Entanglement Polytopes

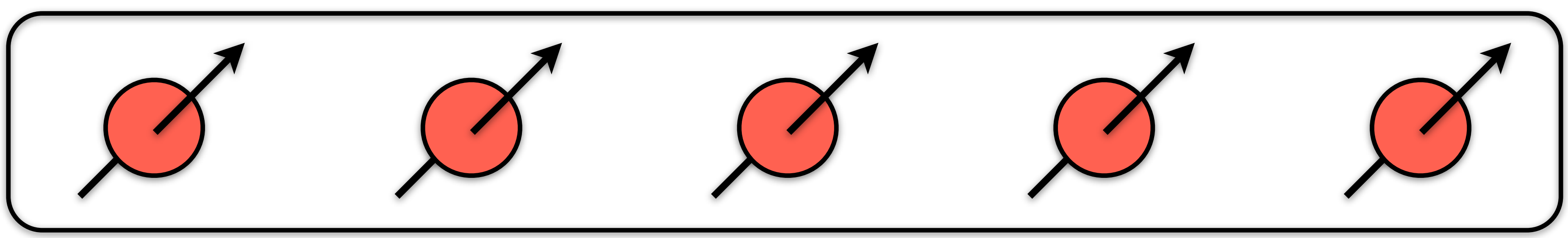
Multi-Particle Entanglement from Single-Particle Information

Michael Walter

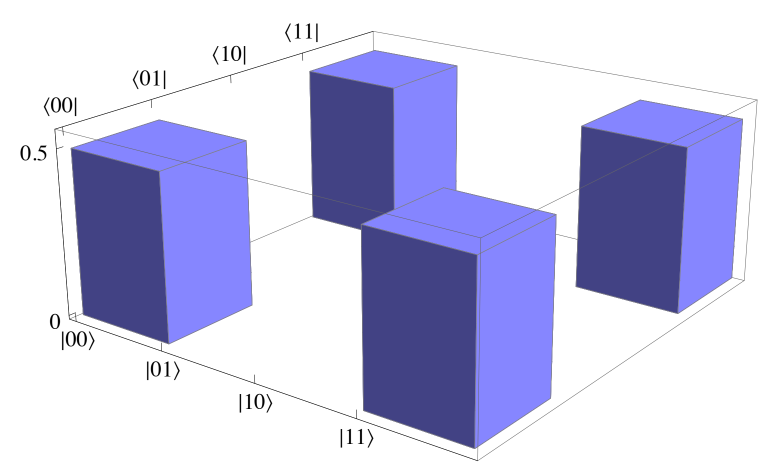
joint work with Matthias Christandl, Brent Doran (ETH Zürich), and David Gross (Univ. Freiburg)

Multi-Particle Entanglement

How entangled is a given multi-particle quantum state prepared in the laboratory?



ρ



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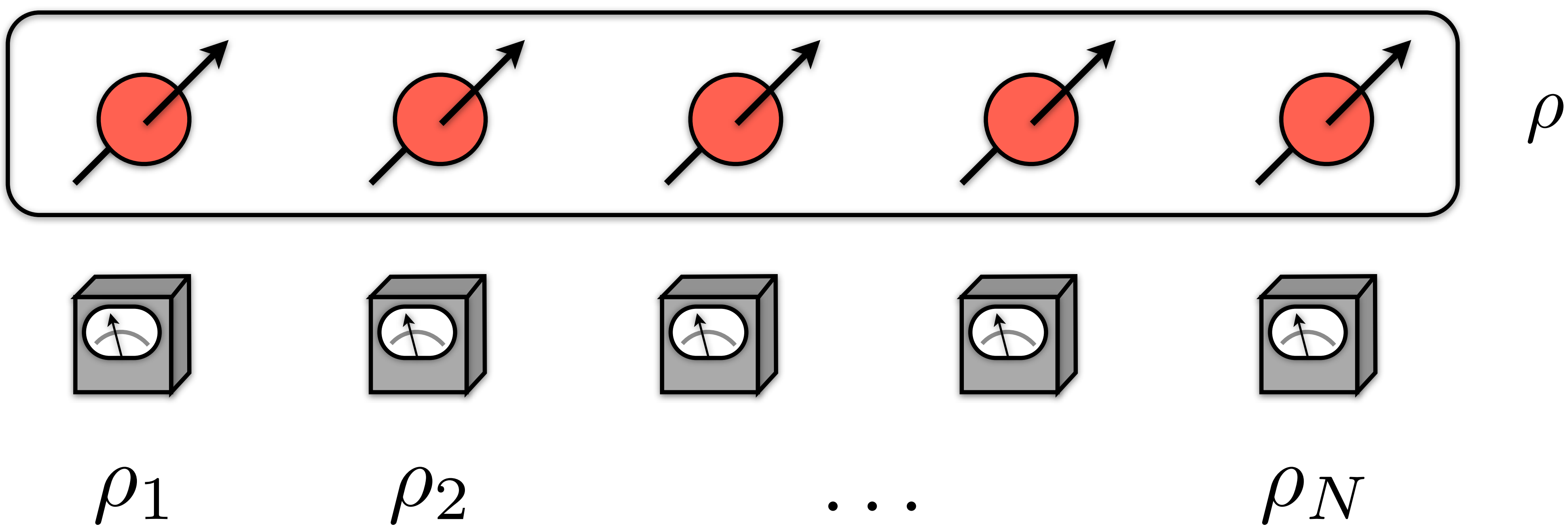

What (if anything) can be said using ***local tomography****?*

efficient!

