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2013

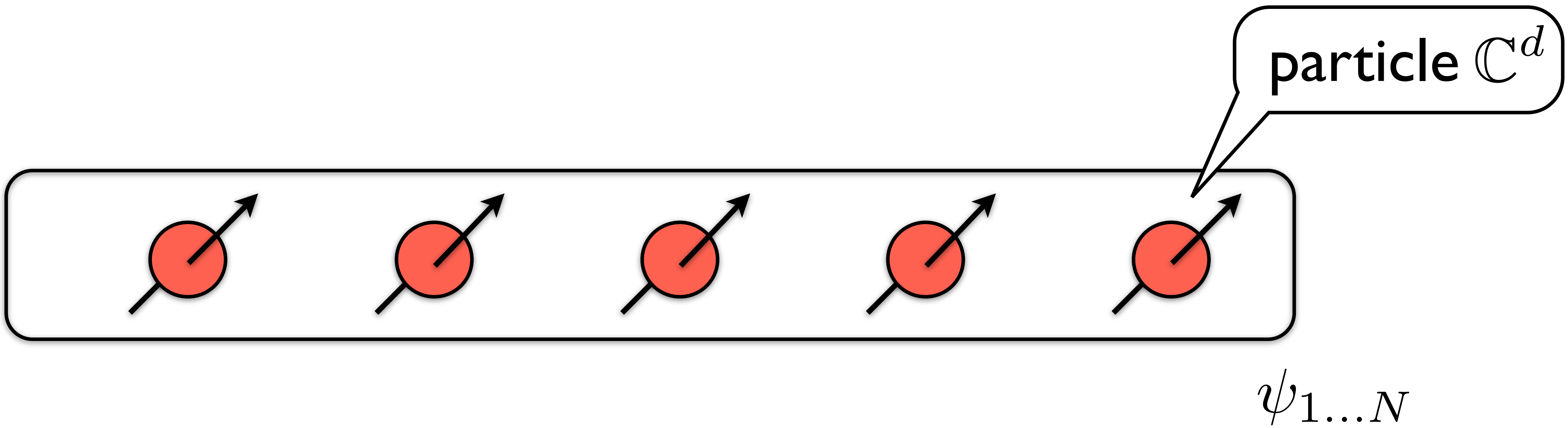
Entanglement Polytopes

Detecting Genuine Multi-Particle Entanglement by Single-Particle Measurements

Michael Walter

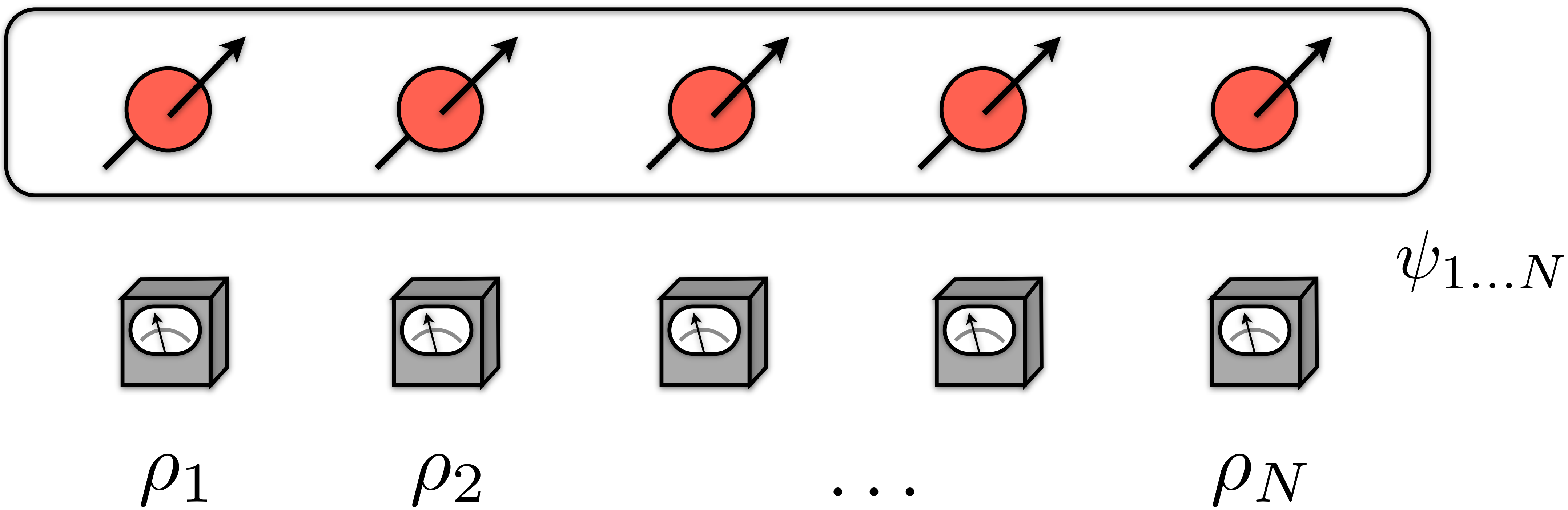
joint work with Matthias Christandl, Brent Doran (ETH Zürich), and David Gross (Univ. Freiburg)

Multi-Particle Entanglement



How entangled is a given multi-particle quantum state $\psi_{1\ldots N}$ (prepared in the laboratory)?

Multi-Particle Entanglement via Single-Particle Measurements



What can be said about the multi-particle entanglement of a <u>pure</u> state given its one-body density matrices only?

