditions are equivalent:

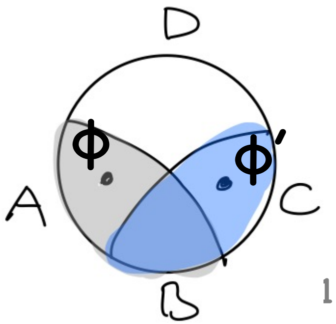
1) There is a channel $D_{A\to a}$ such that: $D(\rho_A) = \rho_a$ for all ρ_{ab}

2) For all ϕ_a, exists O_A such that: $V\phi_a = O_A V$ and $V\phi_a^\dagger = O_A^\dagger$
V
3) For all ϕa, and XB: $[\phi a, V^\dagger XBV]=0$

4) **Decoupling:** $I(a':b'B) = 0$ i.e. $\Omega_{A'b'B} = \Omega_{A'} \otimes \Omega_{b'B}$

Aside: 2) allows computing correlation functions – even if we use different subsystems for each operator:

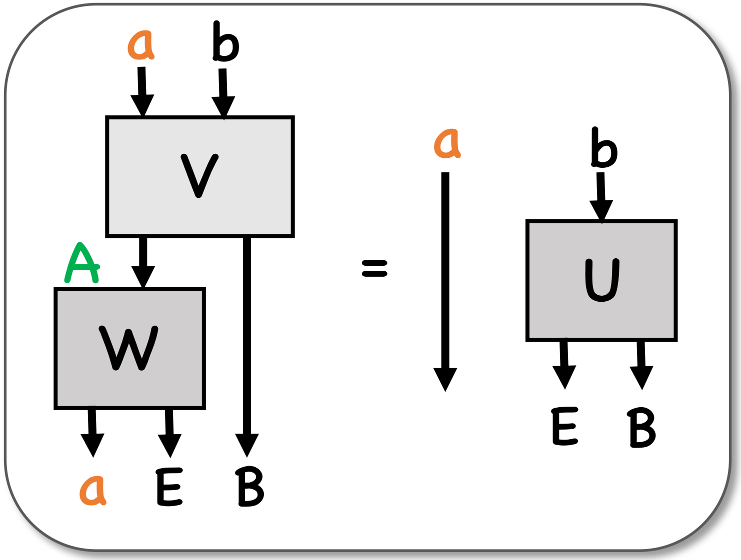
$$\langle \phi_{ab}\, \phi'_{bc} \rangle = \langle O_{AB}\, O'_{BC} \rangle$$



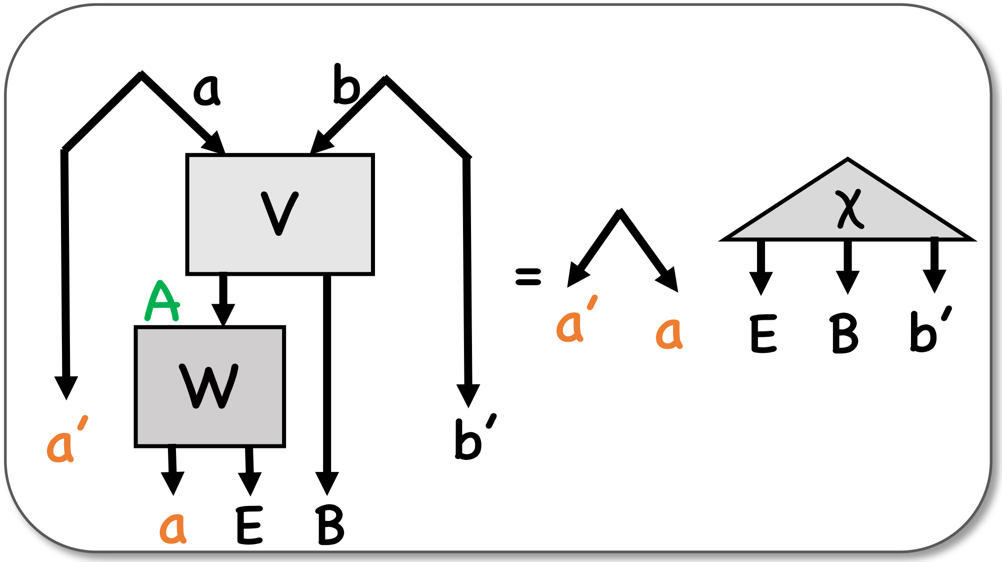
Proof sketch of equivalence

1) There is a channel $D_{A\to a}$ such that: $D(\rho_A) = \rho_a$ for all ρ_{ab}