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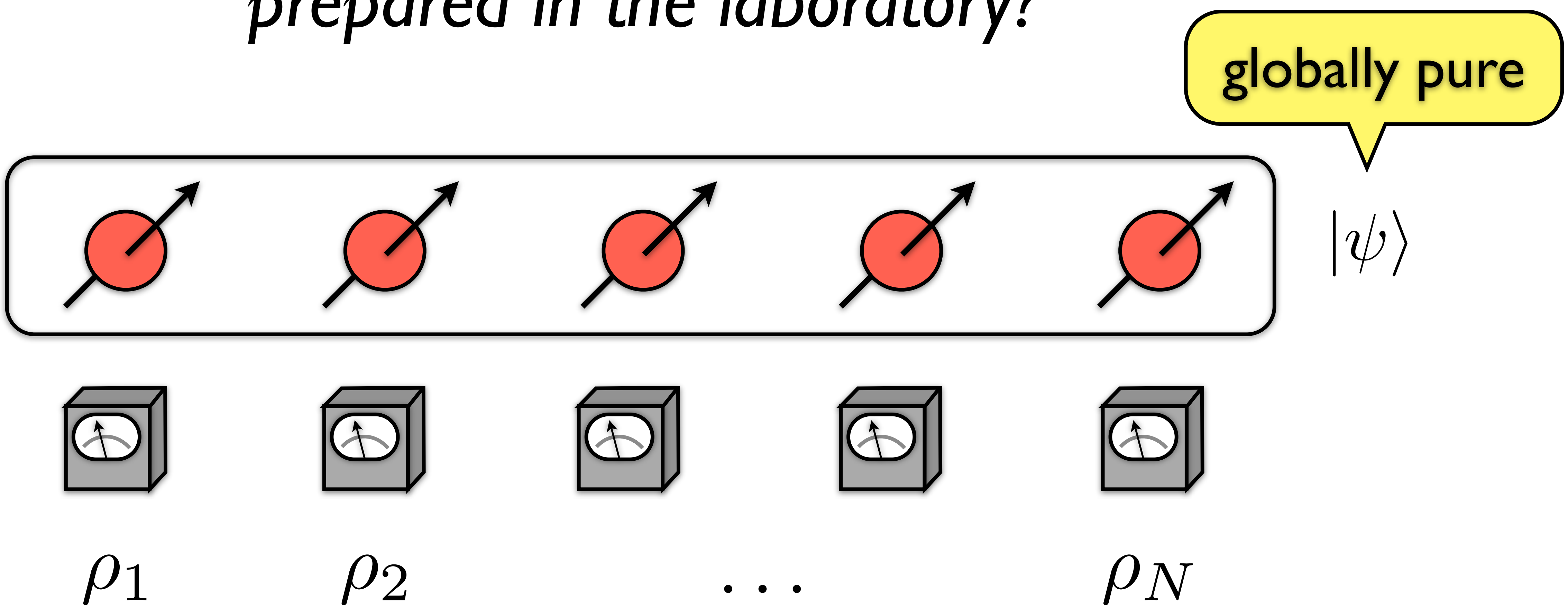

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Pure-State Entanglement

A pure state $\rho = |\psi\rangle\langle\psi|$ is entangled if and only if

$$|\psi\rangle \neq |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle$$

Equivalent:

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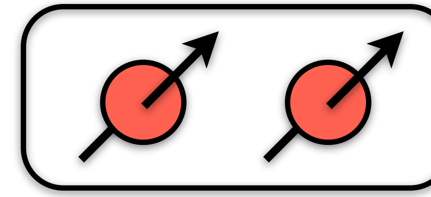
Two Qubits



$\mathbb{C}^2 \otimes \mathbb{C}^2$

Schmidt decomposition

$$|\psi\rangle = \sqrt{\lambda}\;|00\rangle + \sqrt{1-\lambda}\;|11\rangle \qquad (0.5 \le \lambda \le 1)$$

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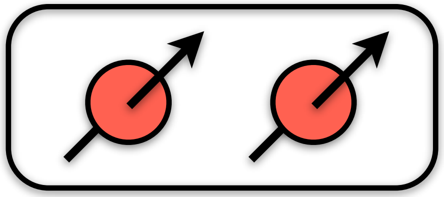

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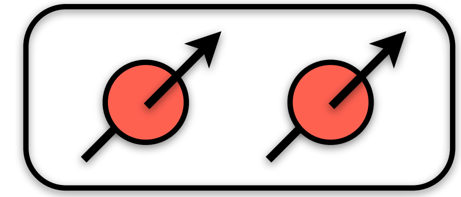

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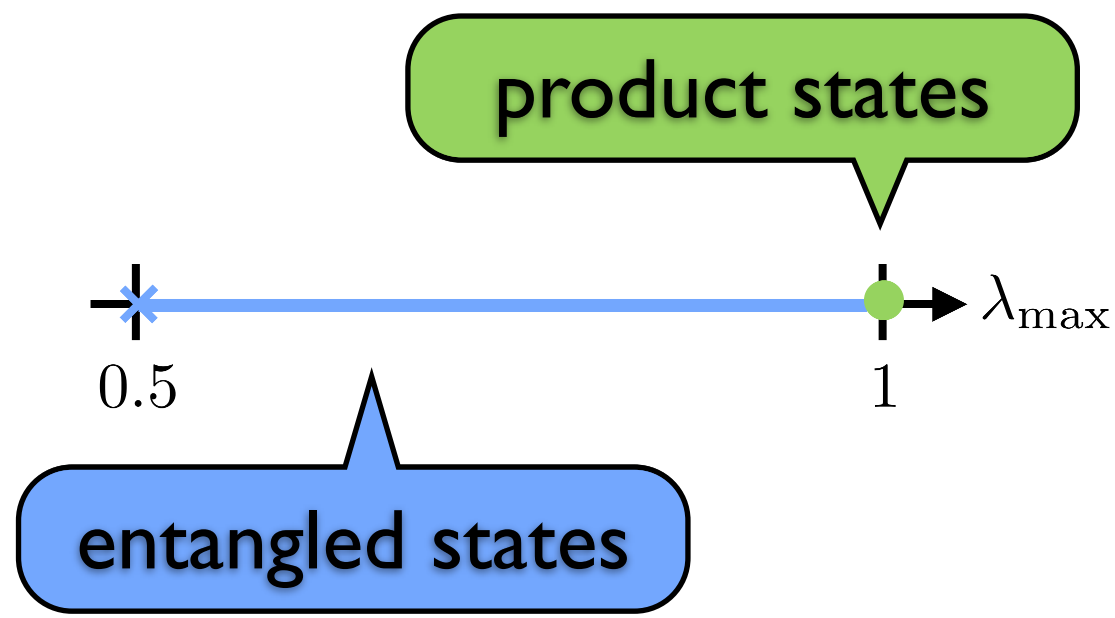

× $\sqrt{0.5}\,(|00\rangle + |11\rangle)$

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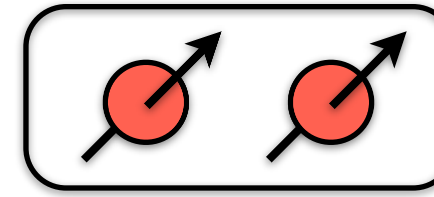

