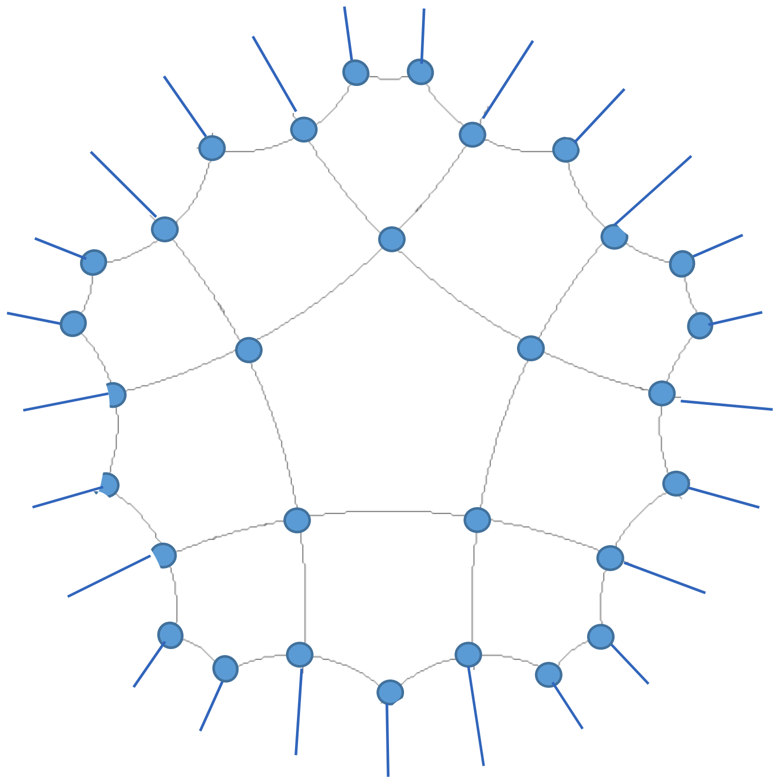


Random tensor networks & holographic duality

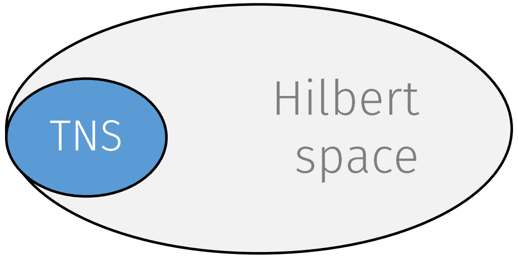
Michael Walter
Stanford University

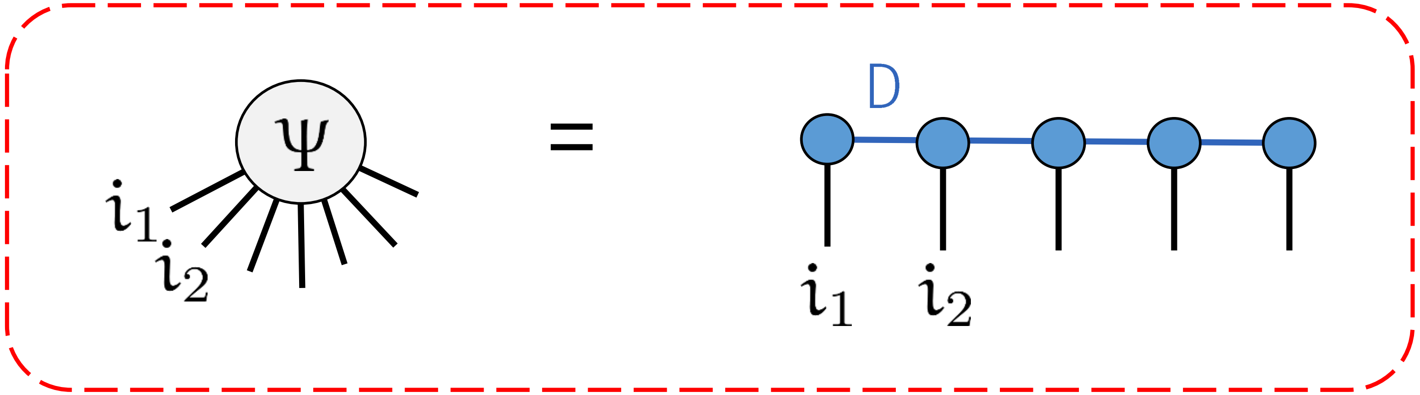
joint with P. Hayden, S. Nezami, X. Qi, N. Thomas, Z. Yang

Condensed Matter Theory Seminar, Cologne - March 2016

Tensor network states

$$|\Psi\rangle = \sum_{i_1,\dots,i_N} \boxed{\Psi_{i_1,\dots,i_N}} |i_1,\dots,i_N\rangle$$





example: matrix product state

Efficient variational classes & useful theoretical formalism

ground states of quantum matter

quantum phases

RG circuits

topological order

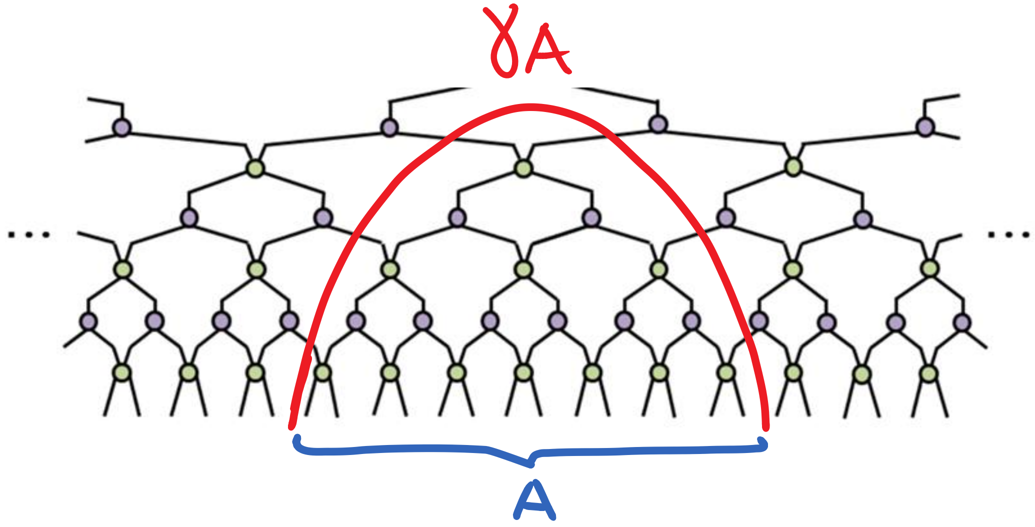
...

Tensor network kinematics (or: how to choose your corner in Hilbert space)

Fundamental bound on ent. entropy:

$$S(A) \leqslant \log D \, |\gamma_A|$$

where $S(A) = -\operatorname{tr} \rho_A \log \rho_A$.



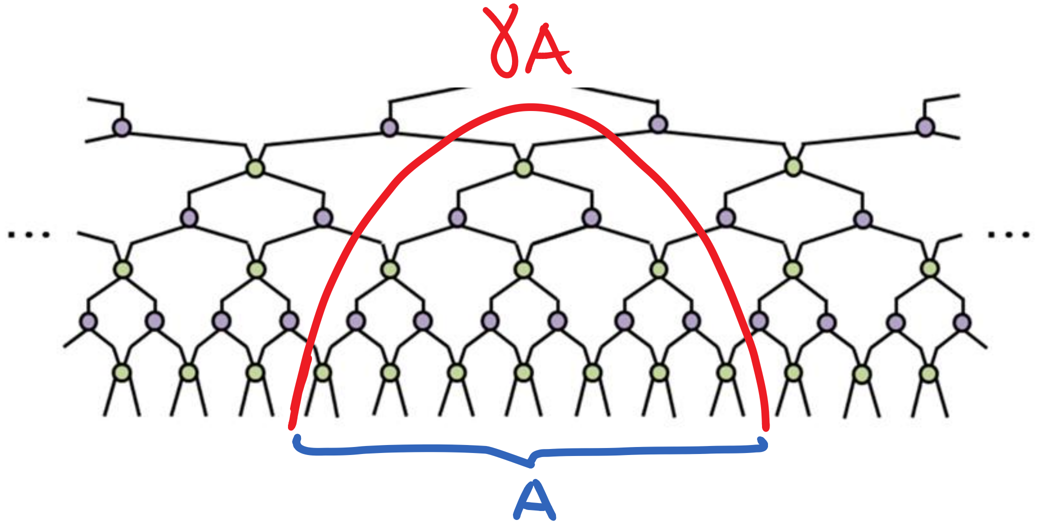
Figures from [Vidal]

Tensor network kinematics (or: how to choose your corner in Hilbert space)

Fundamental bound on ent. entropy:

$$S(A) \leqslant \log D\,|\gamma_A|$$

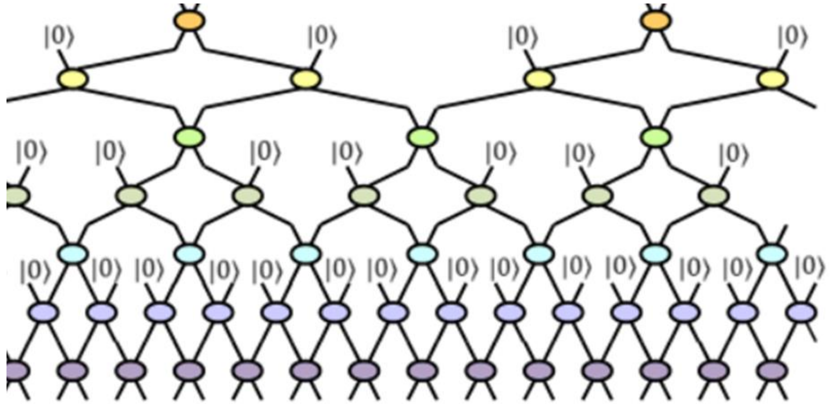
where $S(A) = -\operatorname{tr} \rho_A \log \rho_A$.



