Approximate QCAs and a converse to the Lieb-Robinson bounds

Michael Walter

joint work with Daniel Ranard (MIT) and Freek Witteveen (Copenhagen)

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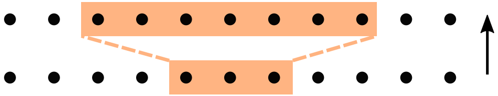
Motivation

Quantum cellular automata model strictly local dynamics. However:

Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.



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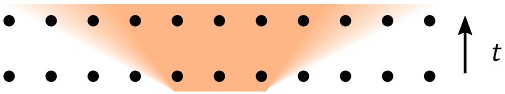

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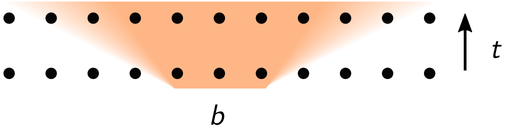

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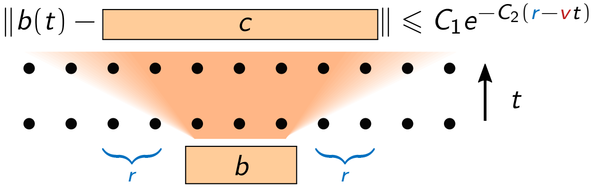

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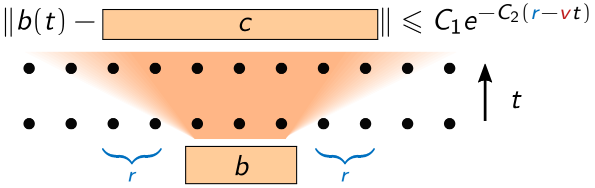

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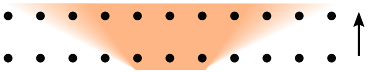

Physics question: Are local dynamics generated by local Hamiltonians?

- That is, can we find converse to Lieb-Robinson bounds?
- How about lattice translations?
- Or boundary dynamics generated by bulk local Hamiltonians?

Mathematics question: Classify approximately local dynamics.

Our results: Approximately local dynamics in 1D have structure & index theory similar to QCAs. In particular, obtain a converse to LR bounds.

The methods that we use might also be useful for other "approximate structures" on QI.

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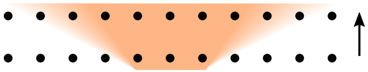

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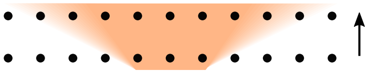

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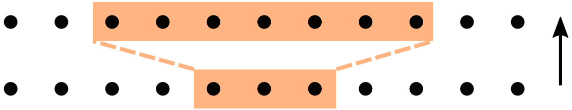
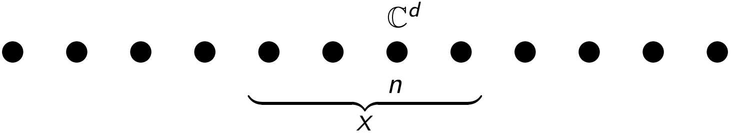
Quantum Cellular Automata



Setup: Infinite spin chains



It is convenient to work in the Heisenberg picture:

$$\mathcal{A}_n = \mathrm{Mat}(d) \quad \rightsquigarrow \quad \mathcal{A}_X = \bigotimes_{n \in X} \mathcal{A}_n \quad \rightsquigarrow \quad \mathcal{A}_{loc} = \bigcup_{X \text{ finite}} \mathcal{A}_X$$

Quasi-local C^*-algebra:

$$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|} = \text{“}\bigotimes_{n \in \mathbb{Z}} \mathcal{A}_n\text{”}$$

We can also define $\mathcal{A}_{\geqslant n} = \mathcal{A}_{\{n, n+1, \ldots\}} \subseteq \mathcal{A}$, etc.

Local dynamics are naturally modeled by automorphisms $\alpha \colon \mathcal{A} \to \mathcal{A}$.