Multi-Particle Entanglement

Operational approach: equivalent entanglement if related by <u>operations which cannot "create entanglement"</u>

$$\psi_{1\ldots N} \longleftrightarrow \psi'_{1\ldots N}$$

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SLOCC

Stochastic Local Operations and Classical Comm.:

- non-zero success probability
- convenient mathematical characterization:

$$\psi'_{1\ldots N} = (A_1 \otimes \ldots \otimes A_N)\, \psi_{1\ldots N}$$

invertible local operators

Dür, Vidal & Cirac (2000)

Multi-Particle Entanglement

Entanglement class of a pure state $\psi_{1\dots N}$:

$$\mathcal{C}_\psi = \{\ \psi' \underset{\text{SLOCC}}{\longleftrightarrow} \psi\ \} = G \cdot |\psi\rangle\langle\psi| \subseteq \mathbb{P}(\mathcal{H})$$

$G = SL(d) \times \ldots \times SL(d)$

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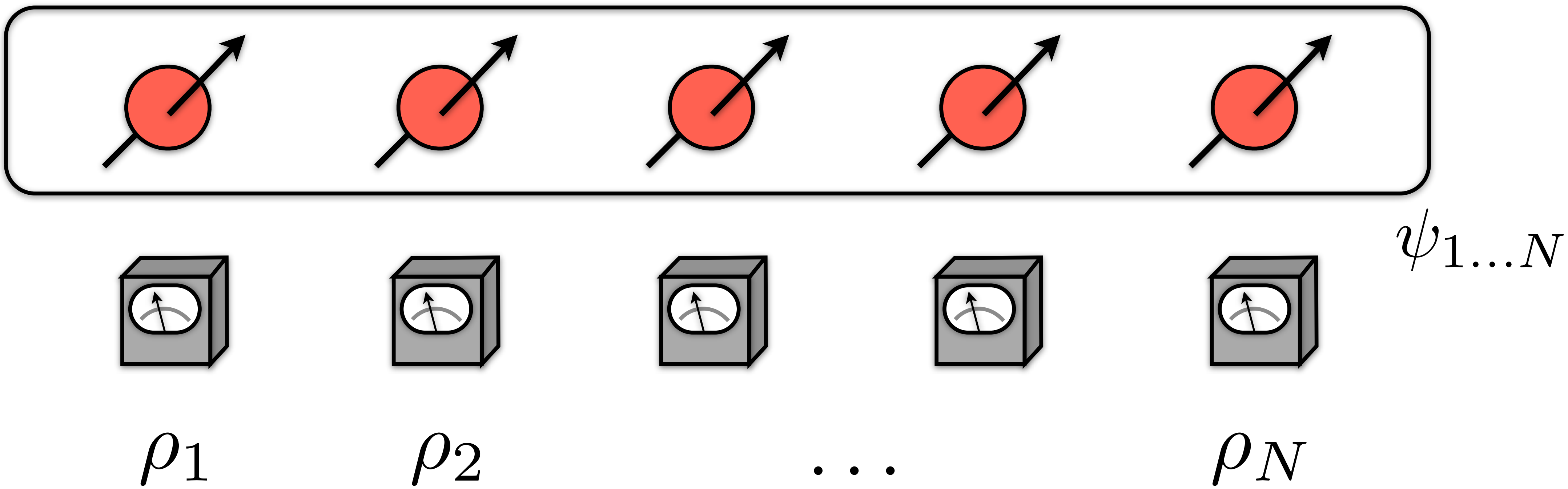
Three Qubits: six classes (GHZ, W, bi-separable, product)

Generically: infinitely many classes, labeled by exp(N) many continuous parameters

Single-Particle View of Multi-Particle Entanglement



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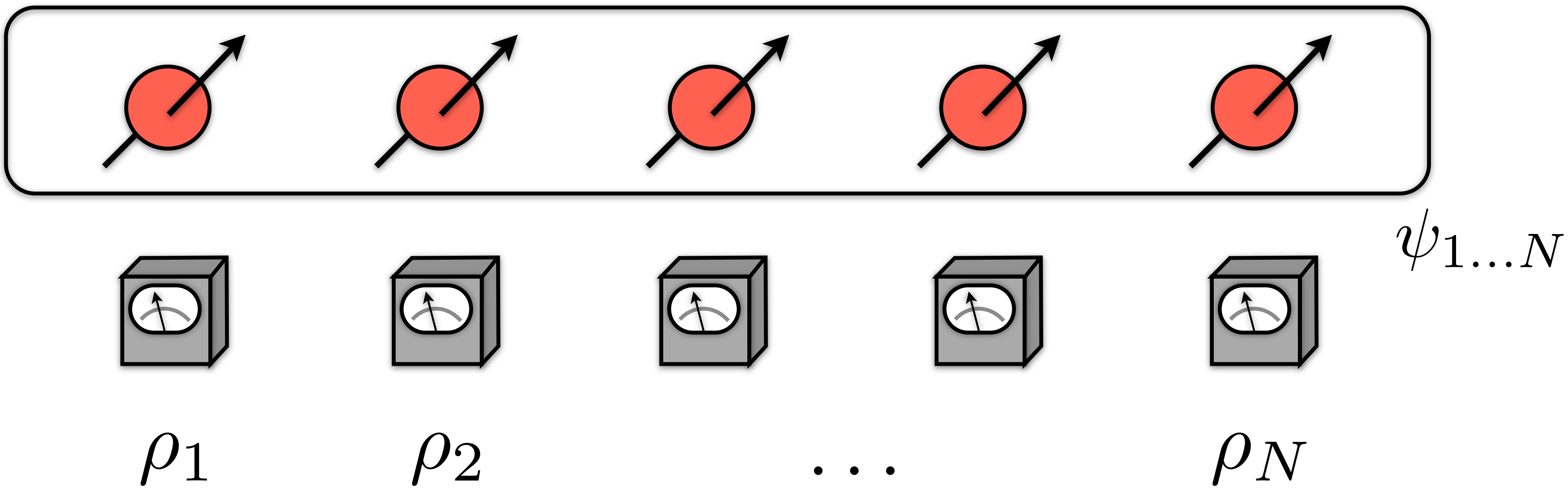

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only depends on local eigenvalues