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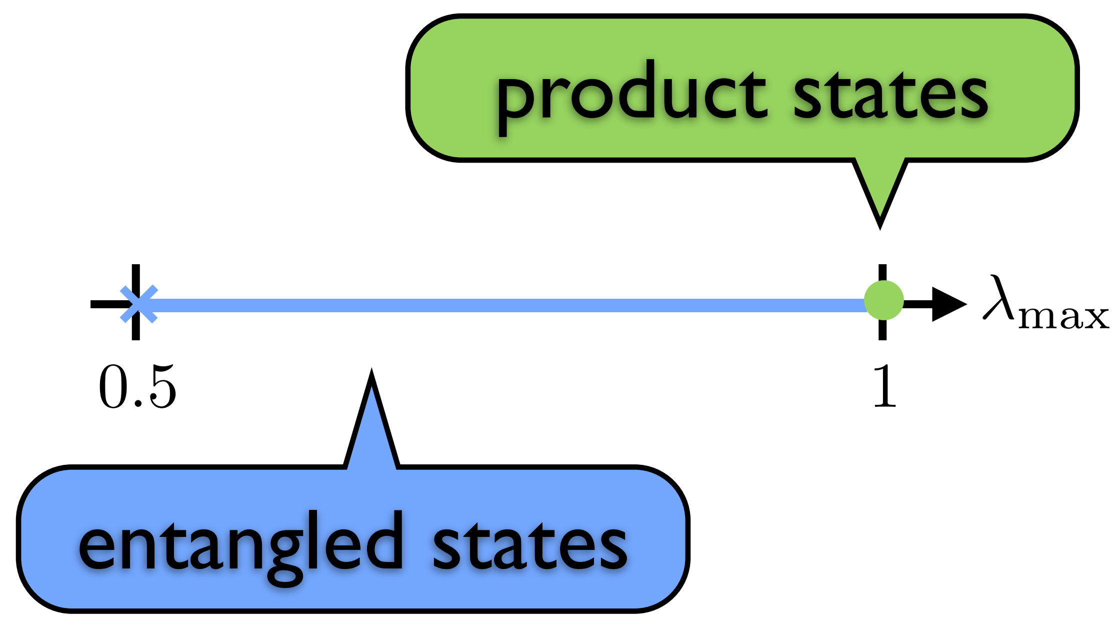
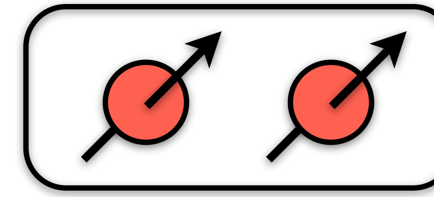


can be converted into $\times\ \sqrt{0.5}\,(|00\rangle + |11\rangle)$ by local operations and post-selection (SLOCC)

Two Qubits



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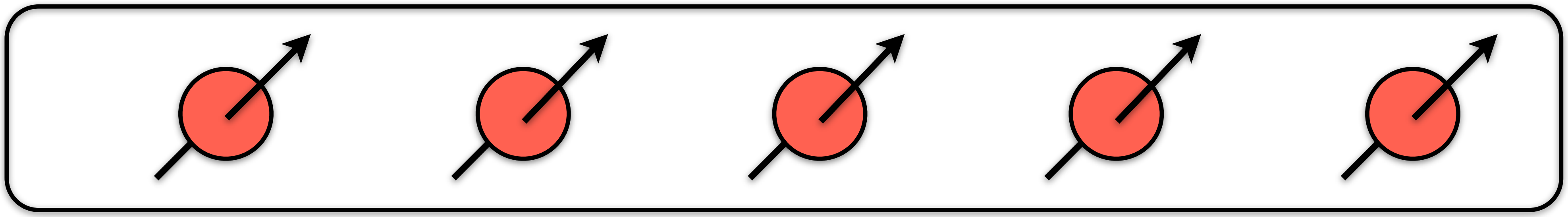
Two classes

product states

entangled states

Eigenvalues of reduced density matrices characterize entanglement of global state.

Multi-Partite Systems


$|\psi\rangle$

- no Schmidt decomposition
- rank of reduced density matrices not enough
- generically: infinitely many classes, labeled by exp(N) many continuous parameters

~ full tomography

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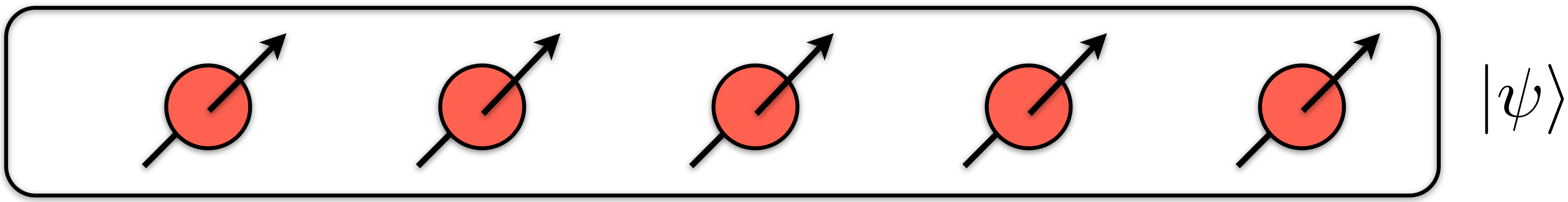

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Eigenvalues of reduced density matrices can still give useful information!

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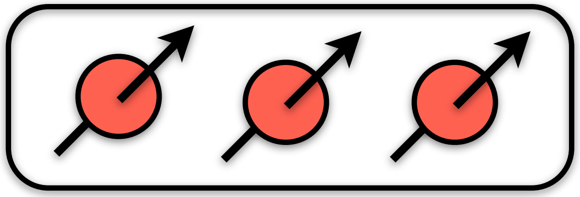

$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

Six classes

$$|GHZ\rangle = |000\rangle + |111\rangle$$

$$|W\rangle = |100\rangle + |010\rangle + |001\rangle$$

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$$|B2\rangle, |B3\rangle$$

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Han, Zhang & Guo (2004)
Botero & Mitchison (p.c.)
Sawicki, W. & Kus (2012)

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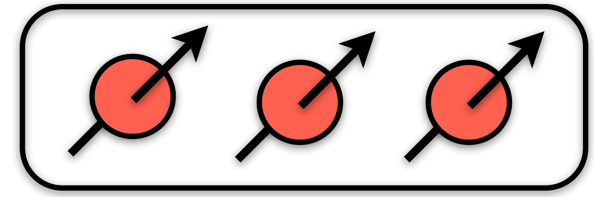