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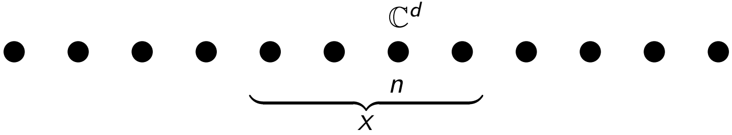

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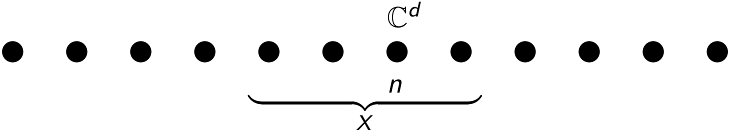

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Quantum cellular automata (QCAs)

[Margolus, Schumacher-Werner, . . .]

Quasi-local algebra on infinite 1D lattice:

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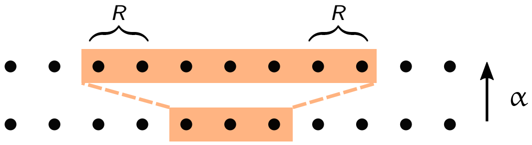
An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is a **quantum cellular automaton (QCA)** or locality preserving unitary (LPU) with radius $R > 0$ if:

$$\alpha(\mathcal{A}_n) \subseteq \mathcal{A}_{\{n-R,\ldots,n+R\}}$$

That is, the support of *any* local operator grows by at most R:



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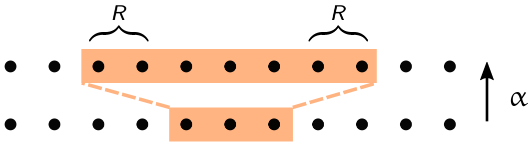
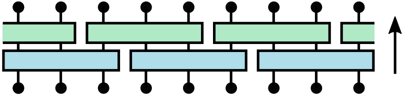


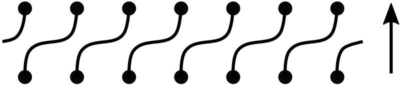
Classification of QCAs in 1D

Examples:

Quantum circuit



Shift



Theorem (Gross-Nesme-Vogts-Werner, GNVW):

- Any QCA is a composition of a circuit and a shift.
- Shifts cannot be implemented by circuits.
- QCAs modulo circuits are classified by quantized index.

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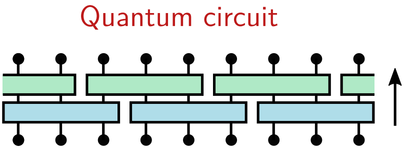

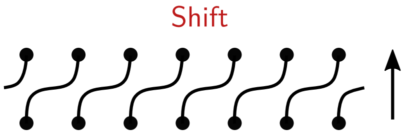

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GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

$$\begin{aligned}\text{index} &= \text{amount of quantum information flowing right} \\ &\quad - \text{amount of quantum information flowing left}\end{aligned}$$

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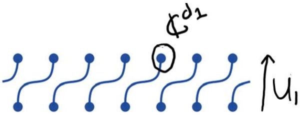

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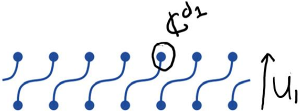

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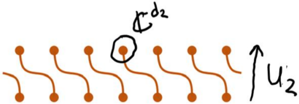

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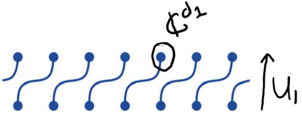

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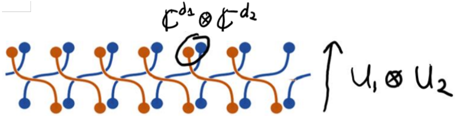

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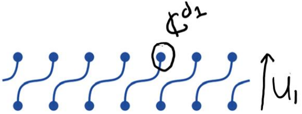