Entanglement Polytopes

$$\Delta_{\mathcal{C}} = \{\vec{\lambda} = (\vec{\lambda}_1, \ldots, \vec{\lambda}_N) \text{ for } \psi \in \overline{\mathcal{C}}\}$$

eigenvalues of $\rho_1, \ldots, \rho_N$

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entanglement polytope

<u>Our Main Result:</u>

- convex polytope!
- finite hierarchy
- algorithm to compute using computational invariant theory (difficult)

using results from Brion (1987), Kempf & Ness (1979)
algebraic geometry / GIT

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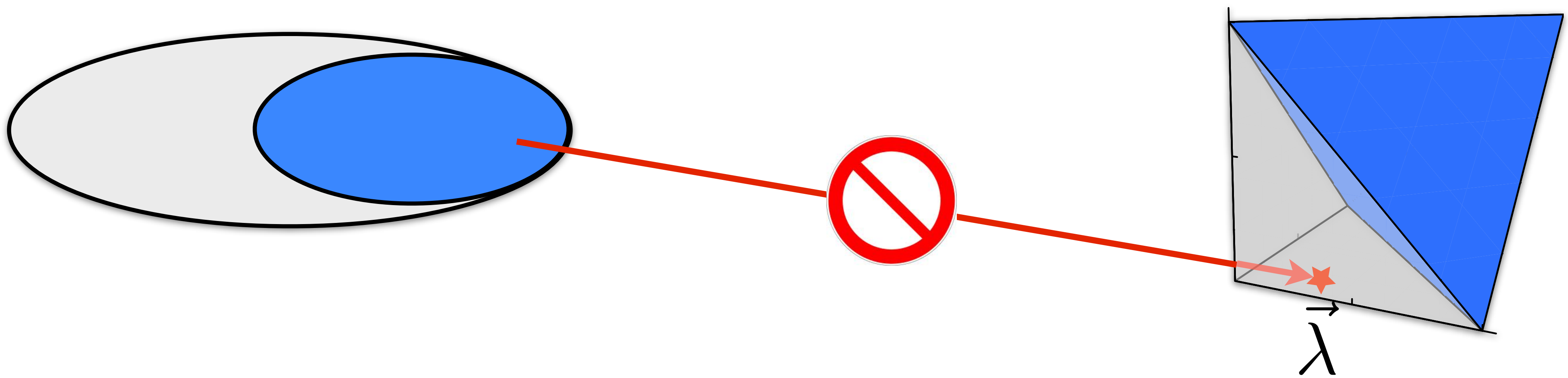
cf. Quantum Marginal Problem

Christandl-Mitchison (2004)
Klyachko (2004)
Daftuar-Hayden (2004)

Entanglement Criterion



$$\vec{\lambda} \notin \Delta_{\mathcal{C}} \implies \psi \notin \mathcal{C}$$

- efficient, requires only linearly many measurements
- robust against experimental noise ($\psi \approx$ pure)

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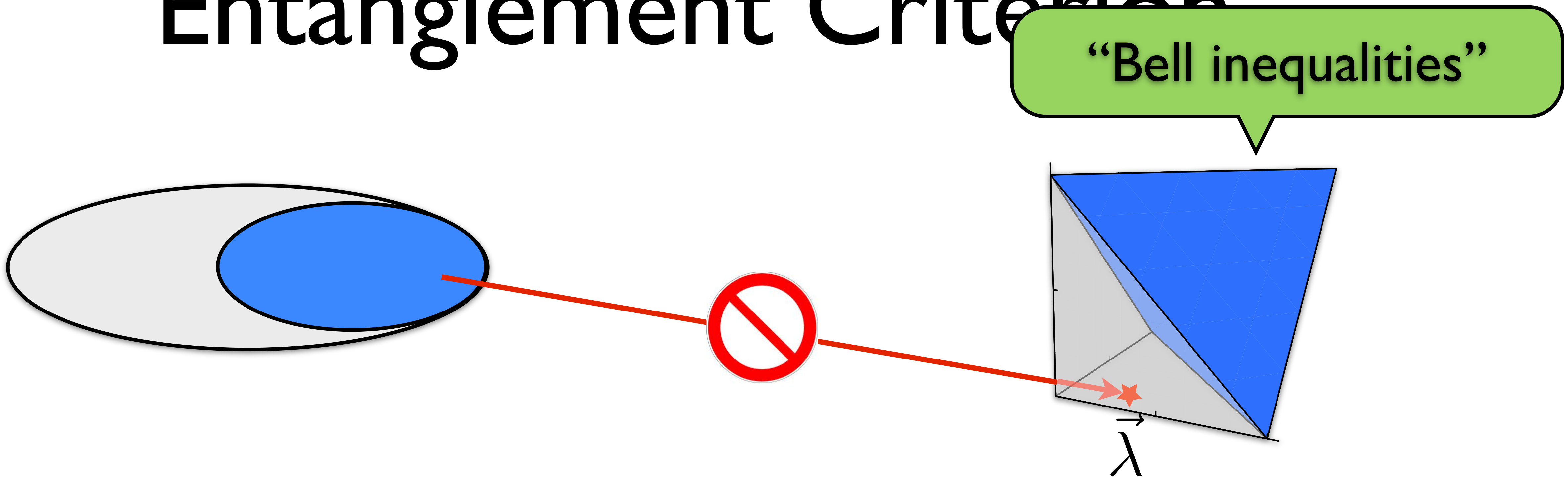

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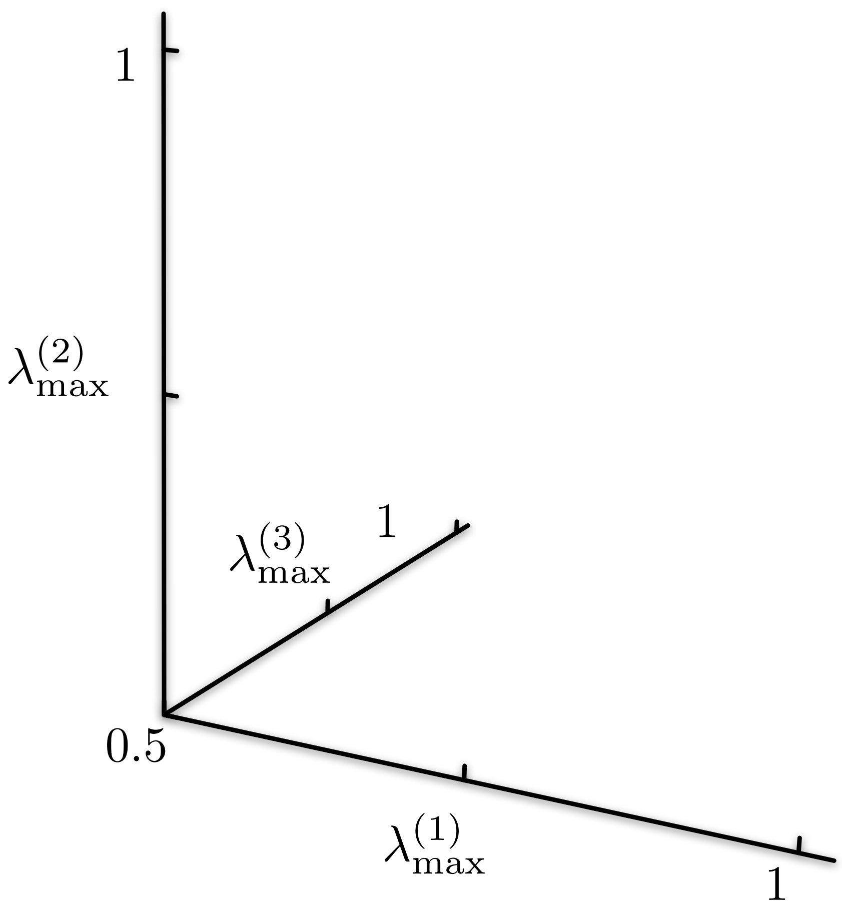
violation of “Bell inequality”