Bulk-boundary dualities: lift physics to the virtual level, e.g.

entanglement Hamiltonian
[Cirac et al]

MERA as a RG circuit
[Evenbly-Vidal]



Figures from [Vidal]

Organization of quantum information? Properties of the bulk theory?

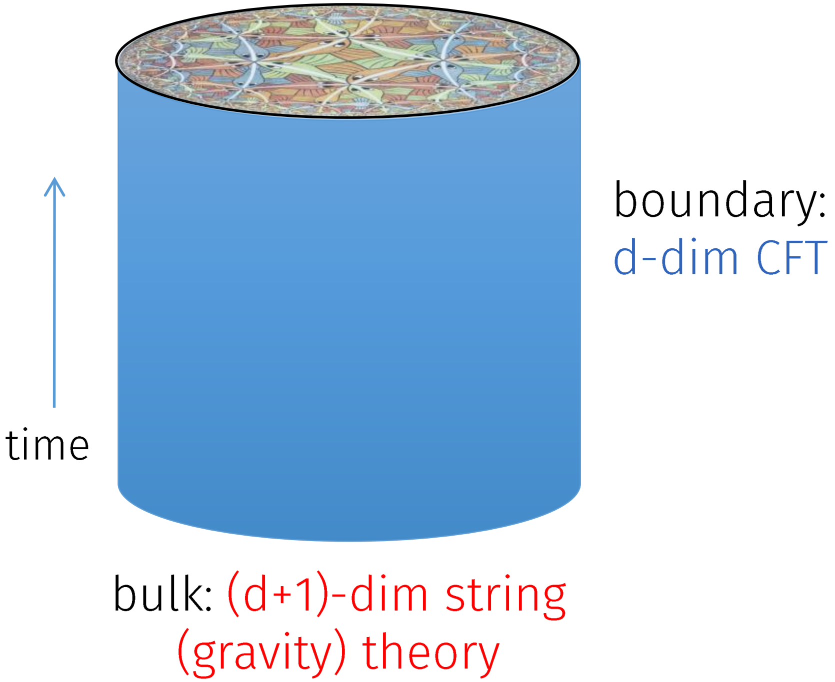
Motivation: Quantum Gravity

[Bekenstein-H.]; [Susskind], [t'Hooft]; [Maldacena]

Holographic principle: All information in a region of space can be represented as a "hologram" living on region's boundary.

$$S_{BH} = \frac{A}{4G_N}$$

AdS/CFT duality: conjectural realization



What is the basic mechanism? Fine-tuned or typical phenomenon?

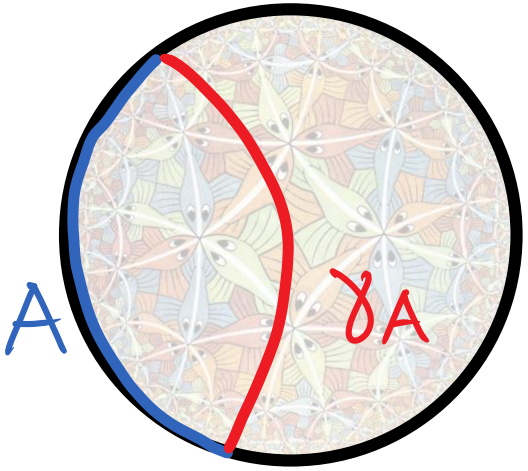
Motivation: Quantum Gravity

[Bekenstein-H.]; [Susskind], [t'Hooft]; [Maldacena]

Holographic principle: All information in a region of space can be represented as a "hologram" living on region's boundary.

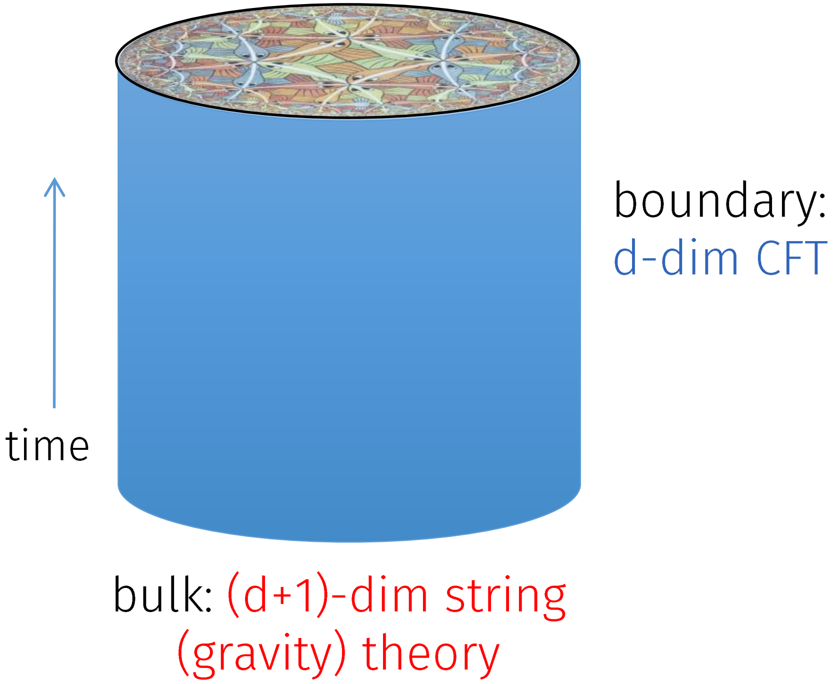
$$S_{BH} = \frac{A}{4G_N}$$

AdS/CFT duality: conjectural realization



$$S(A) = \frac{1}{4G_N} \min |\gamma_A|$$

[Ryu-Takayanagi]



What is the basic mechanism? Fine-tuned or typical phenomenon?

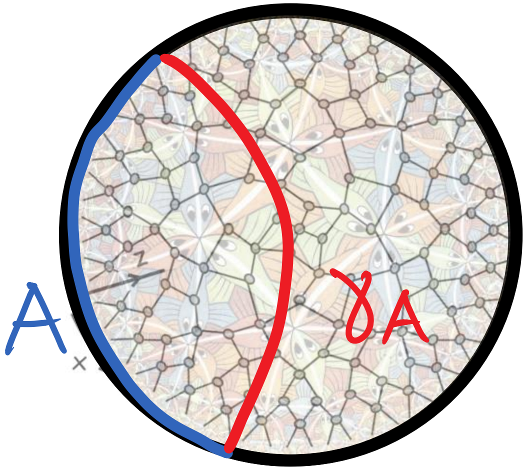
Motivation: Quantum Gravity

[Bekenstein-H.]; [Susskind], [t'Hooft]; [Maldacena]

Holographic principle: All information in a region of space can be represented as a "hologram" living on region's boundary.