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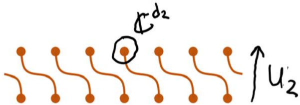

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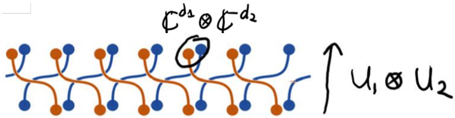

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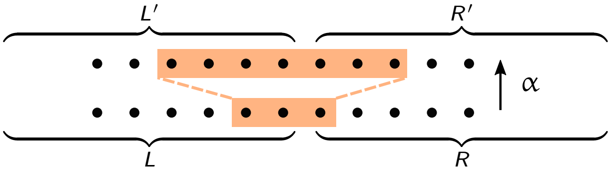
This intuition can be made precise...

(Re)defining the index



Cut chain in halves and consider corresponding Choi state $\rho_{LRL'R'}$. Then:

$$\text{index}\,\alpha = \frac{1}{2}\big(I(L:R') - I(L':R)\big)$$

where $I(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$ is the quantum mutual information.

Properties:

- quantized: $\text{index}\,\alpha \in \mathbb{Z}[\{\log p_i\}]$, p_i = prime factors of local dimension
- additive: $\text{index}\,\alpha \otimes \beta = \text{index}\,\alpha + \text{index}\,\beta$
- robust: if $\alpha \approx \beta$ then $\text{index}\,\alpha = \text{index}\,\beta$

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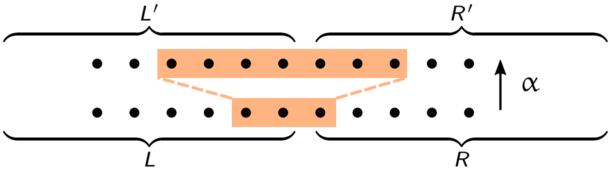

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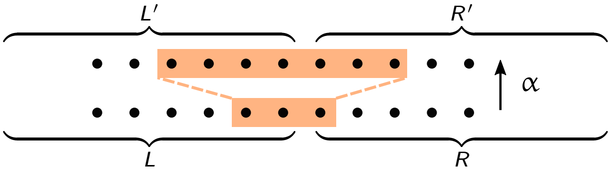

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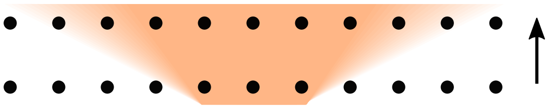
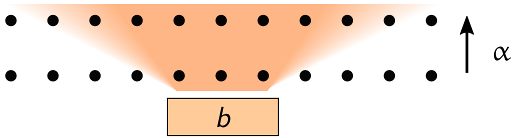
Approximately Locality-Preserving Unitaries



Approximately locality-preserving unitaries (ALPUs)

Idea: Replace strict locality $\to$ Lieb-Robinson type bounds.



An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\forall b \in \mathcal{A}_X\colon \ \exists c \in \mathcal{A}_{r\text{-Neighborhood}(X)}\colon \ \|\alpha(b) - c\| \leqslant f(r)\|b\|$$

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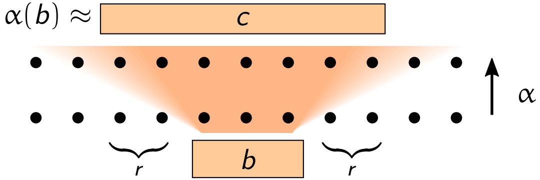

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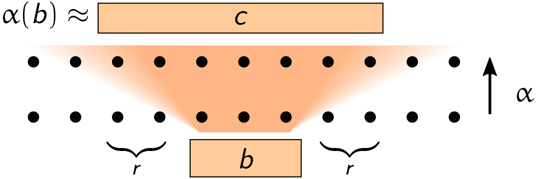

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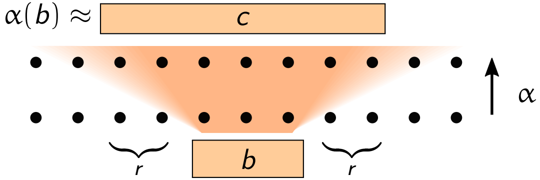

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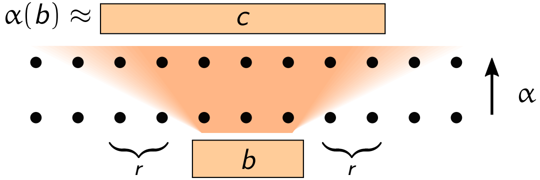

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Why do we care?