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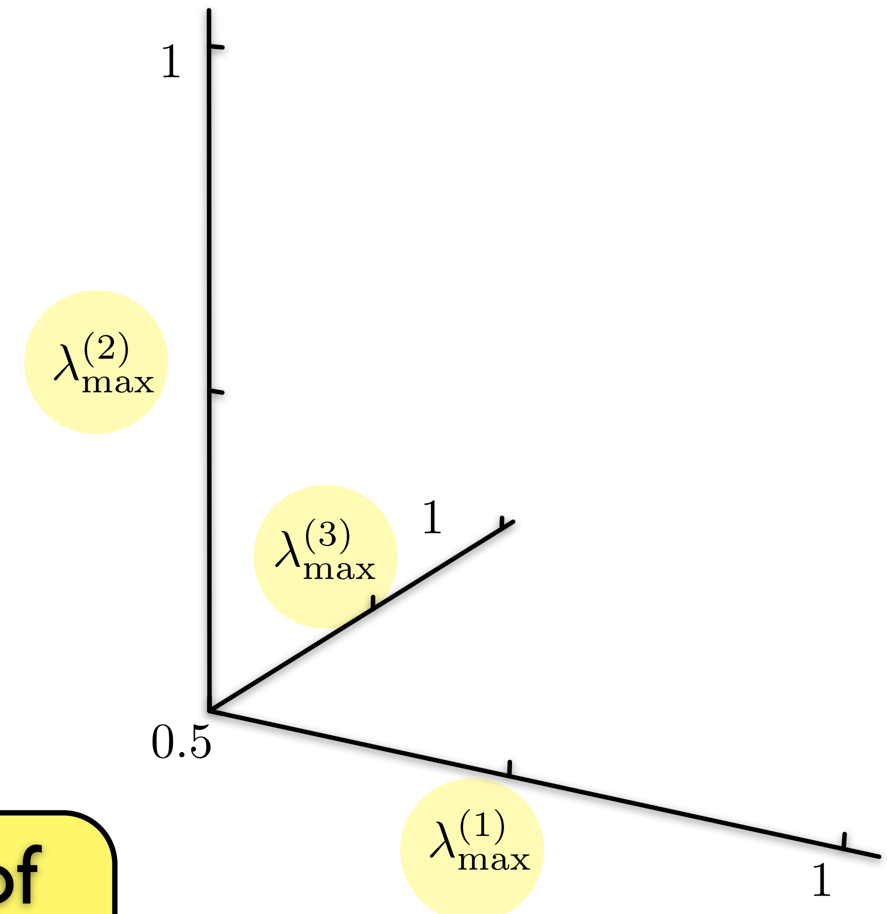

maximal eigenvalues of
ρ_1, ρ_2, ρ_3

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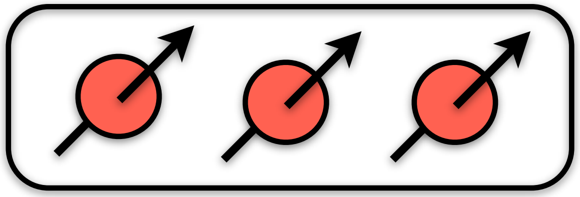

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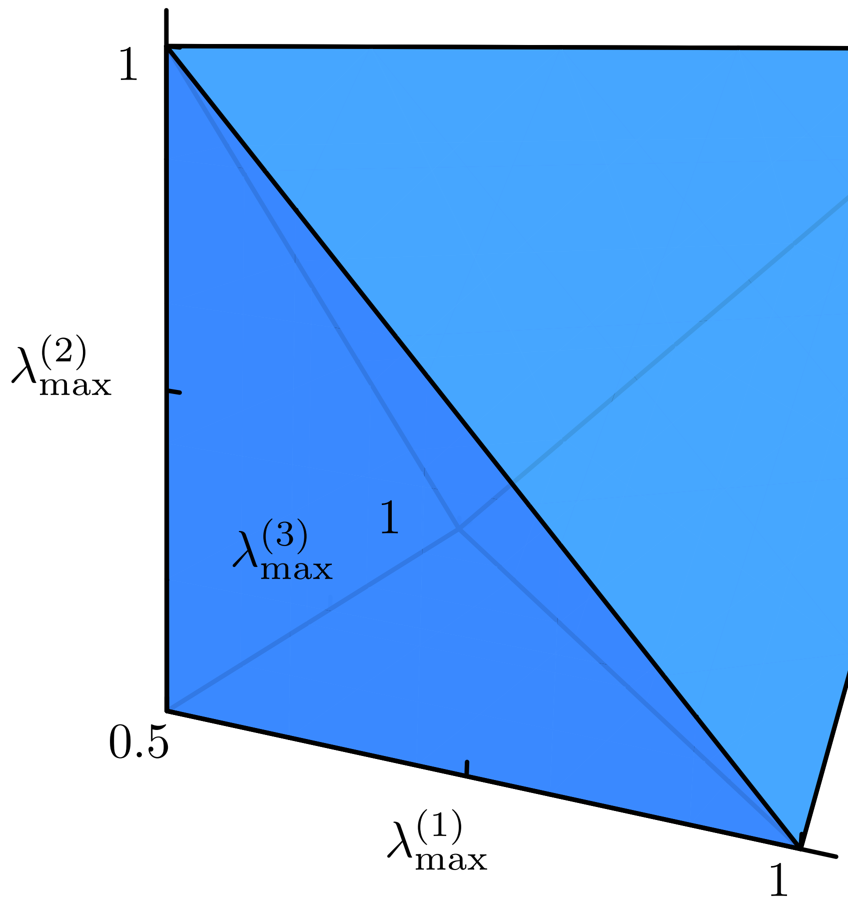

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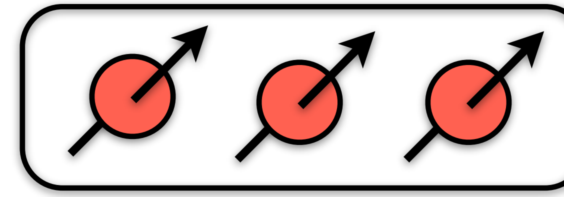

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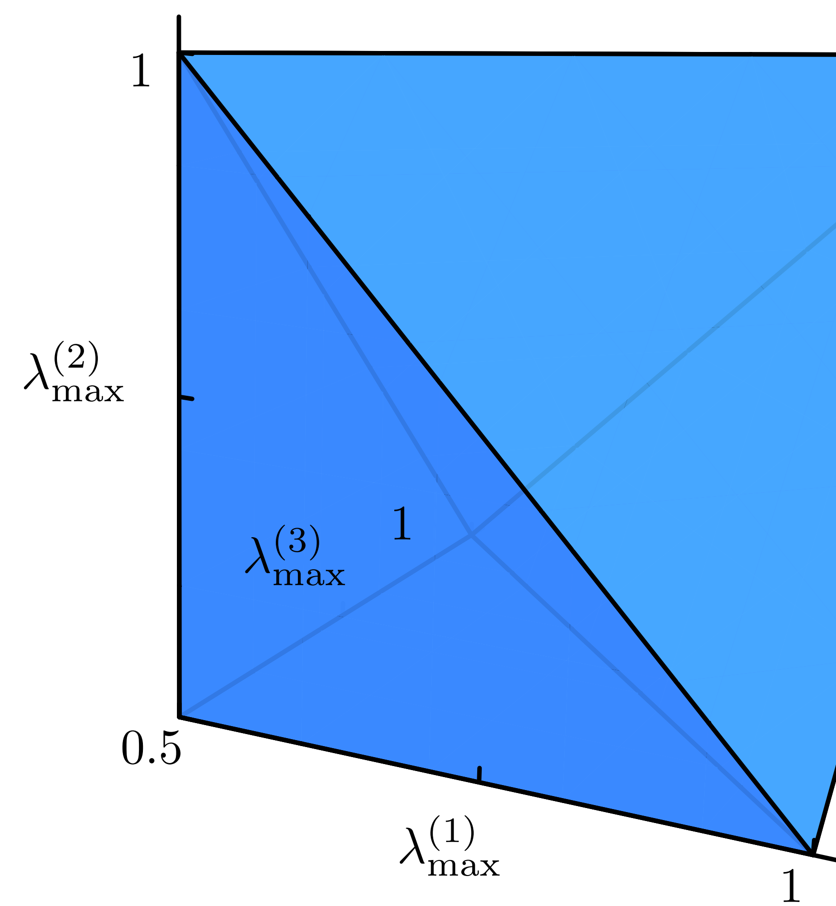
entanglement polytope of GHZ class



