Multiparty entanglement, random codes, and quantum gravity

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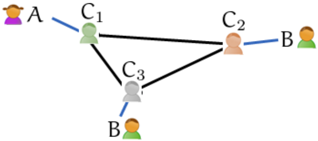
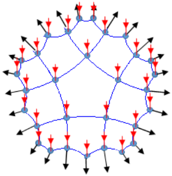
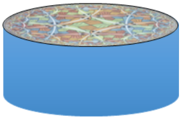
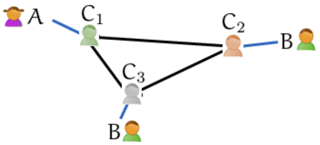






Outline

- Entanglement in random tensor networks
- Proof ingredients – including some new results on stabilizer states
- Quantum gravity interlude
- Random holographic codes



Ning Bao, Sepehr Nezami, Hirosi Ooguri, Bogdan Stoica, James Sully, MW: *The holographic entropy cone* (JHEP, 2015)

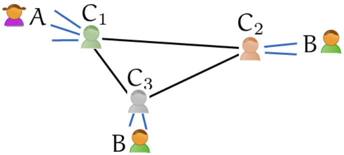
Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, MW, Zhao Yang: *Holographic duality from random tensor networks* (JHEP, 2016)

Sepehr Nezami, MW: *Multipartite entanglement in stabilizer tensor networks* (arXiv:1608.02595)

Sepehr Nezami, MW: forthcoming

Entanglement distillation with a twist

Alice, Bob, Charlies share a graph of maximally entangled pairs.



$$\psi_{ABC_1 \ldots C_N}^{\otimes n} \xrightarrow{LOCC} EPR_{AB}^{\otimes m}$$

Goal: Distill entanglement between Alice and Bob, with help of Charlies.

Optimal rate is entanglement of assistance (Smolin, Verstraete, Winter):

$$E_{\mathrm{assist}}(A : B) = \text{minimal cut} = \text{maximal flow}$$

What if Charlies do not know Alice/Bob assignment?

- Random measurements. Produces a random tensor network!

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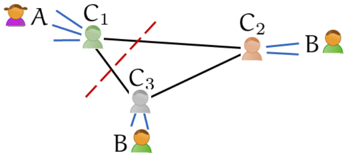

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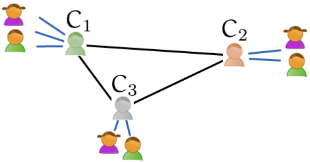

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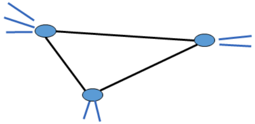

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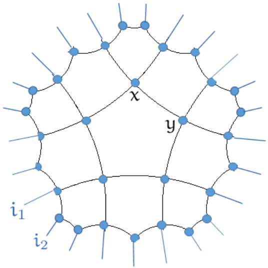
The random tensor network model

Given a graph $G = (V, E)$ and bond dimension 2^N, we consider



$$|\Psi\rangle = \left(\bigotimes_{\langle xy \rangle \in E} \langle xy| \right) \left(\bigotimes_{x \in V} |V_x\rangle \right)$$

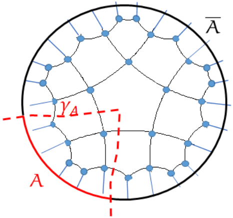
- $|V_x\rangle$ random tensors
- $|xy\rangle = (|00\rangle + |11\rangle)^{\otimes N}$ EPR pairs

We are interested in the behavior for large N.

Prior/related work: Swingle (MERA with expanders), Collins *et al* (random MPS), Hastings (random MERA)

Bipartite entanglement

Fundamental bound: $S(A) \leq N \min|\gamma_A|$



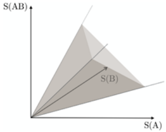
Result

In random tensor networks: $S(A) \simeq N \min|\gamma_A|$ with high probability

Holographic entropy inequalities

Entropy formula has interesting structural properties.

$$S(A) = c \min|\gamma_A|$$



Can be studied systematically via entropy cone formalism:

- many nonstandard entropy inequalities – but finite number for any number of subsystems (with Bao, Nezami, Ooguri, Stoica, Sully)
- can constrain QIT protocols (Czech *et al*, QIP) – but also theories of quantum gravity (Ooguri, Strings)
- ex.: monogamy of mutual information

$$I(A:B) + I(A:C) \leq I(A:BC)$$

is unique additional inequality for three subsystems. But correlations are not in general monogamous – not valid for Shannon, vN entropy.