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- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

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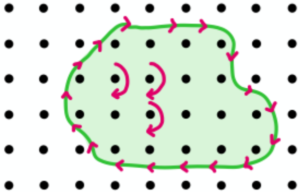

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- Previous techniques sensitive to perturbations.
- Local Hamiltonian dynamics are not quantum circuit.

- Previous definitions of index do not apply to ALPUs.

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  But: Can always approximate by circuits. How about ALPUs?
- Previous definitions of index do not apply to ALPUs.

Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Why not obvious?

- Previous work only treats exact QCAs.
- Previous techniques sensitive to perturbations.
- Local Hamiltonian dynamics are not quantum circuit.
  But: Can always approximate by circuits. How about ALPUs?
- Previous definitions of index do not apply to ALPUs.
  But: Mutual information defn. applies! Does index remain quantized?

Our results: Classification of ALPUs

ALPUs / time-dependent quasi-local Hamiltonian dynamics
$\cong$ QCAs / circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
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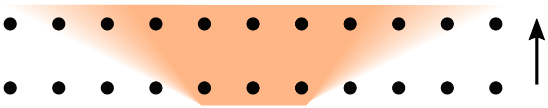
Proof Ideas



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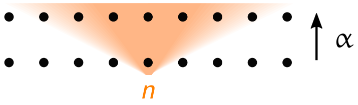

For any *fixed* site n, can truncate tails to obtain *approximate* morphism

$$\mathcal{A}_n \to \mathcal{A}_{\{n-r,\dots,n+r\}}.$$

By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites n, the images of these morphisms need *not* commute → unclear how to **patch together**!

Need a more clever strategy. . .

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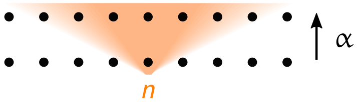

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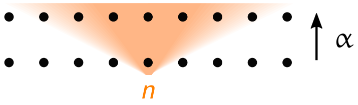

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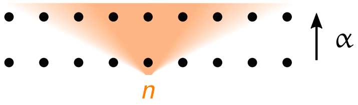

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Main tool: Stability of inclusion

Theorem (Christensen, 80s): If $\mathcal{B} \subseteq_{\varepsilon} \mathcal{C}$ for hyperfinite von Neumann algebras and $\varepsilon < \frac{1}{8}$, then there is a unitary $u \in (\mathcal{B} \cup \mathcal{C})''$ such that

$$u\mathcal{B}u^* \subseteq \mathcal{C} \quad \text{and} \quad \|u - I\| \leqslant 12\varepsilon.$$

We extend this to show that, moreover:

- If $x \in_{\delta} \mathcal{B}'$ and $x \in_{\delta} \mathcal{C}'$, then $\|x - uxu^*\| = O(\delta\|x\|)$.
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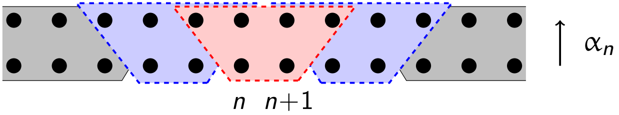

Left and right are decoupled – stronger than what we had before!