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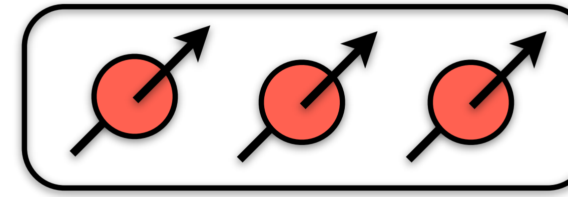

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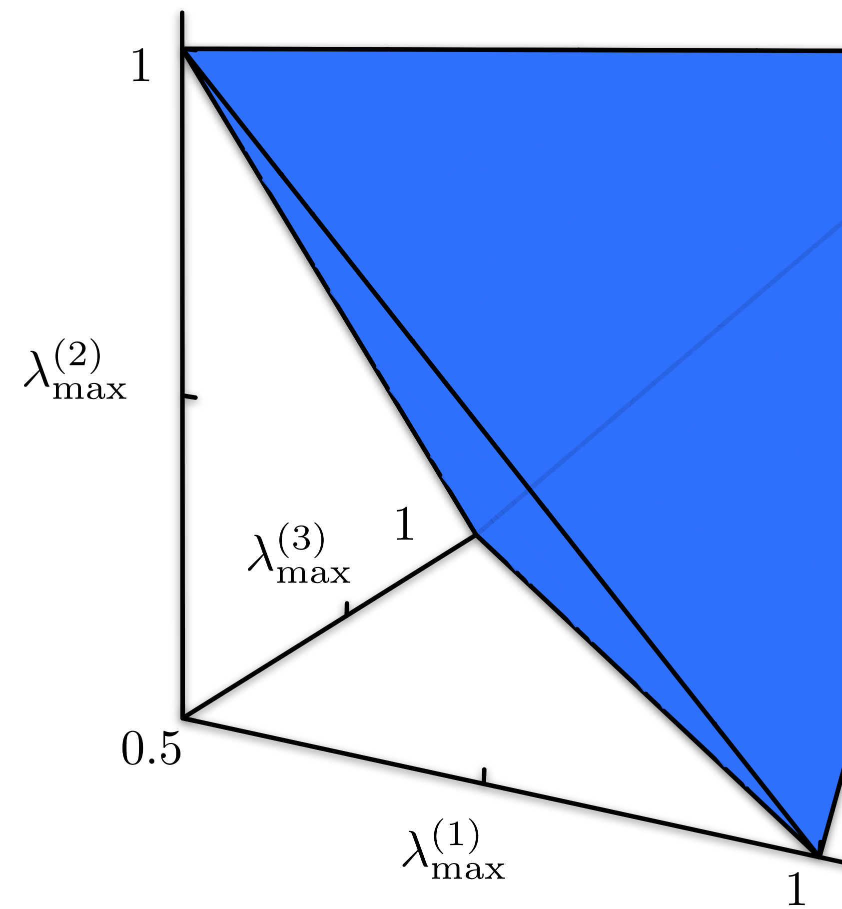

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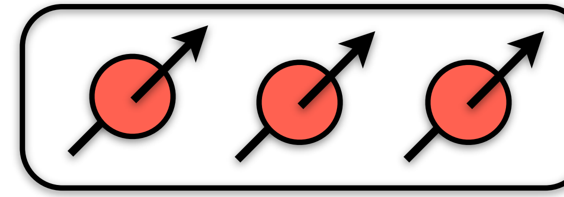

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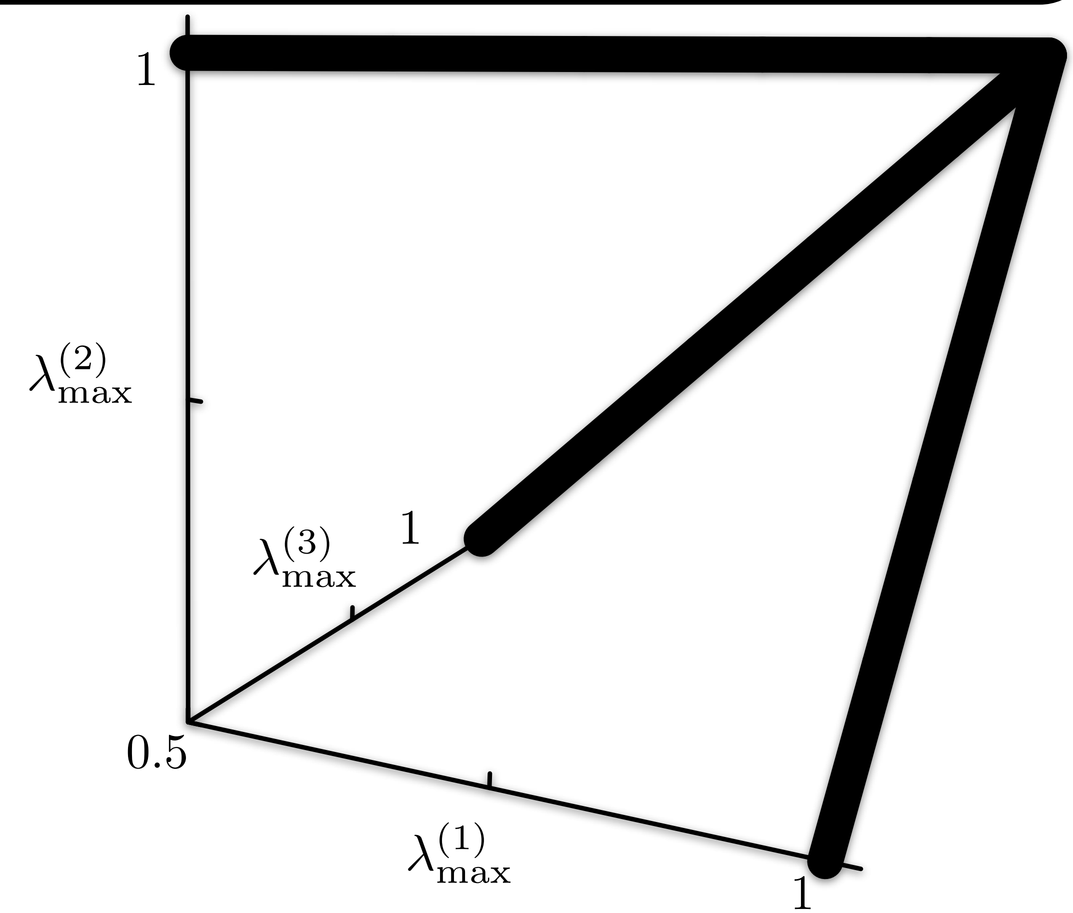
entanglement polytopes of biseparable states