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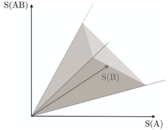

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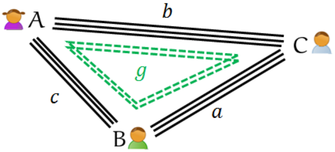
Does the mutual information in these states measure entanglement?

Stabilizer states $D = 2^n$

From now on: we use random stabilizer states as the vertex tensors $|V_x\rangle$.
Then the tensor network state $|\Psi\rangle$ is also a stabilizer state.

Stabilizer states: Eigenvector of maximal subset of Pauli operators.
Ex: $|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle$ is stabilized by $X_1X_2X_3$, Z_1Z_2, Z_2Z_3.

- Useful for codes, efficient random constructions (Friday)
- Reason: 2-design, 3-design for qubits
- Tripartite entanglement structure is simple:



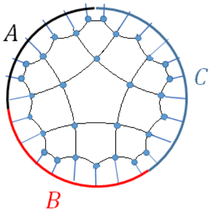
$$I(A:B) = 2c + g$$

where g is the number of GHZ states.

Tripartite entanglement

Result

In random stabilizer network states: $\#\text{GHZ}(A:B:C) = O(1)$ *w.h.p.*



Corollary

Can distill $\simeq \frac{1}{2} I(A:B)$ EPR pairs by local unitaries.

- mutual information is an entanglement measure

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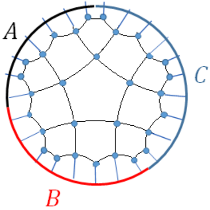

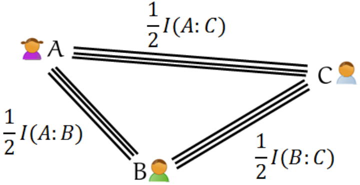

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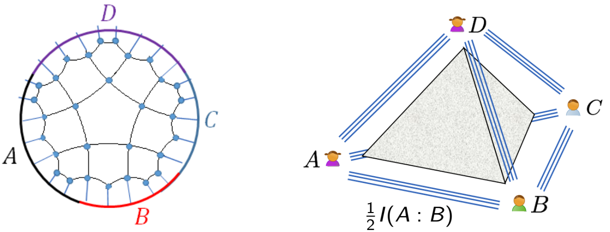
Higher-partite entanglement



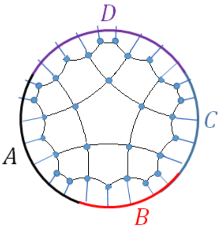
After distilling bipartite EPR pairs, we obtain residual state:

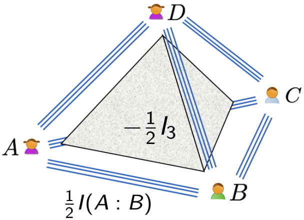
$$S(A), \ldots, S(D) \simeq -\tfrac{1}{2} I_3, \quad S(AB), \ldots, S(CD) \simeq -I_3$$

with the tripartite information $I_3 = I(A:B) + I(A:C) - I(A:BC)$:

- residual state has entropies of perfect tensor
- I_3 is invariant under distillation: can estimate via Ryu-Takayanagi
- another proof that the mutual info is monogamous
- $I_3 < 0$ diagnoses four-partite entanglement

Higher-partite entanglement





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Proof ingredient I: Spin models

Result (Bipartite entanglement)

In random tensor networks: $S(A) \simeq N \min |\gamma_A|$ with high probability

Sketch of proof: Lower-bound $S_2(A) = -\log \operatorname{tr} \rho_A^2$.

- swap trick: $\operatorname{tr} \rho_A^2 = \operatorname{tr} \rho^{\otimes 2}(F_A \otimes I_{\bar{A}})$
- random tensors: $\mathbb{E}[V_x^{\otimes 2}] \propto I_x + F_x$

Ferromagnetic Ising model at $T = 1/N$ with mixed boundary conditions:

$$\mathbb{E}[\operatorname{tr} \rho_A^2] \propto Z_A = \sum_{\{s_x\}} 2^{-N \sum_{\langle xy \rangle} (1 - s_x s_y)/2}$$

- large N: dominated by minimal domain wall

Useful general technique. More precise estimates from geometry of graph.