This allows us to glue different α_n, α_{n+2}, ... together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

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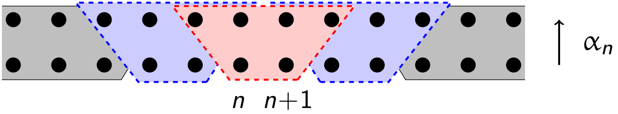

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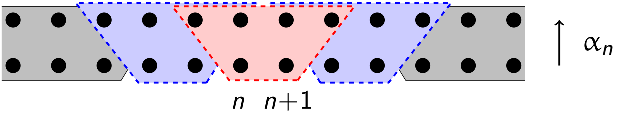

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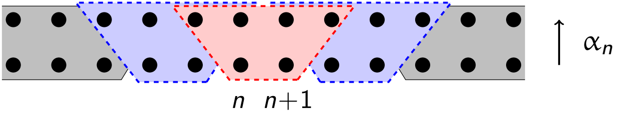

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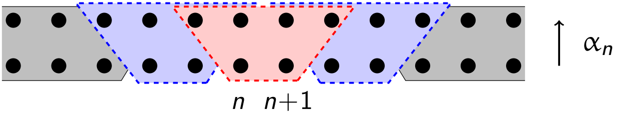

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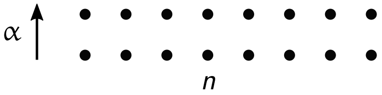

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$$\alpha(\mathcal{A}_{\geqslant n}) \subseteq_{\varepsilon} \mathcal{A}_{\geqslant n-1}$$

By Christensen's theorem, we can find unitary $u \approx I$ s.th.

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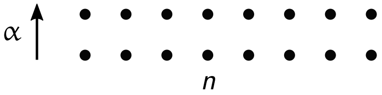

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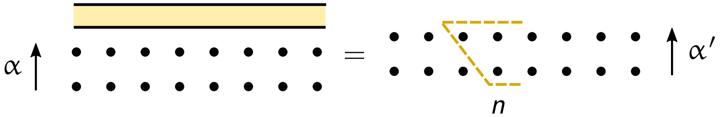

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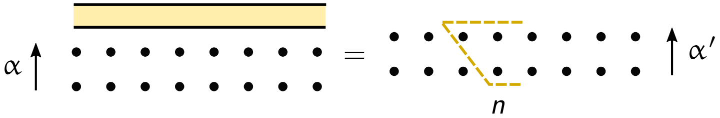

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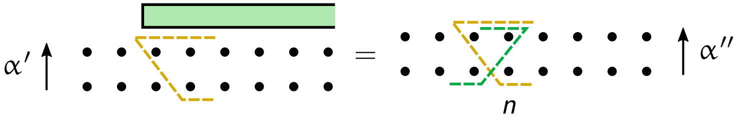

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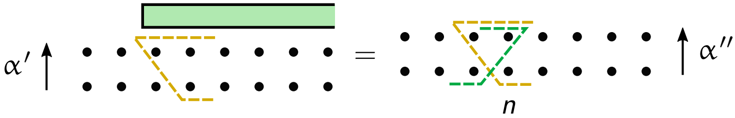

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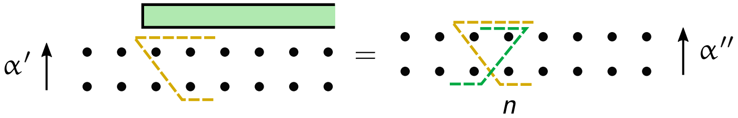

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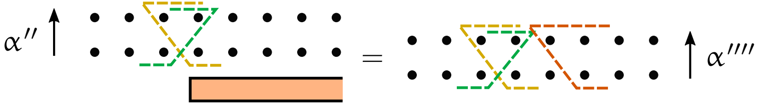

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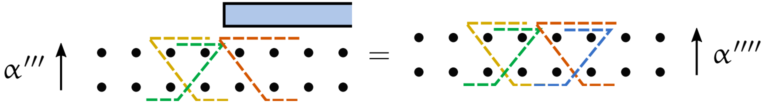