⇔ Stinespring extension:



⇔ Choi state:



$O_A = W^\dagger \phi_a W$

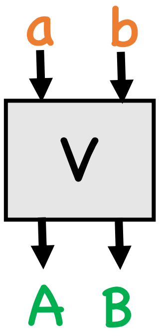
compare purifications

2) $V\phi_a = O_A V$
$V\phi_a^\dagger = O_A^\dagger V$

3) $[\phi_a, V^\dagger X_B V]=0$

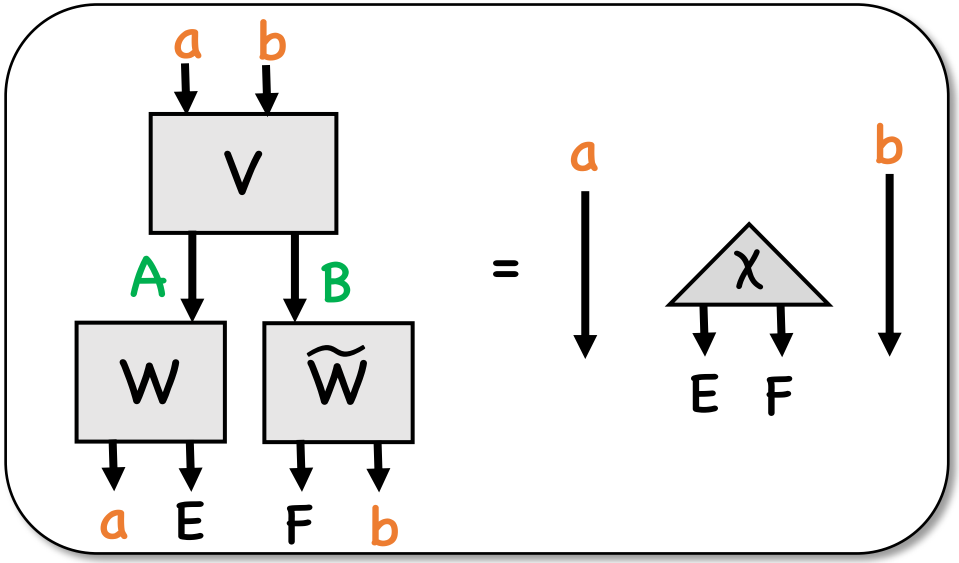
4) $I(a':b'B) = 0$

Complementary recovery



When can we recover a from A and b from B? Result:

Normal form:



Precisely like in toy models!

Ryu-Takayanagi formula:

$S(A) = c + S(a)$ for all ρ_{ab}

$S(B) = c + S(b)$ for all ρ_{ab}

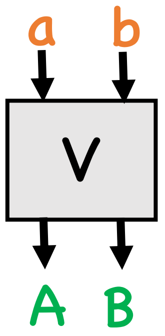
$c = S_E(\chi)$

The punchline:

RT formula is also sufficient for subregion duality.

➔ "proof" that latter holds in AdS/CFT

Bonus: Proof that Ryu-Takayanagi implies complementary recovery



Assume: $\boxed{S(A) = c + S(a) \text{ for all } \rho_{ab}}$

Use 1st law: $\mathrm{tr}[K_a\delta\rho_{ab}] = \delta S(a) = \delta S(A) = \mathrm{tr}[K_A\delta\rho_{AB}] = \mathrm{tr}[V^\dagger K_A V\delta\rho_{ab}]$

➔ $\boxed{V^\dagger K_A V = c + K_a}$ for modular Hamiltonians $K_a = -\log \rho_a$, $K_A = -\log \rho_A$

$D(\rho_A \| \sigma_A) = -\mathrm{tr}[\rho_A K_{\rho A}] - \mathrm{tr}[\rho_A K_{\sigma A}] = -\mathrm{tr}[\rho_a V^\dagger K_{\rho A} V] - \mathrm{tr}[\rho_a V^\dagger K_{\sigma A} V] = \ldots$

➔ $\boxed{D(\rho_A \| \sigma_A) = D(\rho_a \| \sigma_a) \text{ for all } \rho_{ab} \text{ and } \sigma_{ab}}$

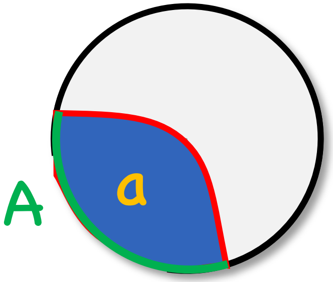
$\rho_{ab} = e^{i\phi bs}\,\sigma_{ab}\,e^{-i\phi bs}$

$\boxed{[\phi_b, V^\dagger X_A V]=0}$

Use Petz map to obtain decoder $D_{A\to a}$

Homework: Fill in the details.