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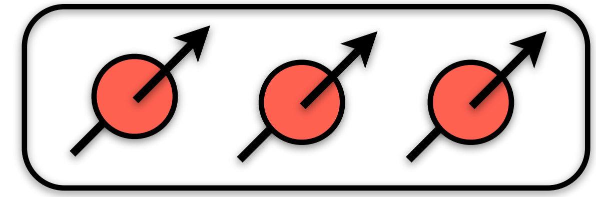

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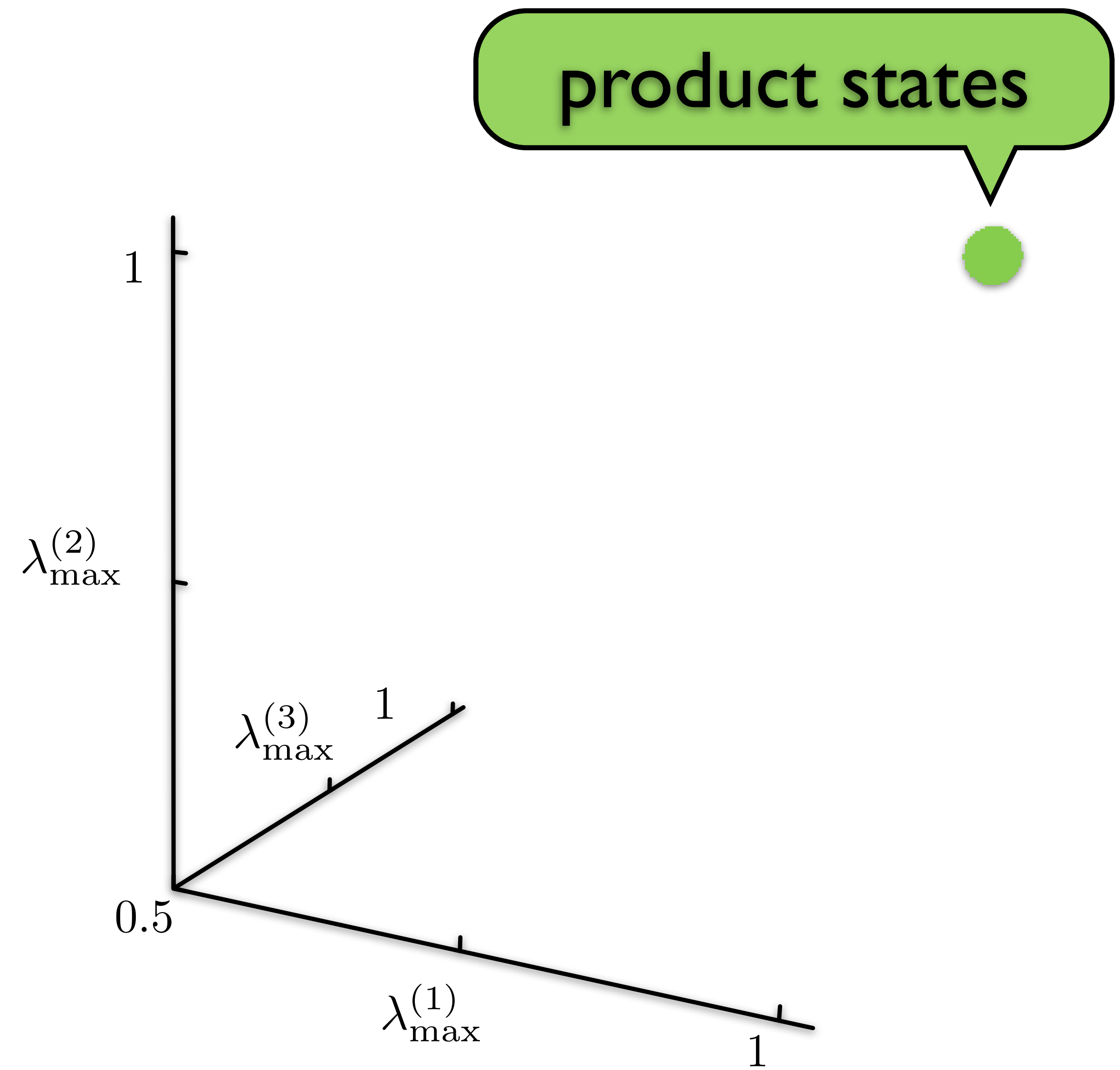

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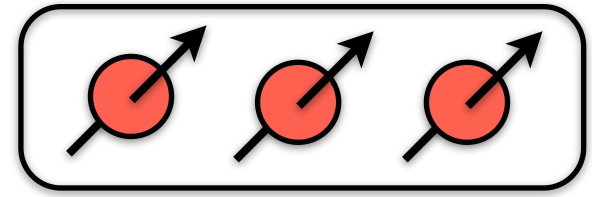