$$S_{BH} = \frac{A}{4G_N}$$

AdS/CFT duality: conjectural realization

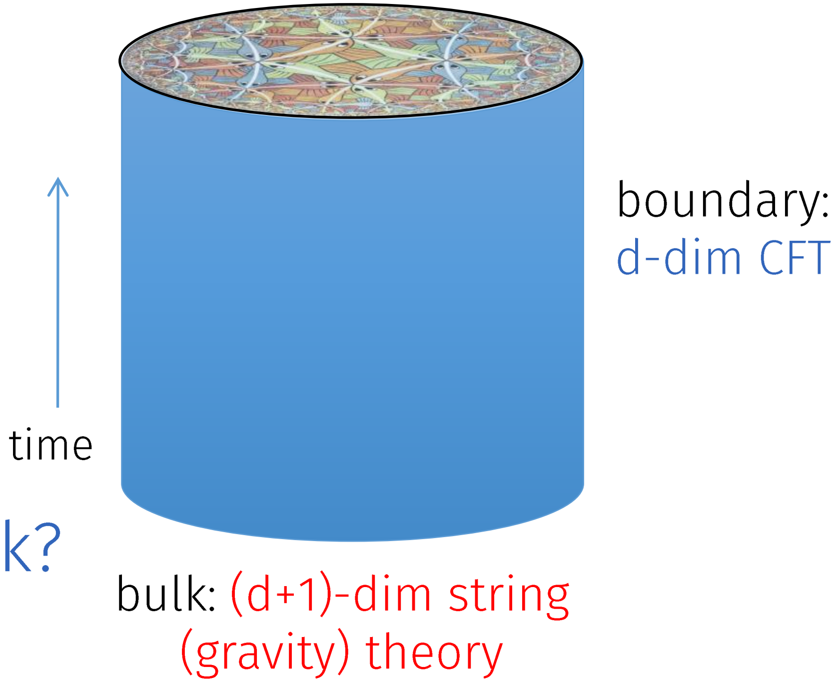


$$S(A) = \frac{1}{4G_N} \min |\gamma_A|$$

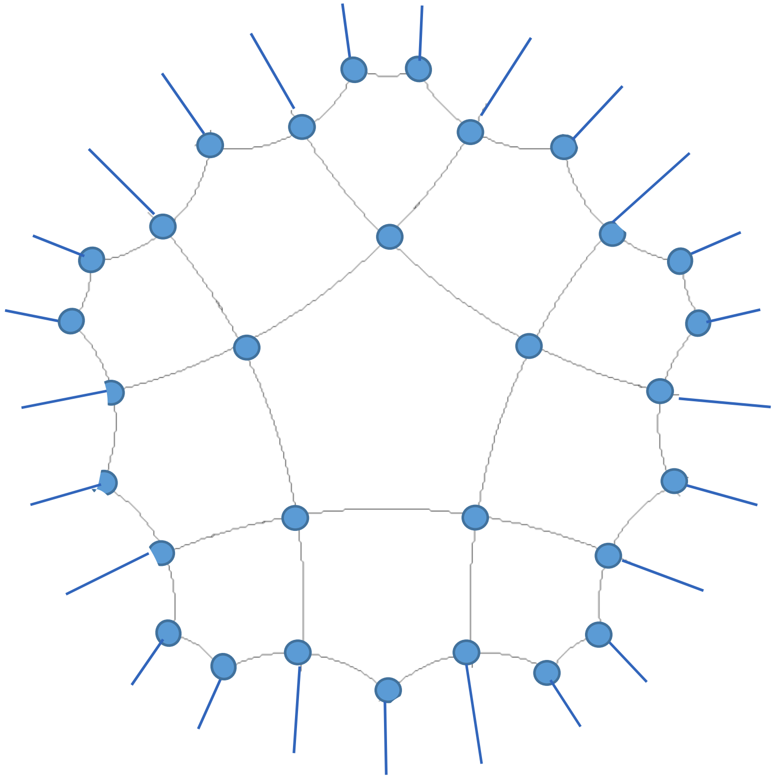
[Ryu-Takayanagi]

Space-time as a tensor network?

[Swingle]

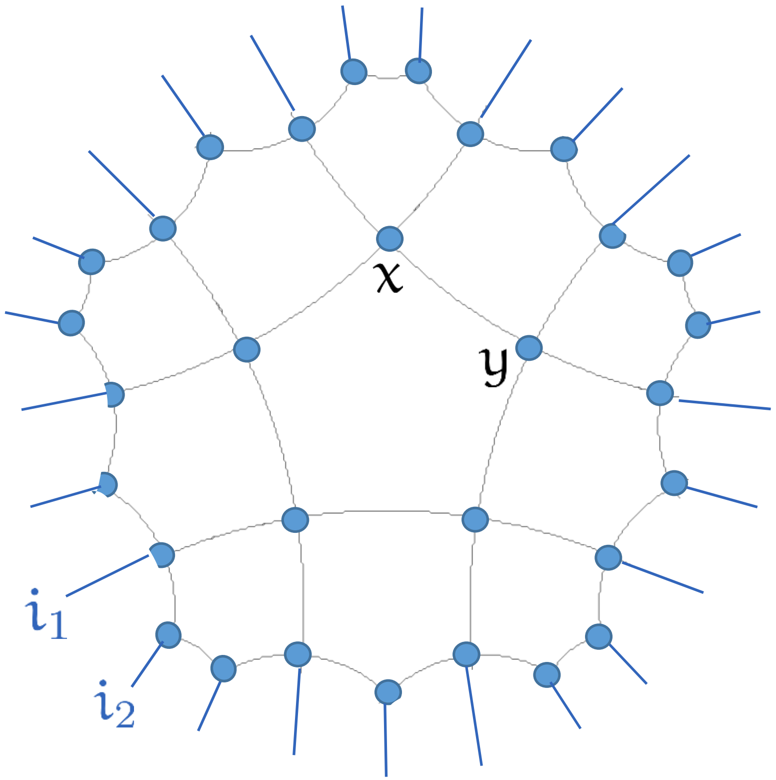


What is the basic mechanism? Fine-tuned or typical phenomenon?



Random tensor network states

Random tensor network states

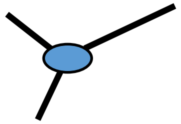


$$|\Psi\rangle = \Big(\bigotimes_{\langle x,y\rangle} \langle xy|\Big)\Big(\bigotimes_{x} |V_x\rangle\Big)$$

max. entangled states

$$|xy\rangle = \sum_{\mu=1}^{D} |\mu,\mu\rangle$$

random tensors

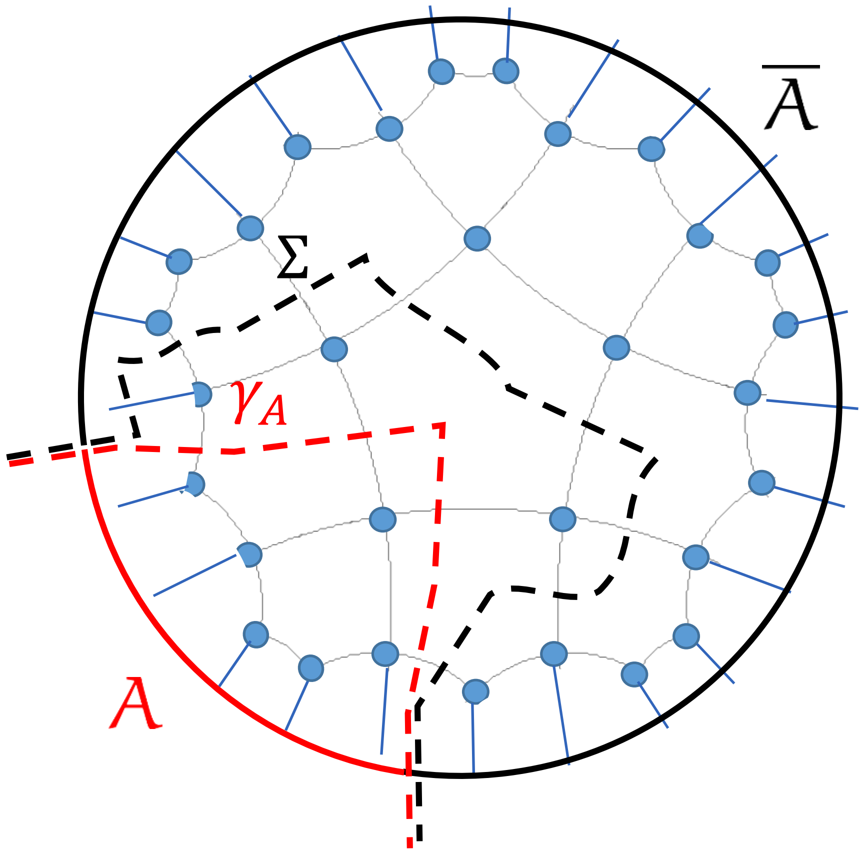


Random tensor network state on "boundary" of graph

Arbitrary lattice or graph. Tensors are chosen i.i.d. from Haar measure.

Entanglement entropy

$$S(A) = -\operatorname{tr} \rho_A \log \rho_A$$



In any tensor network:

$$S(A) \leqslant \log D \, |\gamma_A|$$

Goal: Show that saturated in random tensor networks with large bond dimension D.

Strategy: Lower bound via Renyi entropy

$$S_2(A) = -\operatorname{tr} \log \rho_A^2$$

Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

$$|\Psi\rangle = \Big(\bigotimes_{\langle x,y\rangle} \langle xy|\Big)\Big(\bigotimes_{x} |V_x\rangle\Big)$$

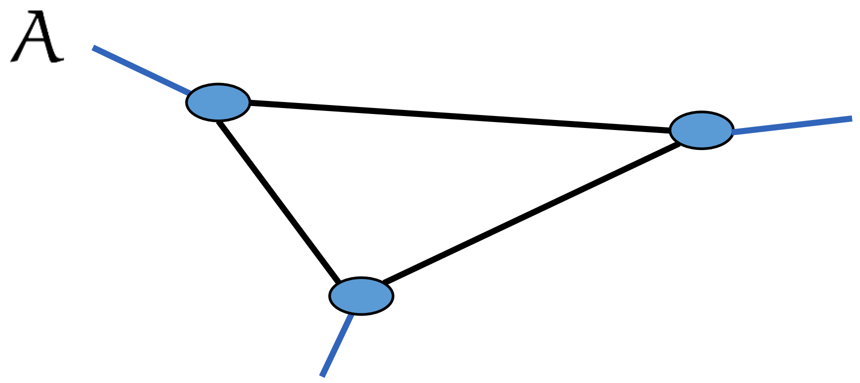
Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

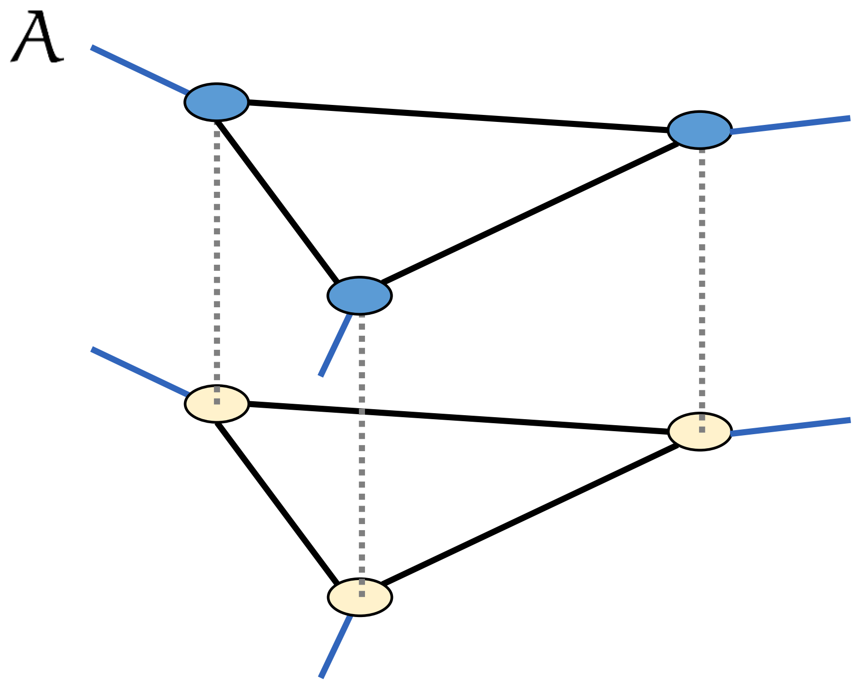


Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y \rangle} \langle xy| \right) \left(\bigotimes_{x} |V_x\rangle \right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