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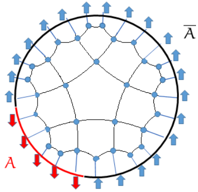

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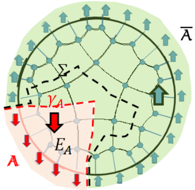

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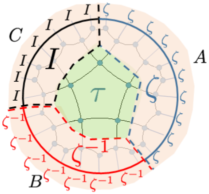

- random tensors: $\mathbb{E}[V_x^{\otimes 3}] \propto \sum_{\pi \in S_3} \pi_x$
- ferromagnetic spin model with variables $\pi_x \in S_3$, cyclic boundary conditions
- minimal energy configuration displayed on the right

$$\#\,\mathrm{GHZ} \sim \text{ground state degeneracy}$$

- three-fold degenerate for every residual region (independent of large N)

Proof ingredient II: Higher moments $D = 2^N$

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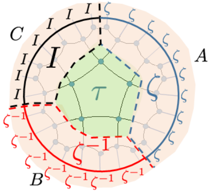

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Third moments of stabilizer states $\quad D = 2^N$

Fact

$$\mathbb{E}[\psi^{\otimes 3}] \propto \sum_{\pi \in S_3} r(\pi)^{\otimes N},$$

where $r(\pi)\,|\vec{y}\rangle = |\pi \vec{y}\rangle$ is a permutation operator on $(\mathbb{C}^2)^{\otimes 3}$.

For $p = 2$, stabilizer states form a 3-design.
For $p > 2$, not the case!

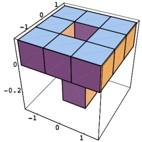
Third moments of stabilizer states $D = p^N$

Result

$$\mathbb{E}[\psi^{\otimes 3}] \propto \sum_{T \in \Sigma_3(p)} r(T)^{\otimes N},$$

where $r(T) = \sum_{(\vec{x},\vec{y}) \in T} |\vec{x}\rangle\langle\vec{y}|$ is an operator on $(\mathbb{C}^p)^{\otimes 3}$.

- $\Sigma_3(p)$: collection of $2p+2$ many 3-dimensional subspaces $T \subseteq \mathbb{F}_p^3 \oplus \mathbb{F}_p^3$
- $\pi \in S_3$ permutation $\rightsquigarrow$ $T_\pi = \{(\pi\vec{y}, \vec{y})\} \in \Sigma_3(p)$



Applications:

- spin model for GHZ content
- new 3-designs (e.g., Clifford orbit of non-stabilizer state for qutrits, cf. Friday)

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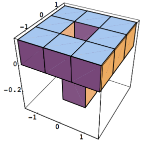

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Higher moments of stabilizer states $D = p^N$

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$$\mathbb{E}[\psi^{\otimes t}] \propto \sum_{T \in \Sigma_t(p)} r(T)^{\otimes N},$$

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- $\Sigma_t(p)$: collection of $\prod_{i=0}^{t-2}(p^i + 1)$ many t-dimensional subspaces $T \subseteq \mathbb{F}_p^t \oplus \mathbb{F}_p^t$
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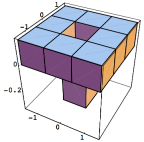
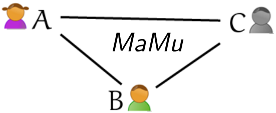


In fact: $\{r(T)^{\otimes N}\}$ are a basis of the commutant of $\{U_{\text{Cliff}}^{\otimes t}\}$

Aside: GHZ distillation and algebraic complexity theory

Random tensors are very natural from a tensor network point of view. But our original motivation was distillation!



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- $MaMu \overset{SLOCC}{\rightarrow} GHZ$ at rate 2; $GHZ \overset{SLOCC}{\rightarrow} MaMu$ famously unknown
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How many GHZ states can be distilled in an assisted scenario? Work in progress.

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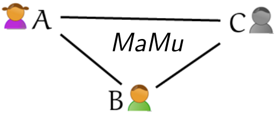

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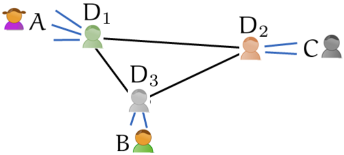

And now for something different: Quantum Gravity

Black hole entropy law: $S_{BH} \sim area$

Holographic principle (Susskind, 't Hooft): All information in a region of space can be represented as a "hologram" living on region's boundary

AdS/CFT duality (Maldacena): quantum gravity in bulk, quantum field theory on boundary

Ryu-Takayanagi formula: $S(A) \sim \min|\gamma_A|$

- spacetime as a tensor network? (Swingle)
- entanglement as the glue for spacetime? (Van Raamsdonk)
- "ER = EPR" (Maldacena, Susskind)

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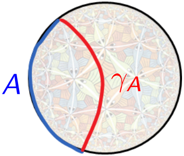

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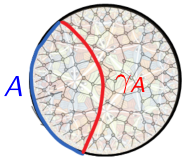

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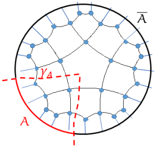
Quantum gravity and tensor networks

Our random tensor network model provides evidence for this picture:

- shows that Ryu-Takayanagi formula fundamentally compatible with QM
- proposes a simple QIT mechanism



But, wait. AdS/CFT is a duality of physical theories:

- a whole dictionary, mapping states & observables. . .