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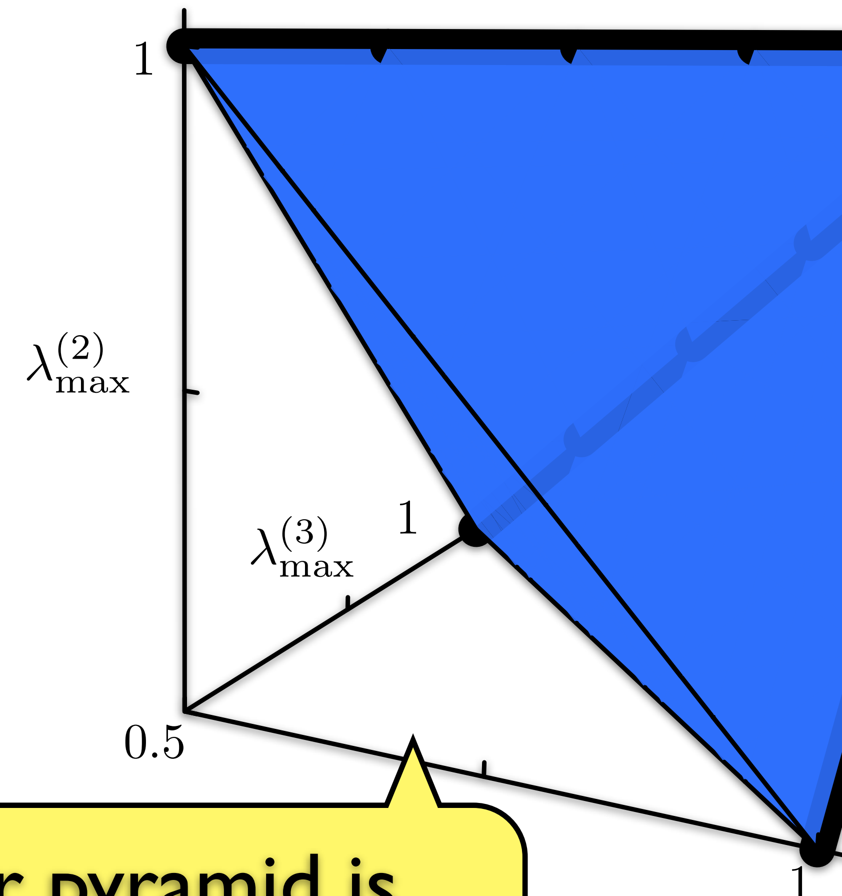

Lower pyramid is witness for GHZ class!

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Entanglement Polytopes

$$\Delta_{\mathcal{C}} = \{\vec{\lambda} = (\vec{\lambda}_1, \ldots, \vec{\lambda}_N) \text{ for } \psi \in \overline{\mathcal{C}}\}$$

eigenvalues of $\rho_1, \ldots, \rho_N$

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<u>Our main results:</u>

- convex polytope!
- finite hierarchy
- algorithm to compute using computational invariant theory (difficult)

using results from Brion (1987), Kempf & Ness (1979)
algebraic geometry / GIT

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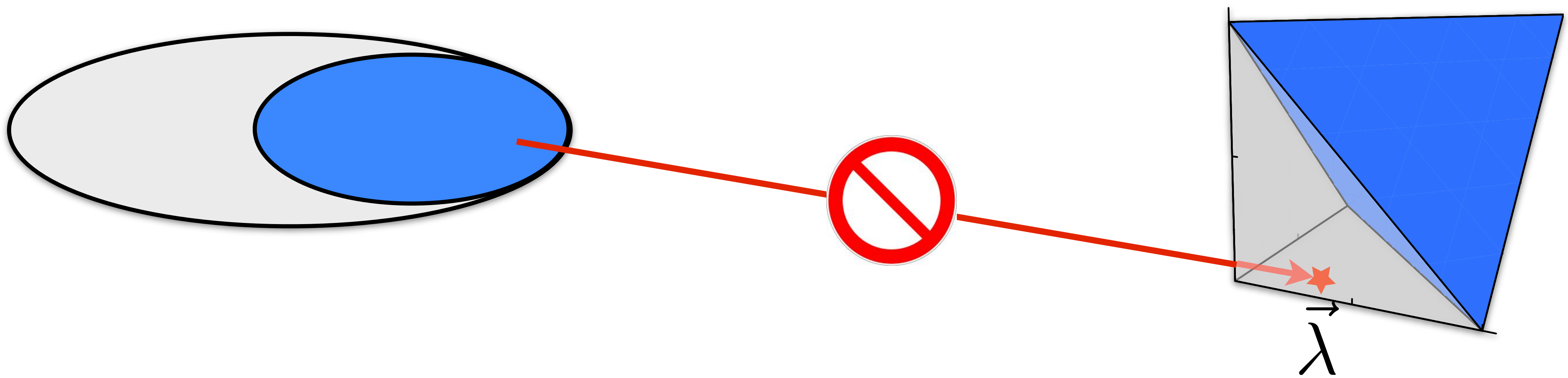
cf. Quantum Marginal Problem

Christandl-Mitchison (2004)
Klyachko (2004)
Daftuar-Hayden (2004)

Entanglement Criterion



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- robust against small noise ($\psi \approx$ pure)

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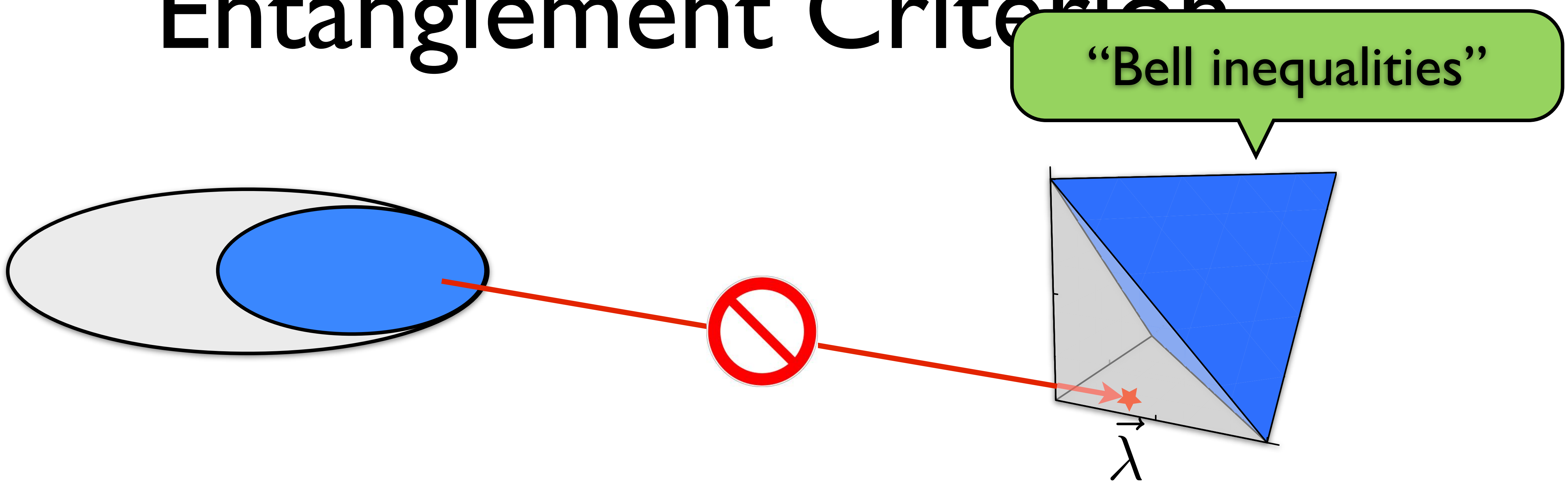