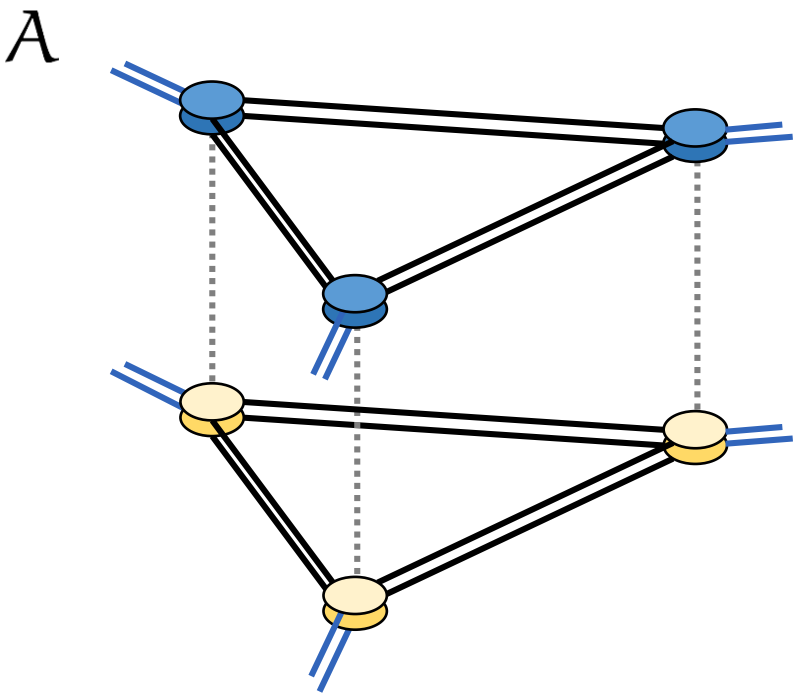


Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_x |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$



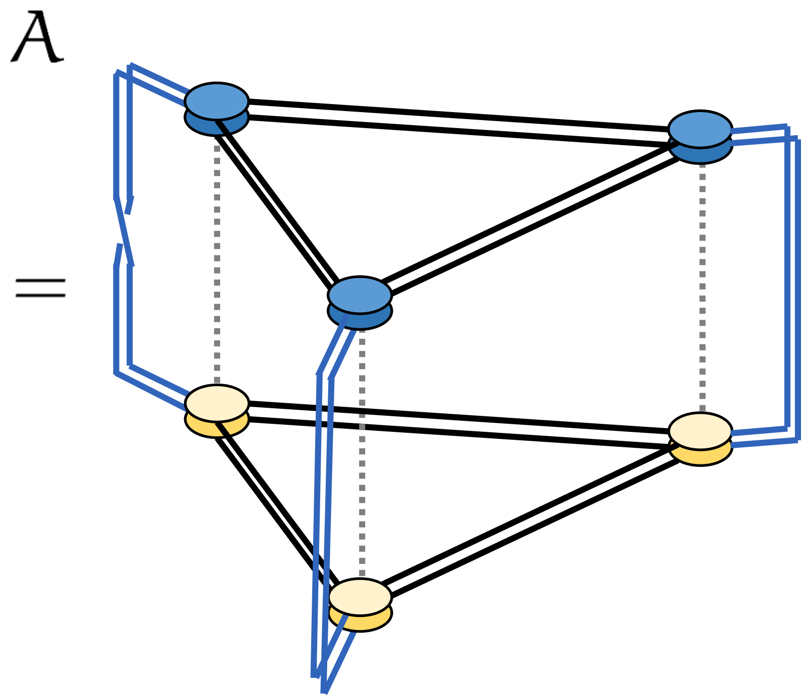
Calculation of Renyi entropy

$$S_2(A) = -\log \operatorname{tr} \rho_A^2$$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

$\operatorname{tr} \rho_A^2 =$



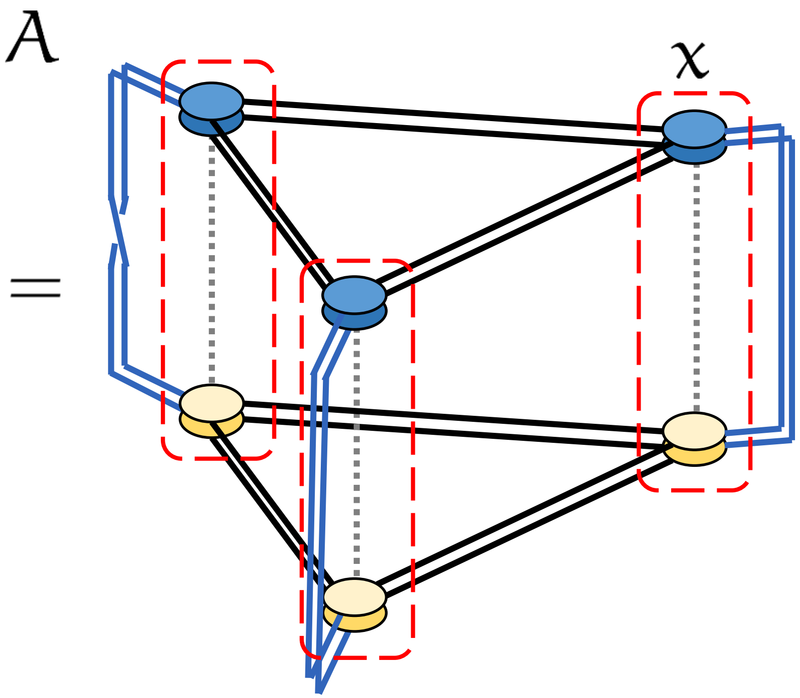
Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

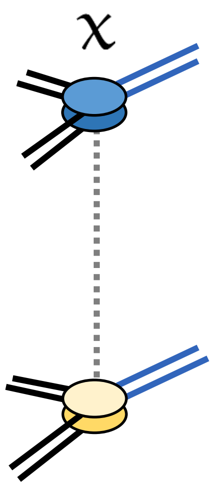
$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

$\operatorname{tr} \rho_A^2 =$



$|V_x\rangle\langle V_x|^{\otimes 2} =$

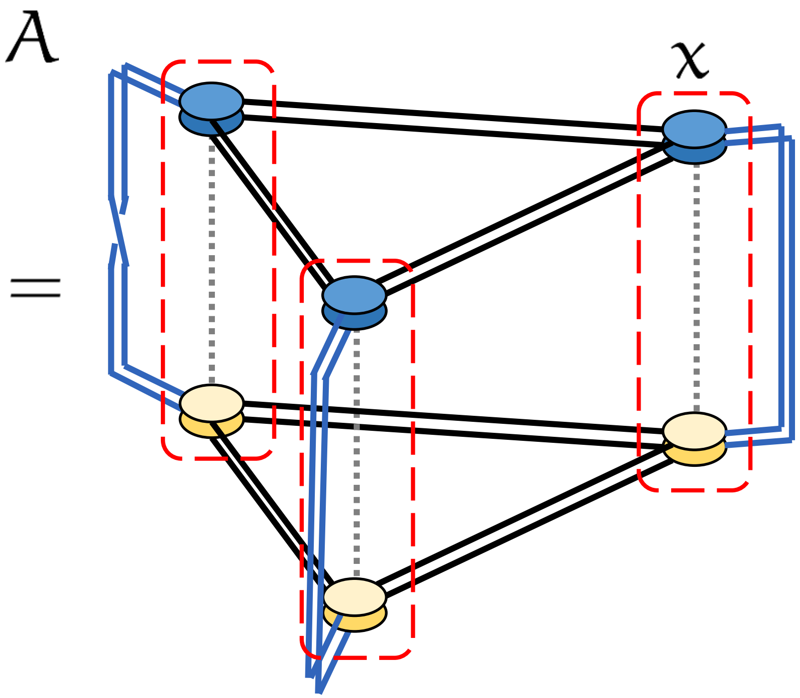


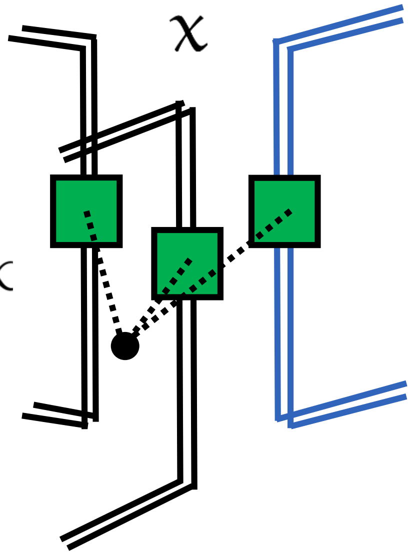
Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_x |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