- efficient, requires only linearly many measurements
- robust against experimental noise ($\psi \approx$ pure)

Three Qubits

$$N = 3$$
$$\mathcal{H} = \mathbb{C}^2$$



<u>Entanglement classes:</u>

$$|GHZ\rangle = |000\rangle + |111\rangle$$
$$|W\rangle = |100\rangle + |010\rangle + |001\rangle$$
$$|B1\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)\,,$$
$$|B2\rangle, |B3\rangle$$
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Han, Zhang & Guo (2004)
Botero & Mitchison (p.c.)
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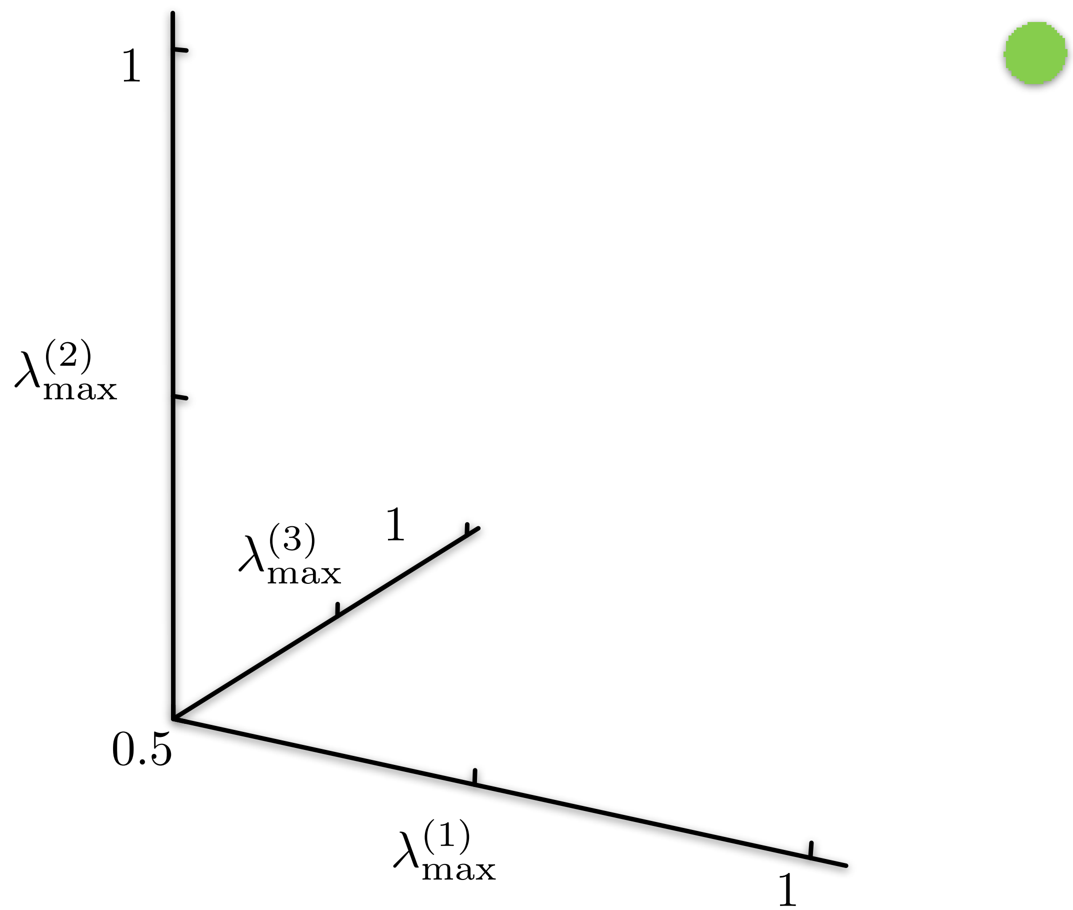

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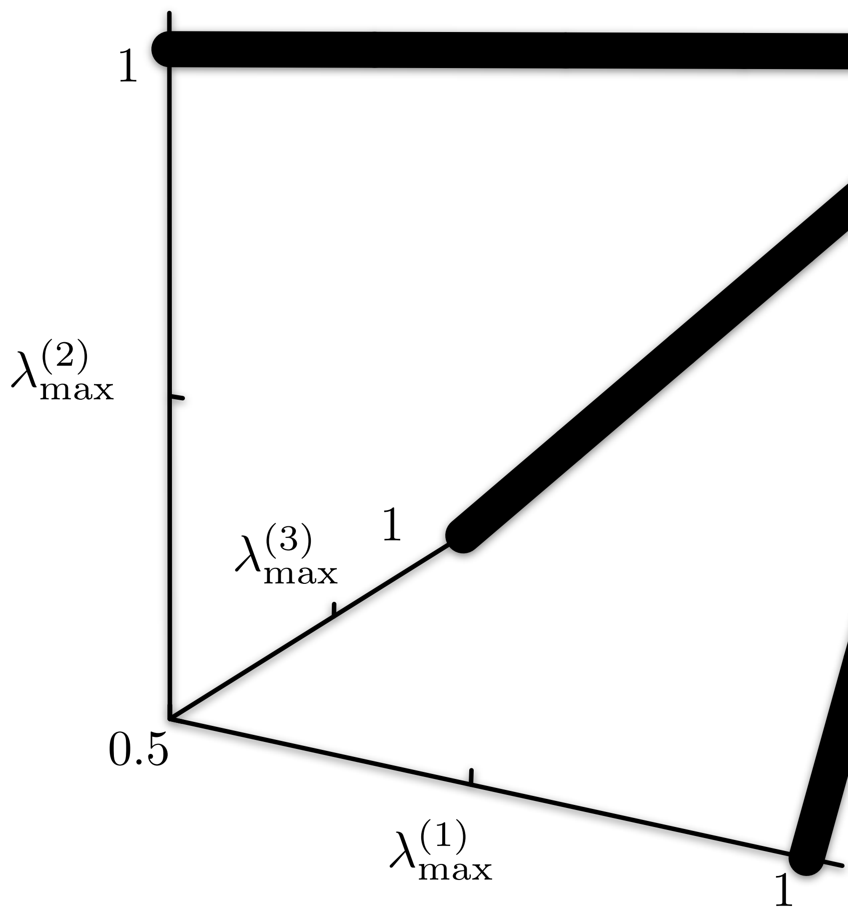