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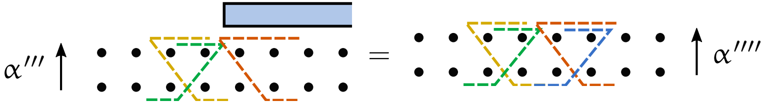

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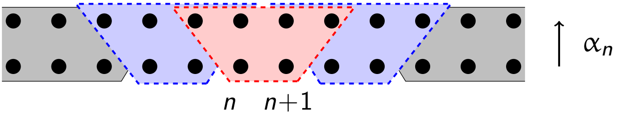

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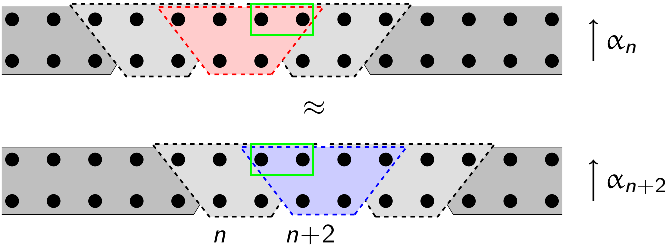
Compare two such local QCAs:



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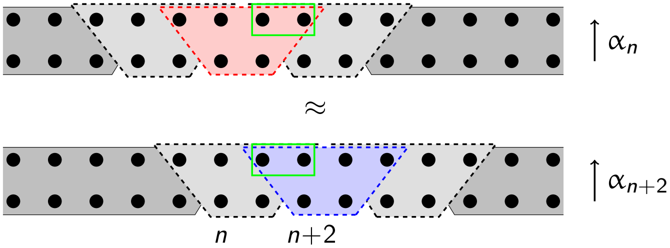

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Index of an ALPU

Thus we proved that any ALPU α in 1D can be approximated by sequence of QCAs β_r (sufficiently fast). This allows us to define the **index**:

$$\operatorname{index} \alpha := \lim_{r \to \infty} \operatorname{index} \beta_r$$

- well-defined, independent of choice of $\{\beta_r\}$
- inherits properties of GNVW index: quantized, additive, continuous, ...

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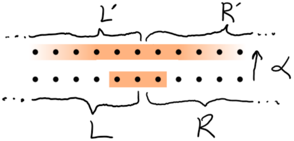

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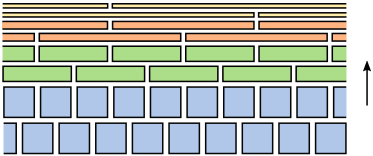

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- Repeat with $\beta_1^{-1}\alpha$.

For an appropriate "schedule", obtain time-dependent Hamiltonian

$$H(t) = \sum_X H_X(t)$$

that is piecewise constant and has geometrically local interactions

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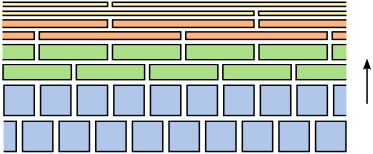

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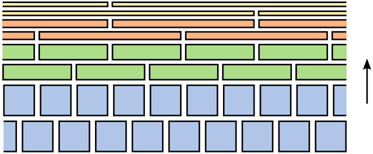
How to obtain time-dependent quasi-local Hamiltonians?



- Start with ALPU of index $\alpha = 0$.
- Approximate α by QCA β_1 of same index. Thus β_1 is circuit and can be implemented by time-dependent local Hamiltonian evolution.
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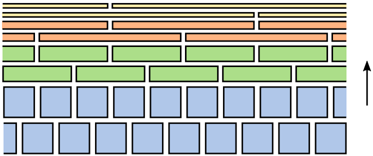

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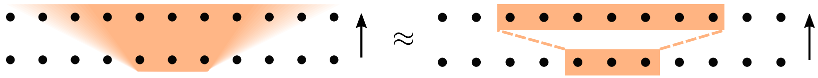
Summary and outlook



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- Periodic chain in 1D? Extension to higher dimensions?
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- Beyond QCAs: Is there an algebra near any almost-algebra?
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- *Other applications of stability results in QI!*

Thank you for your attention!