$\operatorname{tr} \rho_A^2 =$ 

$\overline{|V_x\rangle\langle V_x|^{\otimes 2}} \propto$ 

projector onto symmetric subspace

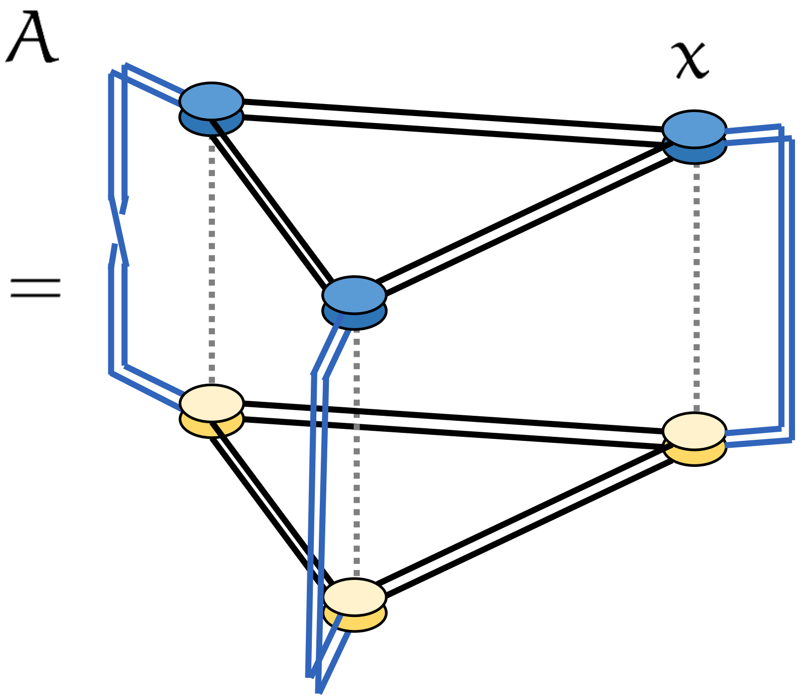
Calculation of Renyi entropy

$S_2(A) = -\log \operatorname{tr} \rho_A^2$

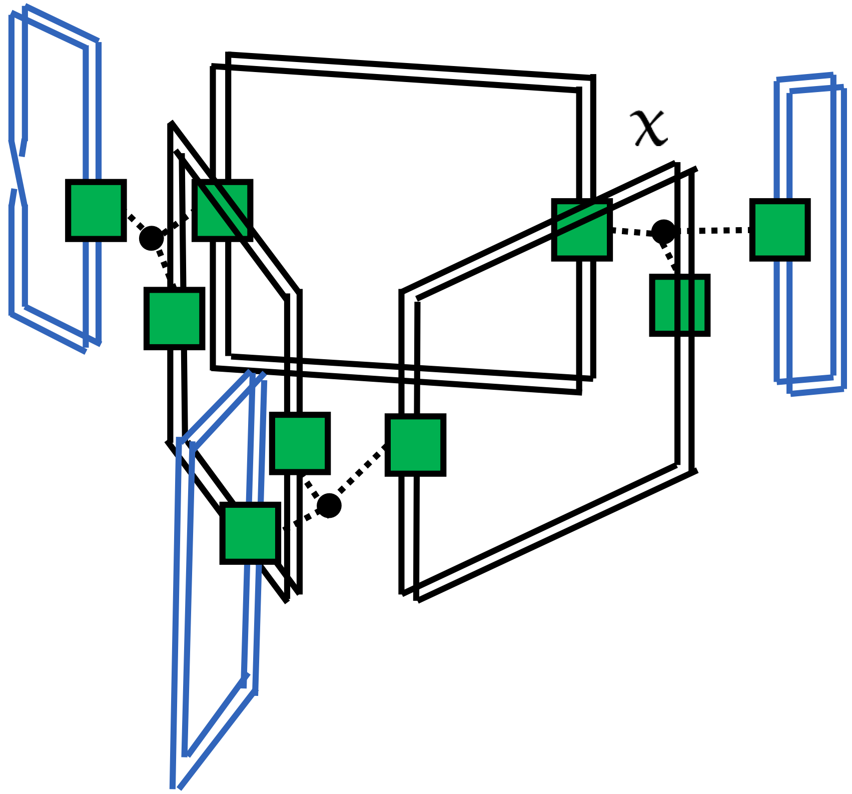
$$|\Psi\rangle = \Big(\bigotimes_{\langle x,y\rangle} \langle xy|\Big)\Big(\bigotimes_x |V_x\rangle\Big)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

$\operatorname{tr} \rho_A^2 =$



$\overline{\operatorname{tr} \rho_A^2} \propto$



each loop is trace: factor D

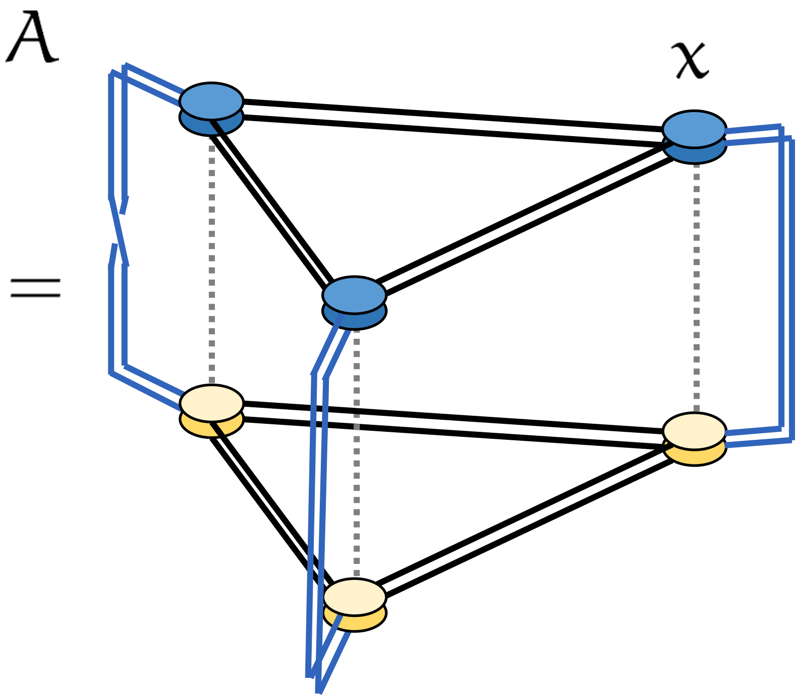
Calculation of Renyi entropy

$$S_2(A) = -\log \operatorname{tr} \rho_A^2$$

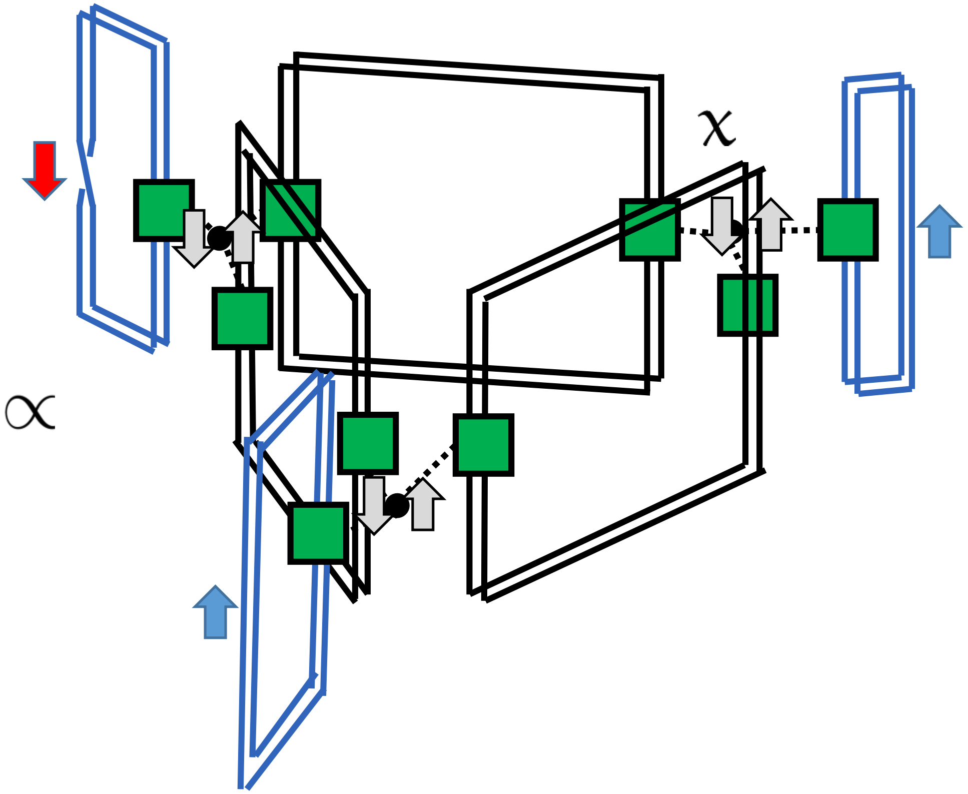
$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

Replica trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr}(\rho \otimes \rho) F_A$

$\operatorname{tr} \rho_A^2 =$



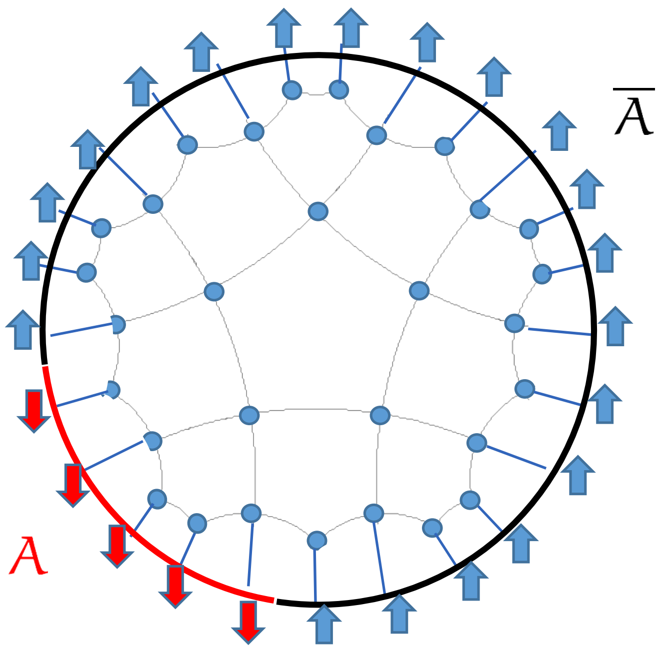
$\overline{\operatorname{tr} \rho_A^2} \propto$



Ising variables & boundary conditions!