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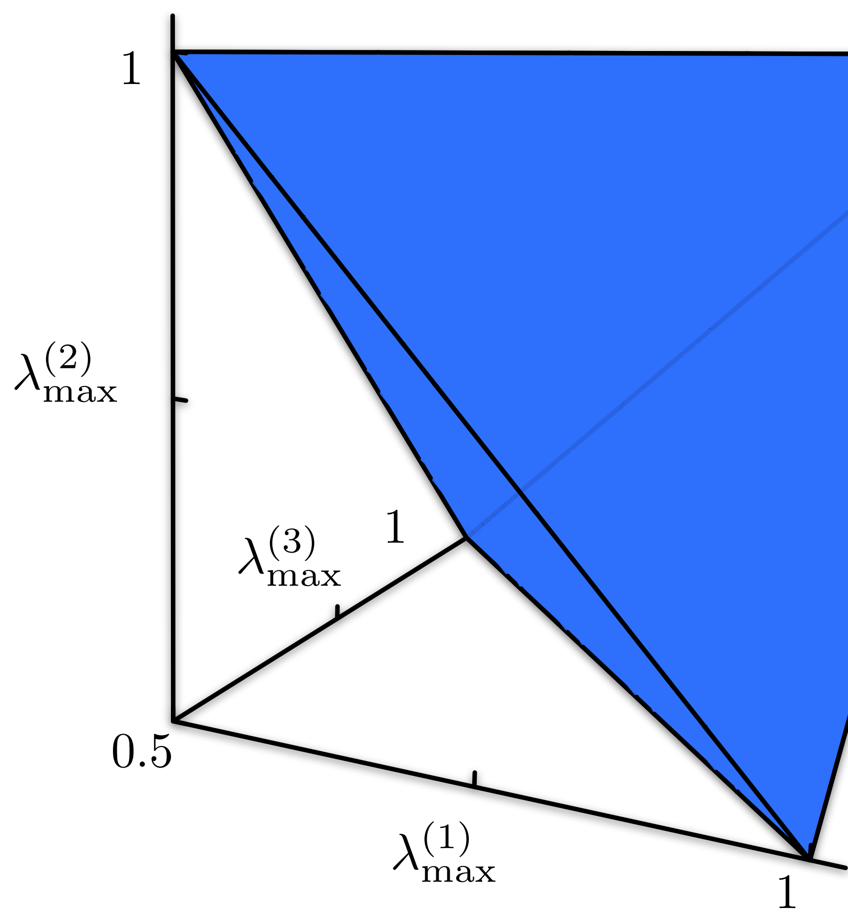

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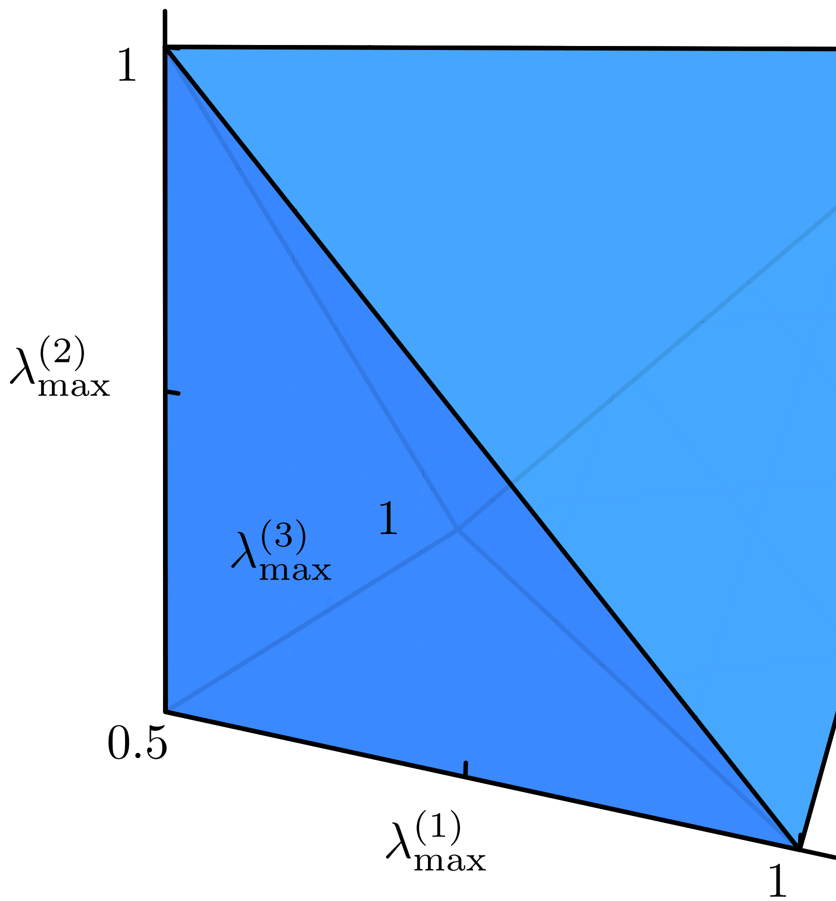