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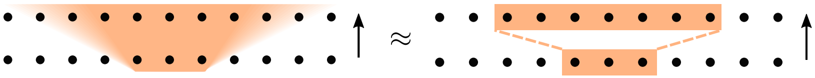

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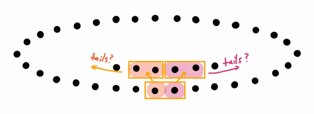
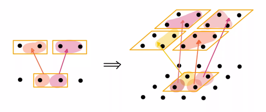
Discussion slides

Some directions for future work

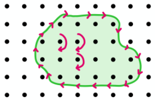
- Finite periodic chain



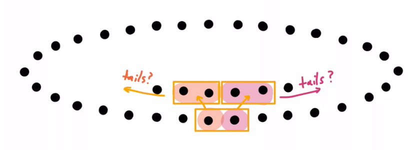
- Higher dimensions



- Condensed matter application: many-body localized Floquet phases



QCA approximation of ALPUs on periodic chain



No clear factorization into left vs right moving algebra → how to decouple?

Potential solution: Cut off in some finite region.

A quantum channel conjecture

The stability result we used is closely related to the following:

Theorem: Let Φ be unital channel s.th. $\|\Phi(xy) - \Phi(x)\Phi(y)\| < \varepsilon\|x\|\|y\|$ for $\varepsilon < \varepsilon_0$ and all operators x, y. Then there exists a morphism Ψ such that $\|\Phi(x) - \Psi(x)\| \leqslant C\varepsilon\|x\|$.

To handle the periodic chain, the following would suffice:

Conjecture: Let $\Phi\colon \mathrm{Mat}(d) \to \mathrm{Mat}(d)$ be unital channel such that $\|\Phi^2(x) - \Phi(x)\| \leqslant \varepsilon\|x\|$ for $\varepsilon < \varepsilon_0$ and all operators x. Then there exists a unital channel Ψ such that $\Psi^2 = \Psi$ and $\|\Phi(x) - \Psi(x)\| \leqslant C\varepsilon\|x\|$.

- **Fact:** Any unital idempotent channel is projection onto subalgebra.
- This would also help to show that strictly-local almost-unitary channels are close to QCAs, and could help extend our results to 2D.

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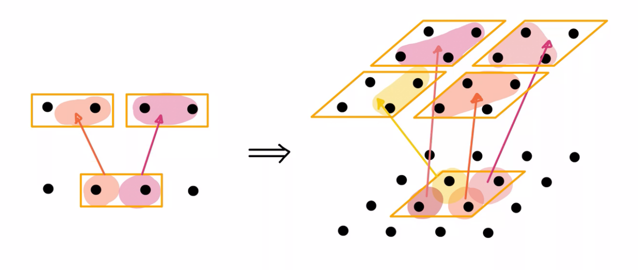
Higher dimensions

In 2D, QCAs have a similar classification:

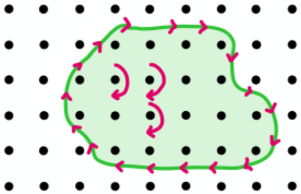


Situation in higher dimensions more mysterious → Jeongwan's talk.

Is it still true that any ALPU can be approximated by sequence of QCAs?

Floquet many-body localization

2D Floquet systems with many-body localization gives rise to a 1D dynamics on the *boundary*:



GNVW index of "boundary QCA" has been proposed as an invariant.

It would be desirable to make this completely rigorous.

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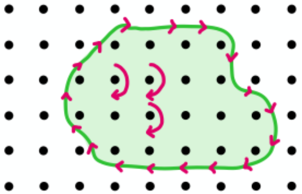

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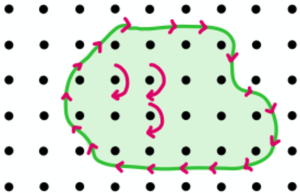

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