Renyi entropy and Ising model

$$S_2(A) = -\log \operatorname{tr} \rho_A^2$$



1/T ferromagnetic Ising action

$$\overline{\operatorname{tr} \rho_A^2} \simeq Z_A = \sum_{\{s_x\}} e^{-\log D \times \frac{1}{2} \sum_{\langle x,y \rangle} (1 - s_x s_y)}$$

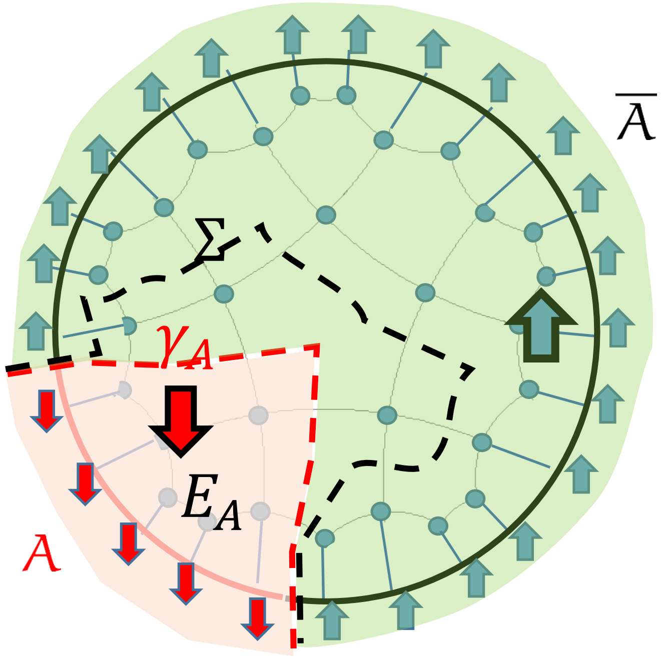
$$S_2(A) \simeq -\log Z_A$$

free energy

… + O(1) if multiple minimal domain walls. NB: Can estimate D_{crit} from Ising physics [Onsager]!
Higher Renyi entropies and fluctuations controlled by higher S_n models.

Renyi entropy and Ising model

$S_2(A) = -\log \operatorname{tr} \rho_A^2$



1/T ferromagnetic Ising action

$$\overline{\operatorname{tr} \rho_A^2} \simeq Z_A = \sum_{\{s_x\}} e^{-\log D \times \frac{1}{2} \sum_{\langle x,y \rangle} (1 - s_x s_y)}$$

$$S_2(A) \simeq -\log Z_A \simeq \log D |\gamma_A|$$

large D / low T

free energy dominated by minimal energy cfg.

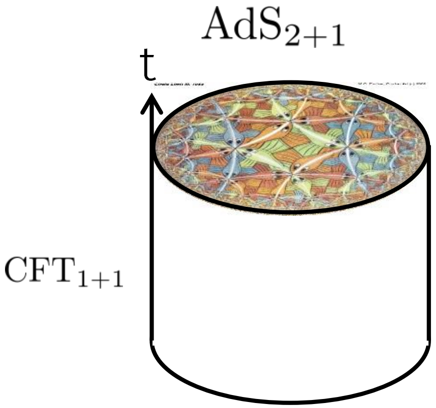
The same is true for the entanglement entropy:

$$S(A) \simeq \log D |\gamma_A|$$

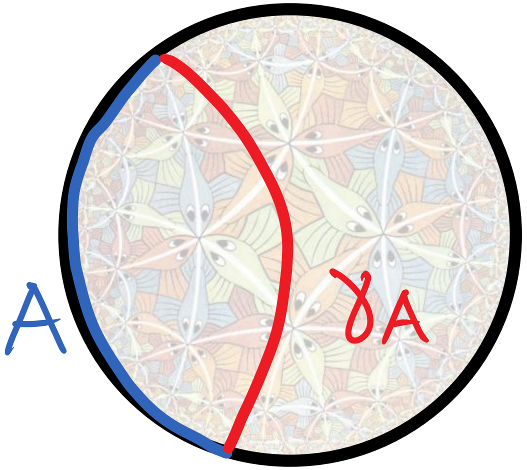
... + O(1) if multiple minimal domain walls. NB: Can estimate D_{crit} from Ising physics [Onsager]!
Higher Renyi entropies and fluctuations controlled by higher S_n models.

Entanglement entropy in AdS/CFT

[Ryu-Takayanagi]



This matches precisely the Ryu-Takayanagi proposal:



$$S(A) = \frac{1}{4G_N} \min |\gamma_A|$$

Possible interpretation: *(1) Start with semiclassical bulk quantum gravity state for which Planckian degrees of freedom satisfy area law.*

(2) Fix bulk Planckian degrees of freedom to typical values.

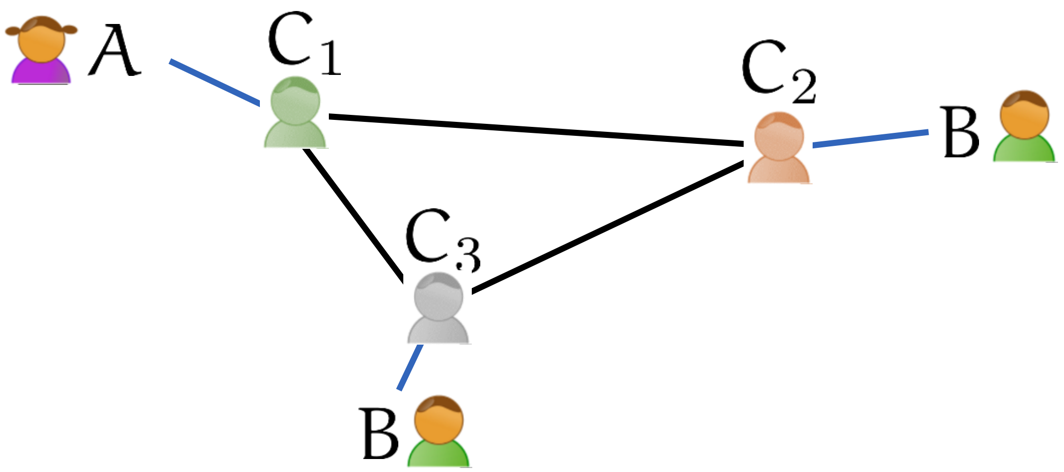
(3) This induces the Ryu-Takayanagi formula on the dual boundary CFT .

Random tensor construction generalizes previous checks [Headrick-Takayanagi]!

QIT interlude I: entanglement of assistance

[Smolin-Verstraete-Winter]
[Hayden-Dutil]

Multiparty entanglement distillation: induce entanglement between Alice and Bob with help of Charlies by measuring & classically communicating results



initial collection of Bell pairs

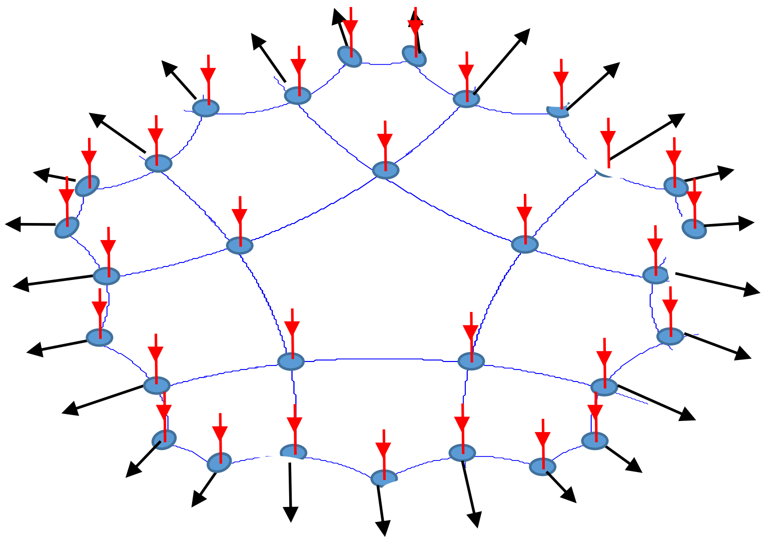
$$|\Psi\rangle = \left(\bigotimes_x \langle V_x|\right)\left(\bigotimes_{\langle x,y\rangle'} |xy\rangle\right)$$

measurement in random basis

optimal! merges state w.h.p.

$$\lim_{n\to\infty} \frac{1}{n} E_{assist}(A^n; B^n) = \min_{M \subset V \setminus AB} S(A \cup M) = S_{RT}(A)$$

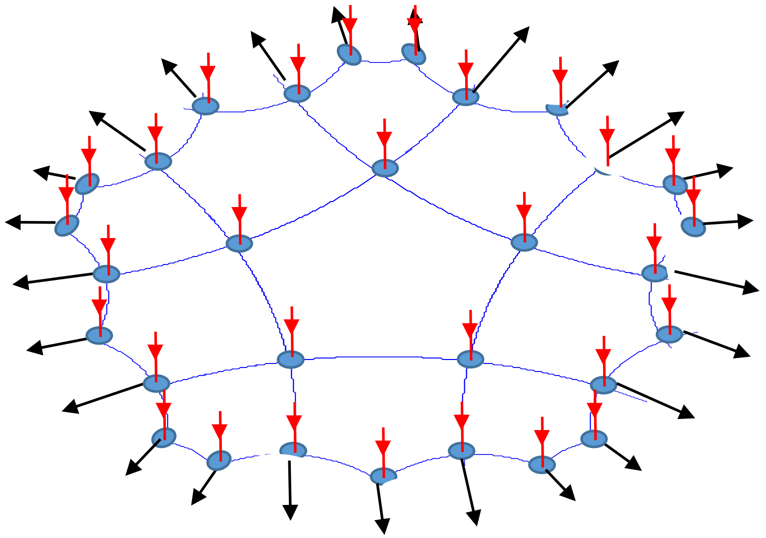
General mechanism for producing Ryu-Takayanagi from area law state!