Decoding the hologram (using error correction)



This proof of "entanglement wedge" reconstruction property is nonconstructive & nonrobust

How to find boundary reconstruction of local bulk operator?

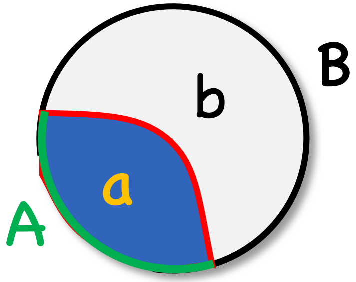
Banks et al, Hamilton et al, Kabat et al, Heemskerk et al, Lin et al, Faulkner-Lewkowycz, ...

Recall: Only understood in special cases.

Recent progress in theory of quantum error correction led to robust proof. More explicit formulas and decoding protocols?

Cotler-...-W, Kitaev-Yoshida, Hayden-Penington

State dependence



Theorem models situation where minimal surface can be considered fixed for all states in code subspace (no backreaction).

In general, state-dependent! "Quantum" minimal surface obtained by minimizing generalized entropy:

$$S(A) = \min \left\{ \frac{|\gamma_A|}{4G} + S(a) \right\}$$

realized in random tensor network model! ✓

This form of subregion duality has featured crucially in very recent research on the black hole information paradox that seeks to give a bulk picture of black hole evaporation.

→ Penington, Almheira et al, lectures by Netta?

Summary

Whirlwind tour through some key concepts and tools of quantum information, motivated by applications to QFT and holography:

- States, Channels, Entropy
- Entanglement of Pure and Mixed States
- Entanglement in Field Theory and Holography
- Toy Models of Holography
- Quantum Error Correction and Decoupling

Page curve

Hayden-Preskill

3-qutrit code

tensor network models

holographic codes

No time for quantum computing: circuits, algorithms, complexity, … ☹

Slides: https://staff.fnwi.uva.nl/m.walter/

The road ahead

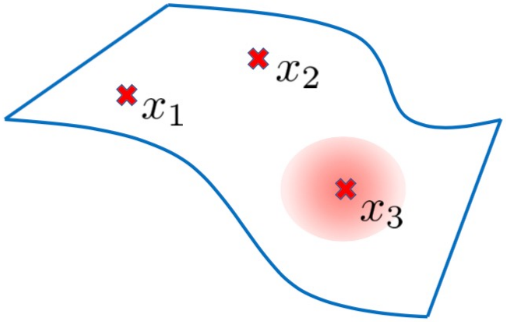
Tensor networks discretize space, but gravity is about space-time: *dynamics, backreaction, causal structure?*

Q. information vs geometry: holography in flat space & de Sitter? superpositions of geometries?

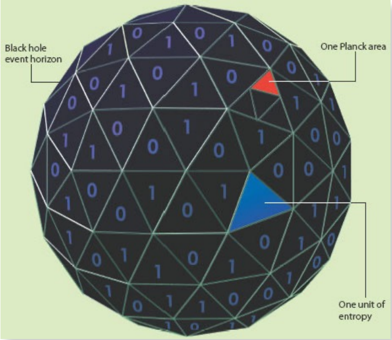
Practical diagnostics for entanglement and correlations

What makes a CFT gravitational?

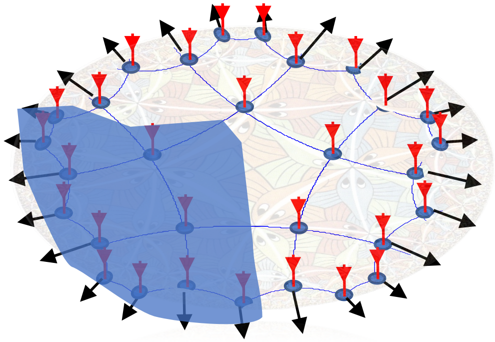
Continuum limits of states and circuits



Summary



Holography predicts remarkable connection between geometry and entanglement



Quantum information offers tools, models, mechanisms from tensor networks to QEC

Ongoing research to exploit connections

Motivation ranges from trying to understand the emergence of space-time from quantum mechanics to learning how dualities can help simulate complex quantum systems on (quantum) computer…

Thank you for your attention!!!

(And thanks to Freek!!!)