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violation of “Bell inequality”

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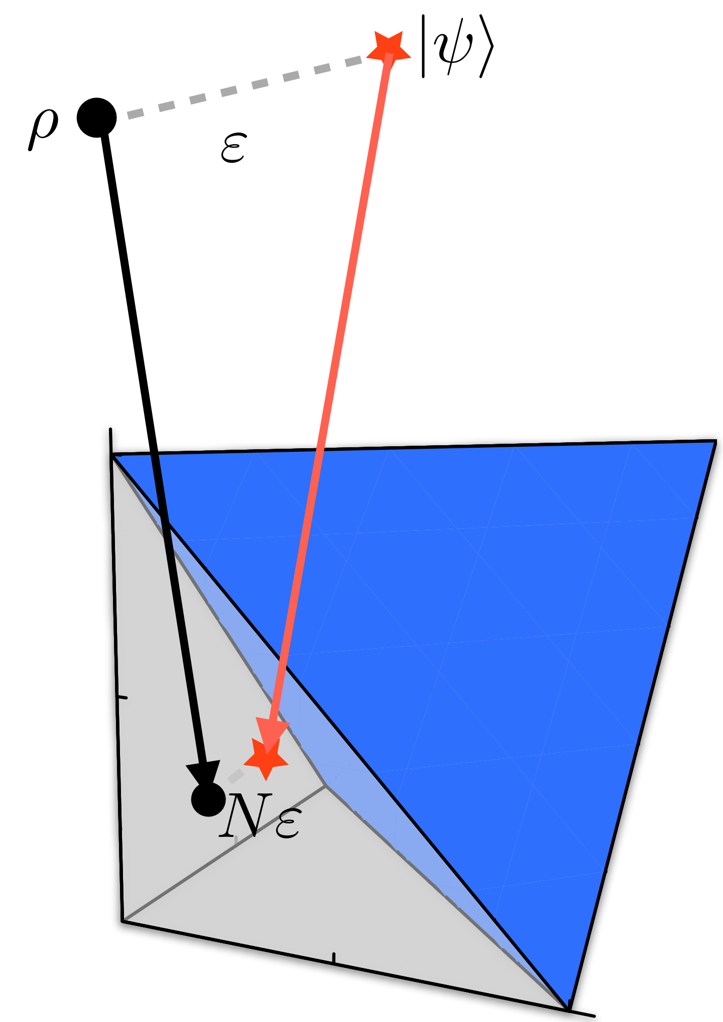
Purity and Noise

Purity: $p = \mathrm{tr}\,\rho^2$

(can be estimated using two-body measurements)

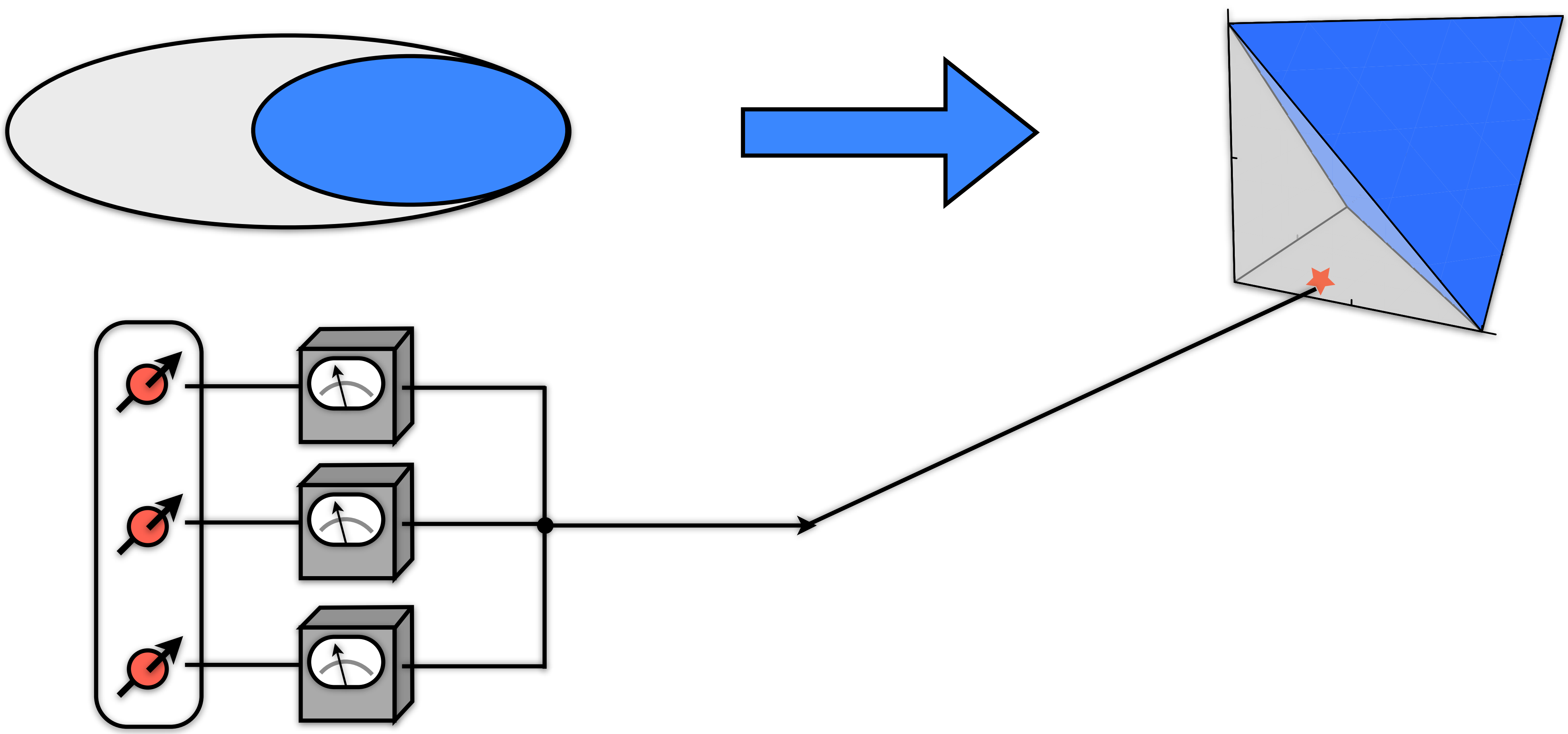
<u>Fact</u>: If $p \geq 1 - \epsilon$ then there exists a pure state $|\psi\rangle$ with $\langle\psi|\rho|\psi\rangle \geq 1 - \varepsilon$ whose local eigenvalues differ by $\lesssim N\varepsilon$.



Impurity enlarges effective error bars!

Thank you!

arXiv:1208.0365



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http://www.itp.phys.ethz.ch/people/waltemic/polytopes