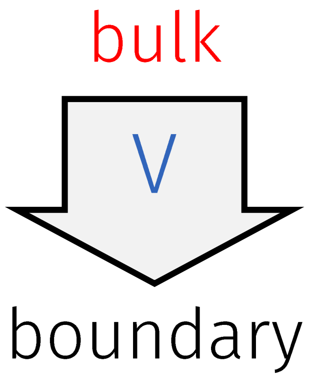
Random tensor networks as holographic mappings

Bulk-boundary mapping from random tensor networks

Tensor network determines “holographic” mapping:



bond dimensions $D_b \ll D$



To study properties, highly useful to consider “fictitious” state $|\Psi_{\text{bulk},\text{boundary}}\rangle$.

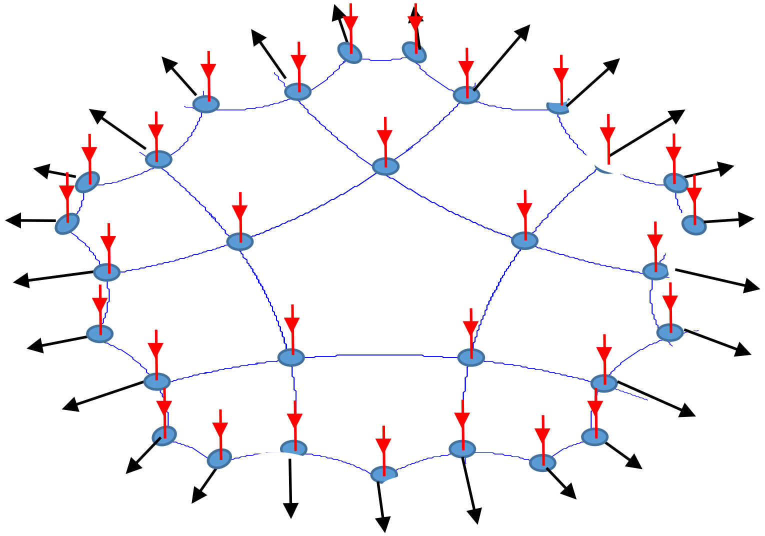
Properties of bulk-boundary mapping

Holographic mapping is isometry (preserves probabilities) if

$$S(\text{bulk}) = N \log D_b$$

That is, if minimal domain wall cuts off bulk legs.

Consequence of Ising calculation!



bond dimensions $D_b \ll D$

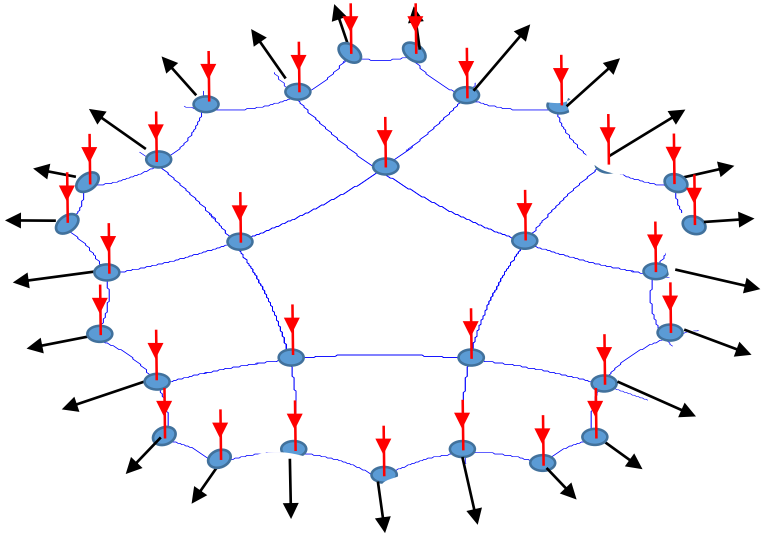
Properties of bulk-boundary mapping

Holographic mapping is isometry (preserves probabilities) if

$$S(\text{bulk}) = N \log D_b$$

That is, if minimal domain wall cuts off bulk legs.

Consequence of Ising calculation!



bond dimensions $D_b \ll D$

Thus: can faithfully map states and operators:

$$|\psi_\partial\rangle = V\,|\psi_b\rangle$$

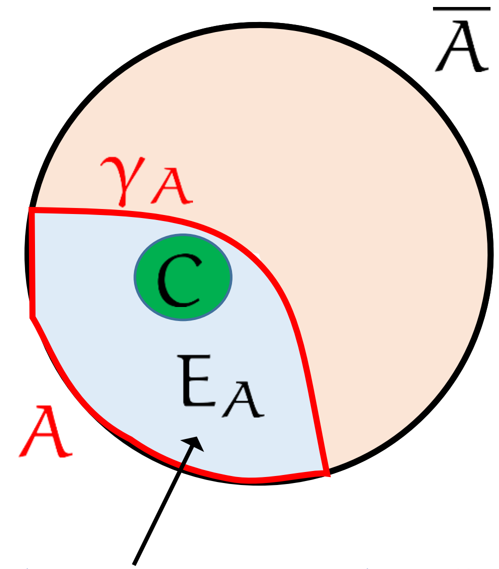
$$O_\partial = V \phi_b V^\dagger$$

All correlation functions are preserved.

Locality of bulk-boundary mapping $O_\partial = V \phi_b V^\dagger$

Given a local bulk operator ϕ_C. How local we can we choose boundary operator O_A?

Answer:



Reduces to an entropy calculation!

$$I(E_A : \bar{A}\,\overline{E_A}) = 0$$

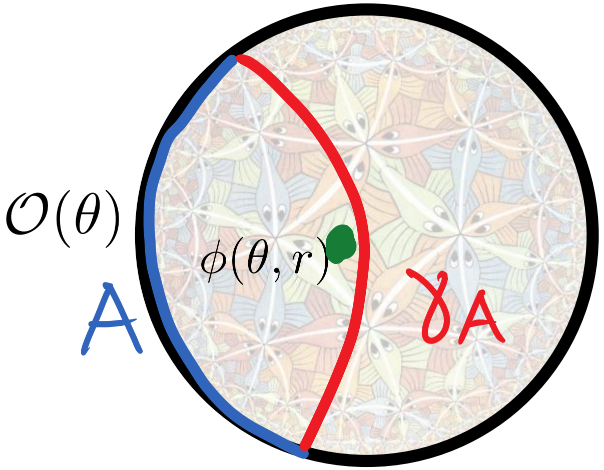
Here, I(X:Y) = S(X) + S(Y) – S(XY) is the quantum mutual information. Can perfectly recover from A if completely decoupled from environment – "no cloning".

Bulk and boundary observables in AdS/CFT

Bulk in terms of boundary: "smearing"

$$\phi(\theta, r) = \int K(\theta, r; \tilde{\theta}, \tilde{t})\, \mathcal{O}(\tilde{\theta}, \tilde{t})\, d\tilde{\theta} d\tilde{t}$$

recovers in the "causal wedge". [Hamilton-Kabat-L-L]



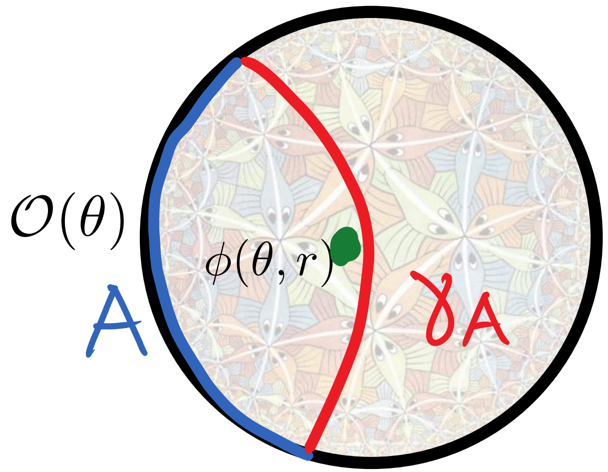
Our result matches the larger, long conjectured "entanglement wedge" reconstruction! [Dong-Harlow-Wall]

Bulk and boundary observables in AdS/CFT

Bulk in terms of boundary: "smearing"