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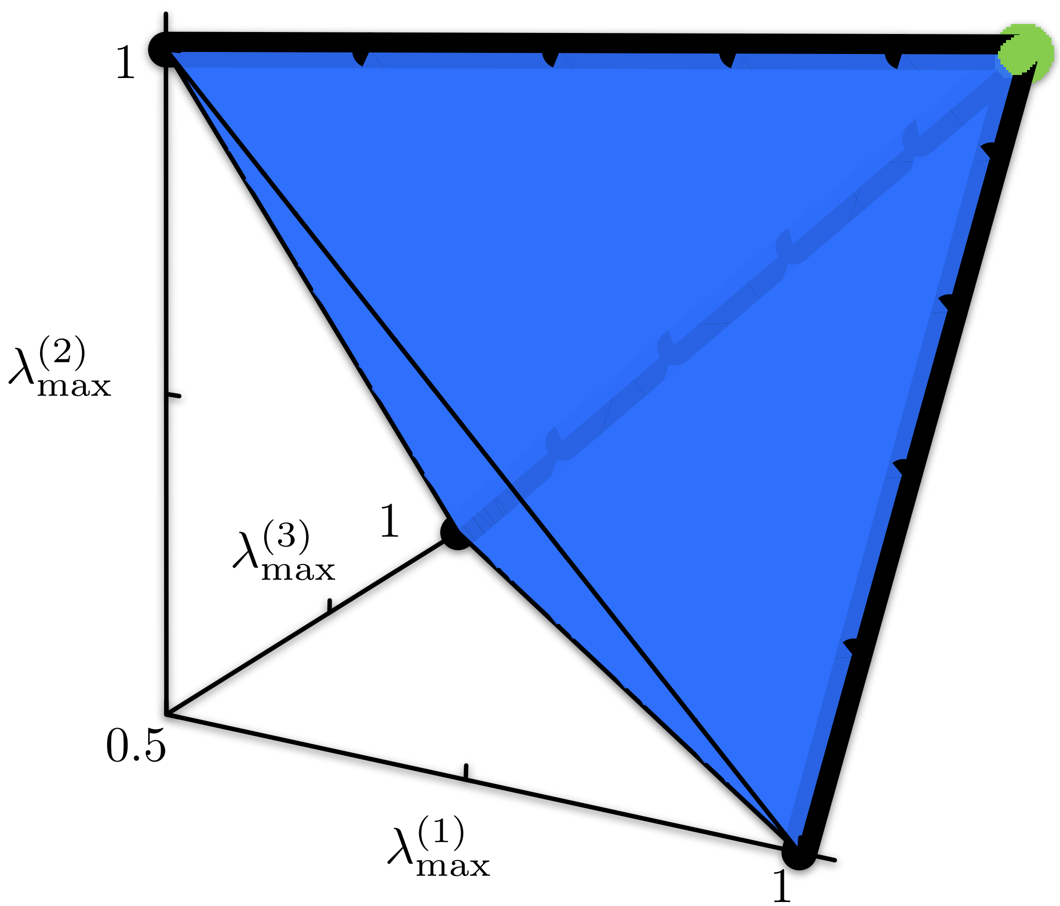

Lower pyramid is witness for GHZ class!

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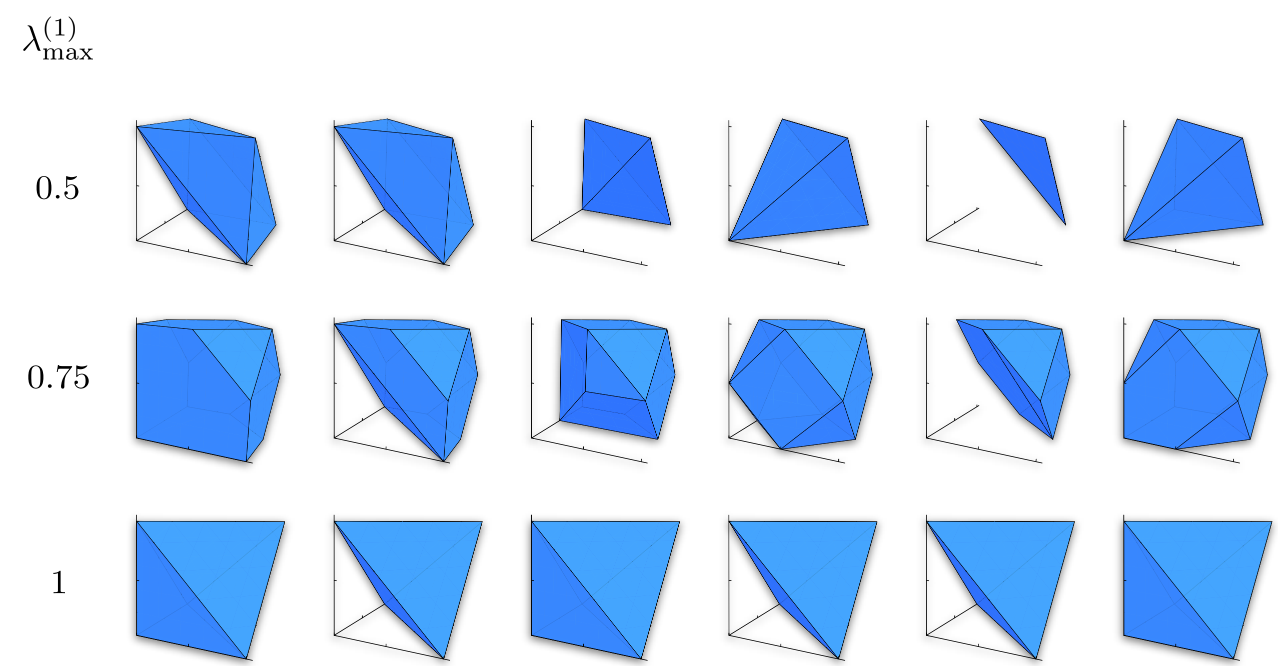
Further Examples

- Four Qubits: six non-trivial entanglement polytopes (3d sections)



- Bosonic qubit systems, fermionic system, ...

$\mathrm{Sym}^N \mathbb{C}^2$

$\wedge^3 \mathbb{C}^6$

http://www.itp.phys.ethz.ch/people/waltemic/polytopes

- Genuine multi-particle entanglement:

$$\Delta \supsetneq \bigcup_{\text{bipartition } S:S^c} \Delta_S \times \Delta_{S^c}$$

Invariant Theory

$$\mathcal{H} = \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$$
$$G = SL(d) \times \ldots \times SL(d)$$

Suppose there exists a G-invariant homogeneous polynomial P on $\mathcal{H}$ with $P(|\psi\rangle) \neq 0$.