Like in a quantum error correcting code (Almheiri, Dong, Harlow)?!

bulk → *boundary* vs. *logical* → *physical*

Quantum gravity and tensor networks

Our random tensor network model provides evidence for this picture:

- shows that Ryu-Takayanagi formula fundamentally compatible with QM
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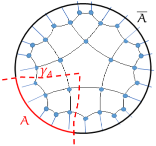

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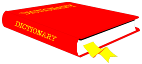

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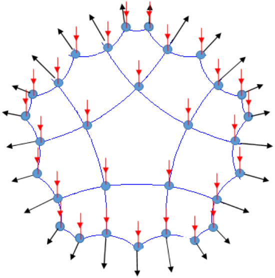
bulk $\rightarrow$ *boundary* vs. *logical* $\rightarrow$ *physical*

Random holographic codes

How to obtain codes from a tensor network?

- red legs = logical qudits
- black legs = physical qudits

We obtain a map *bulk* $\rightarrow$ *boundary*.



bond dimensions D, D_b

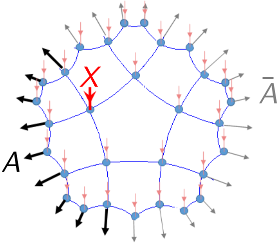
Lemma

If $D \gg D_b$ then we obtain an isometry and hence a stabilizer code *(w.h.p.)*

Holographic codes as erasure codes

When can we decode a logical qudit at X from a subset A of the physical qudits?

That is, can we correct for erasure of $\bar{A}$?



Result

X can be decoded from A if and only if enclosed by minimal cut γ_A *(w.h.p.)*

- erasure codes with nontrivial geometric structure: the deeper in the bulk, the better protected.
- rigorously realizes holographic codes as proposed by Pastawski *et al.*

Many open questions: Optimal parameters (size vs. D vs. D_b)? Precise dependence on graph? Explicit codes?

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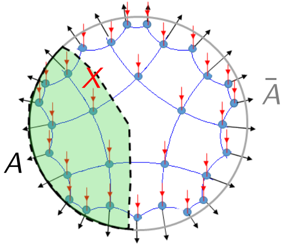

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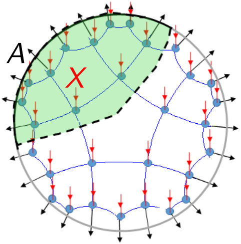

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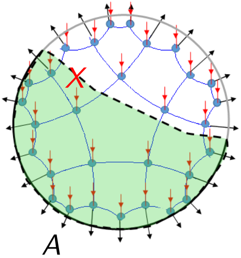

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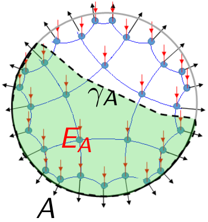
Holographic codes and quantum gravity

Random codes match predictions of quantum gravity (Faulkner *et al*, Dong *et al*):

- local qubits in the *entanglement wedge* E_A are encoded in the physical qubits in A
- entropy of code states:

$$S(A) = N|\gamma_A| + S(E_A)$$

- logical correlations $\rightsquigarrow$ physical correlations



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- minimizing cuts get deformed
- toy model of black hole
- cf. recent work by Verlinde

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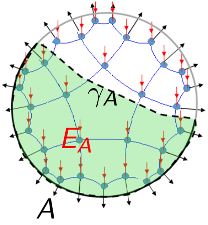

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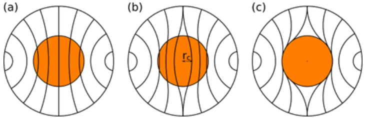
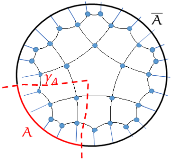
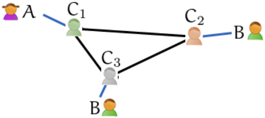
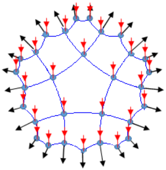


Summary and outlook







Random tensor networks:

- Bipartite & multipartite entanglement properties dictated by geometry
- Toy model & explanation of some structural features of AdS/CFT
- Erasure codes with geometric structure ('holographic' codes)
- Techniques: spin models for random tensor averages; stabilizer states

Outlook:

- What can we do with higher moments of stabilizer states? ($\rightsquigarrow$ Friday)
- QI beyond toy models: design new diagnostics.
- Dynamics, backreaction, superpositions of geometries, ...

Thank you for your attention!