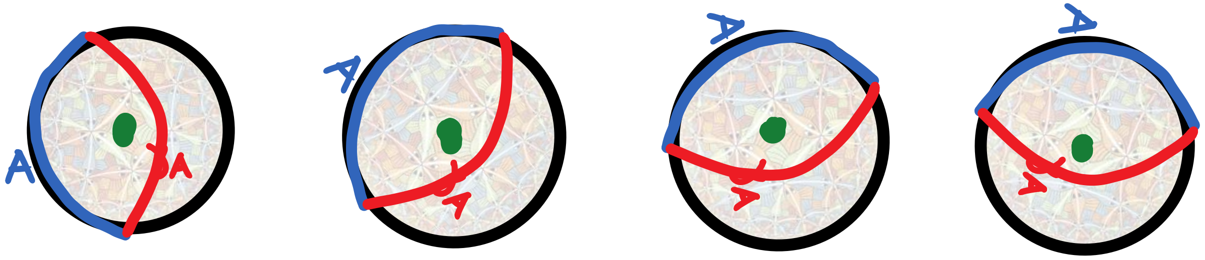
$$\phi(\theta, r) = \int K(\theta, r; \tilde{\theta}, \tilde{t})\, \mathcal{O}(\tilde{\theta}, \tilde{t})\, d\tilde{\theta} d\tilde{t}$$

recovers in the "causal wedge". [Hamilton-Kabat-L-L]



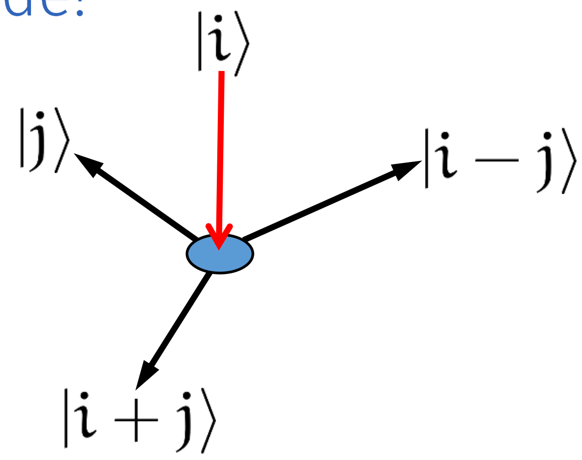
Our result matches the larger, long conjectured "entanglement wedge" reconstruction! [Dong-Harlow-Wall]

For each bulk operator many possible boundary operators. Puzzle? [Almheiri et al]



QIT interlude II: Quantum error correction [Almheiri-Dong-Harlow]

This redundancy is the feature of a quantum error correcting code!



Example: Three-qutrit erasure code

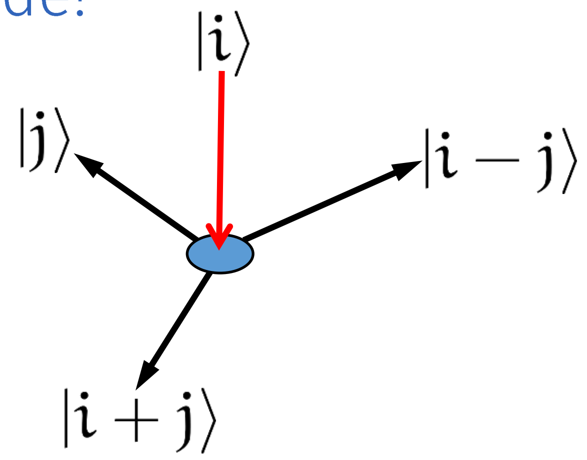
$$V\phi = O_{12}V = O_{23}V = O_{13}V$$

can correct for loss of any single qutrit

QIT interlude II: Quantum error correction

[Almheiri-Dong-Harlow]

This redundancy is the feature of a quantum error correcting code!



Example: Three-qutrit erasure code

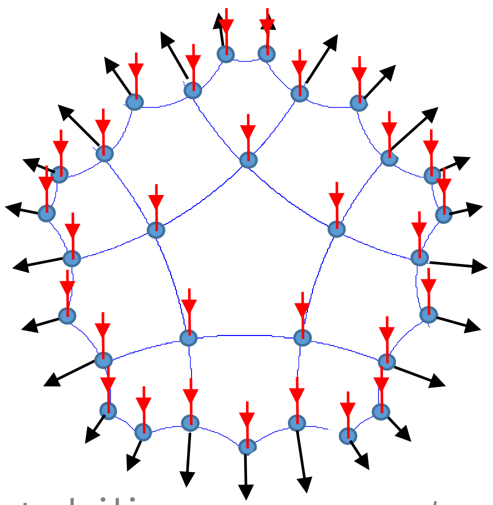
$$V\phi = O_{12}V = O_{23}V = O_{13}V$$

can correct for loss of any single qutrit

→ Tensor network toy models from "perfect tensors"

[Pastawski, Yoshida et al]

Holographic codes are erasure codes with remarkable locality:
Quantum information deeper in the bulk is better protected.



Footnote for David: Entropy formulas will hold exactly (w.h.p.) if we use random stabilizer.

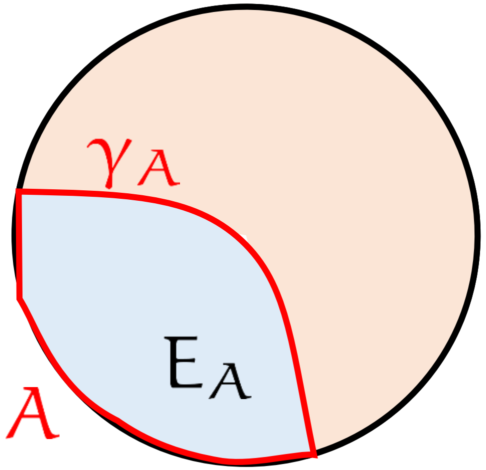
Bulk corrections

$|\psi_\partial\rangle = V|\psi_b\rangle$

What are the typical properties of boundary states (code states)?

Ising action acquires additional "bulk term". Result:

$$S(A) \simeq \min\{\log D|\gamma_A| + S(E_A)_{\rho_b}\}$$



Result matches precisely the corrections to the Ryu-Takayanagi formula in AdS/CFT due to entanglement in bulk quantum fields. [Faulkner et al]

To prove result about entanglement entropy, use decoupling technique from [Dutil-Hayden].

Bulk corrections

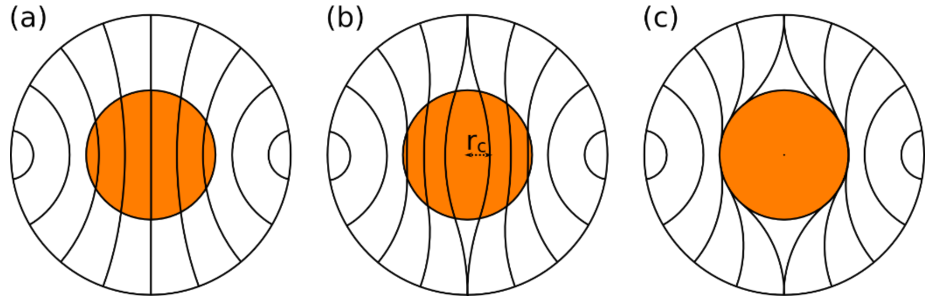
$|\psi_\partial\rangle = V|\psi_b\rangle$

$$S(A) \simeq \min\{\log D|\gamma_A| + S(E_A)_{\rho_b}\}$$

minimal geodesics get deformed (unless $D_b \ll D$)

↓

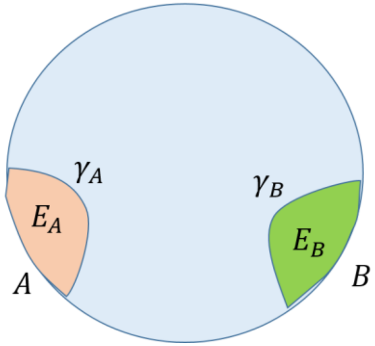
horizons from adding massive amounts of bulk entropy (analog of BH formation)



contribution from bulk entanglement

↓

correlations in bulk state get encoded into boundary state



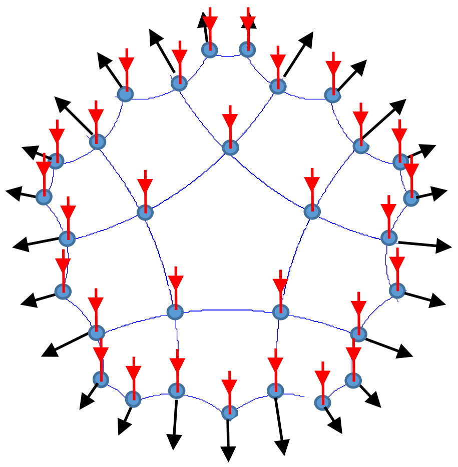
$$I(A:B) = I(E_A:E_B)_b$$

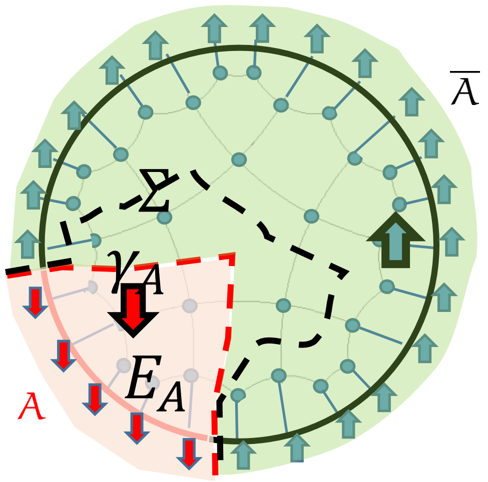
Summary

arXiv:1601.01694

Random tensor networks reproduce several basic structural features of AdS/CFT correspondence:

- entropic structure, boundary reconstruction, bulk corrections
- analog of black hole formation, correlation spectrum, ...
- *not* restricted to AdS





...toy models, emphatically *not* CFT states!

Program: elucidate general mechanism by which holographic properties can arise (typicality)

Thank you for your attention

Work in progress: Dynamics; *bulk* reconstruction; entanglement properties.