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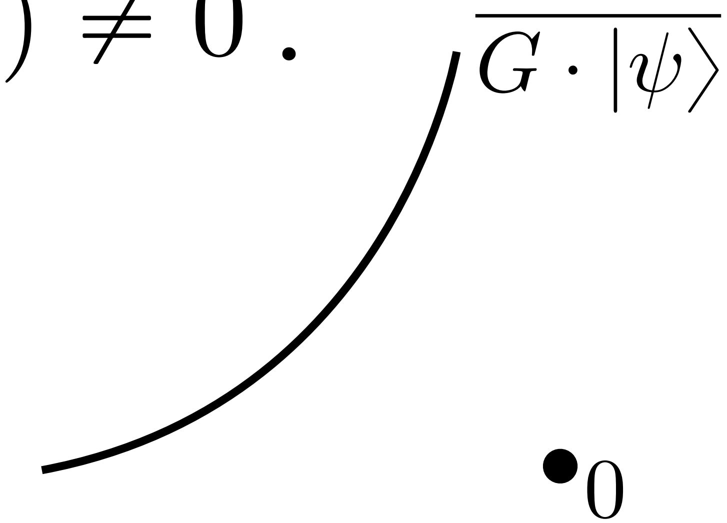
$$\Rightarrow \overline{G \cdot |\psi\rangle} \not\ni 0$$



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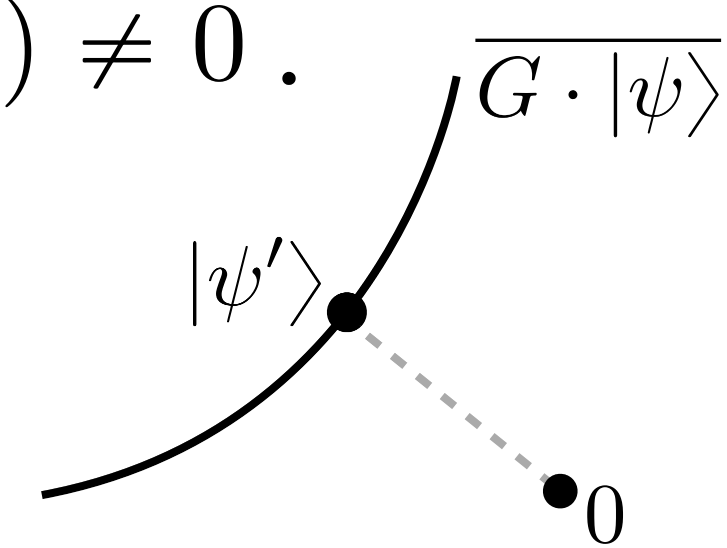

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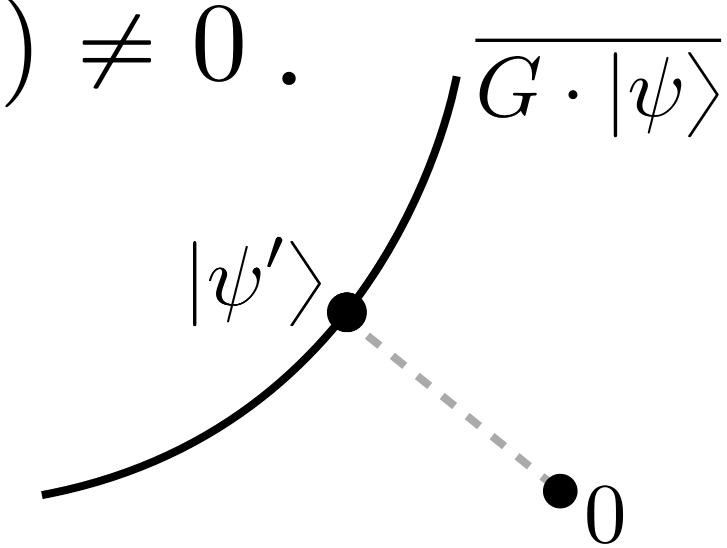

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Let $|\psi'\rangle$ be a vector of minimal length. Then

$$0 = \frac{d}{dt}\Big|_{t=0} \| e^{tX} \cdot |\psi'\rangle \|^2 = 2\langle\psi'|X|\psi'\rangle$$

for all traceless local observables X.

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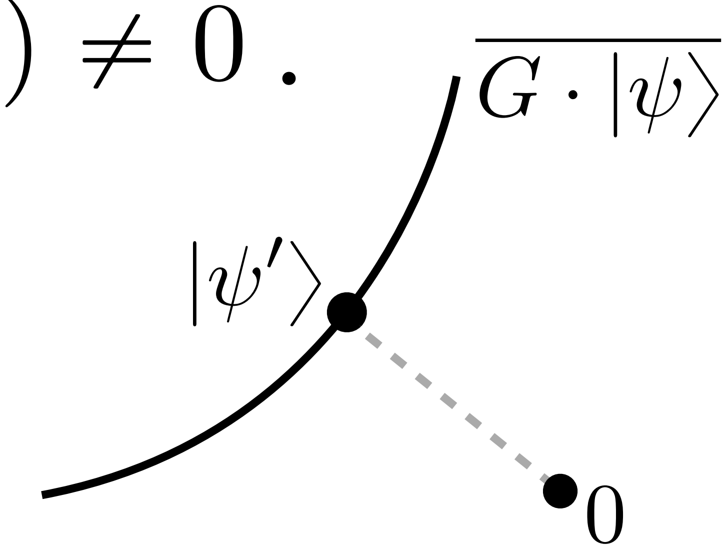

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Thus ψ' is locally maximally mixed!

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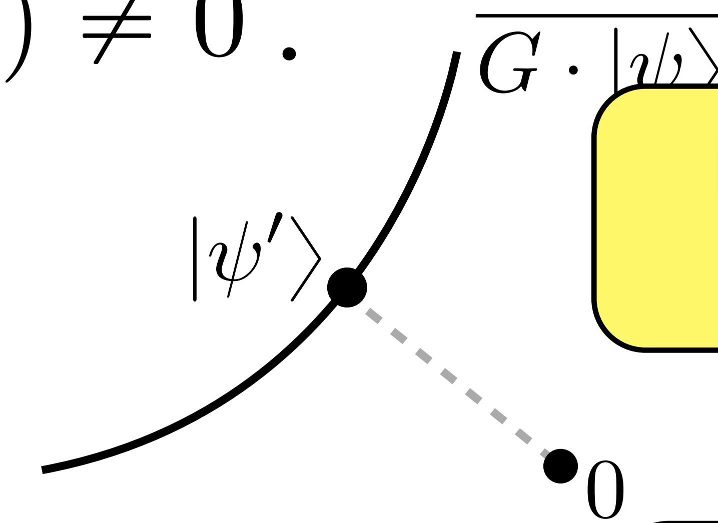

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The converse is also true!

Kempf & Ness (1979)
Klyachko (2004)

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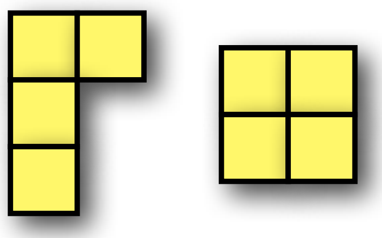
Local eigenvalues

Entanglement Polytopes and Covariants

Covariant: G-equivariant homogeneous polynomial function

$$\Phi : \mathcal{H} \to V_\lambda$$



Irreducible G-representation with highest weight λ

Entanglement Polytopes and Covariants

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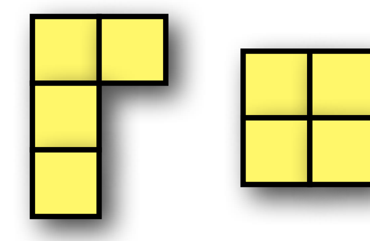

Choose a finite set of generators Φ_k with highest weight λ_k and degree d_k.

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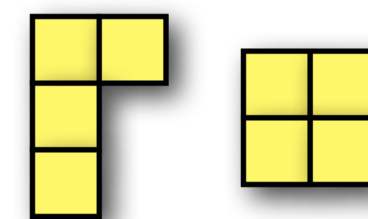

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Theorem: The entanglement polytope of an entanglement class $\mathcal{C}_\psi$ is given by

proof via Brion (1987)

$$\Delta_\psi = \text{conv}\{\lambda_k / d_k : \Phi_k(\psi) \neq 0\}$$

Entanglement Polytopes and Covariants

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computational invariant theory

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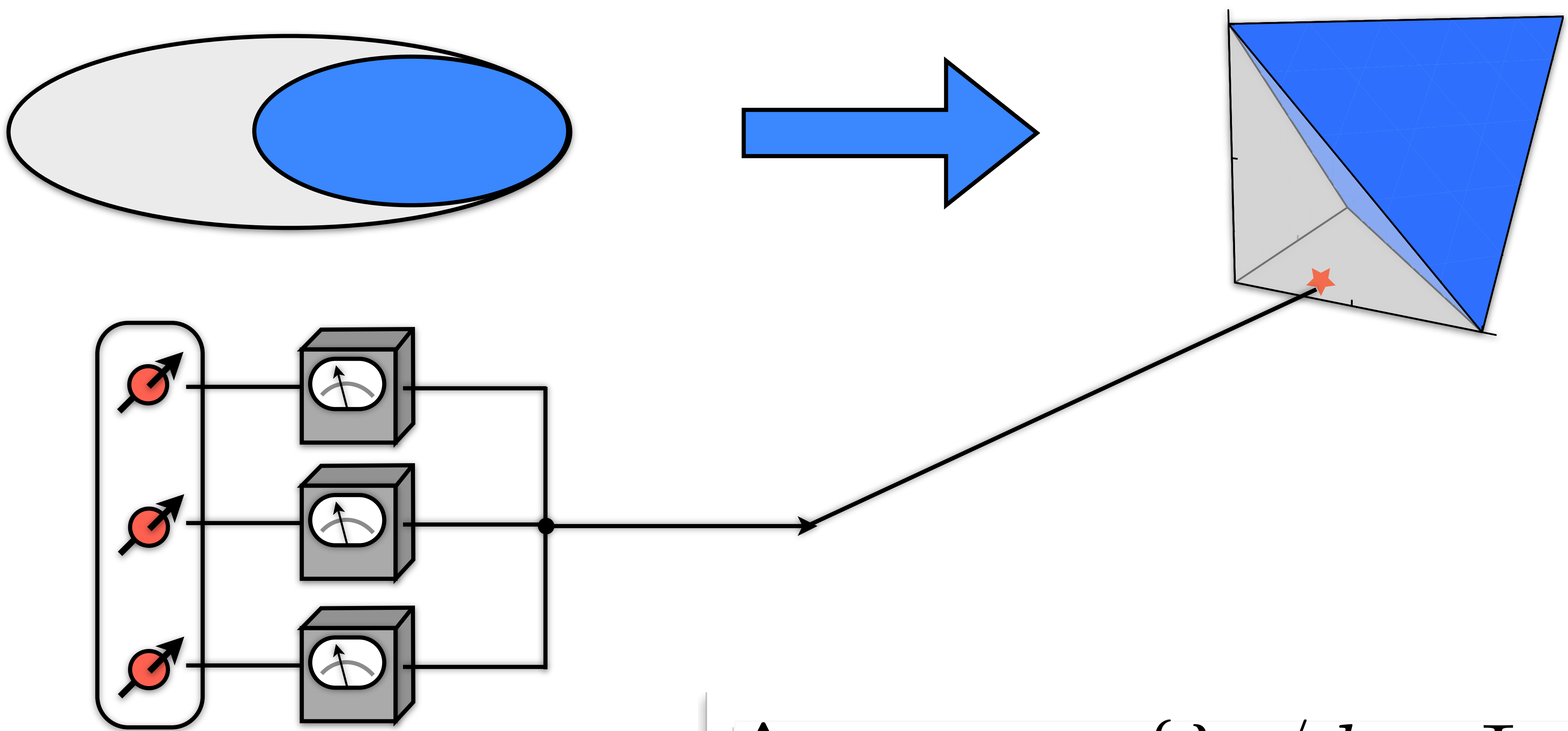
Summary

arXiv:1208.0365



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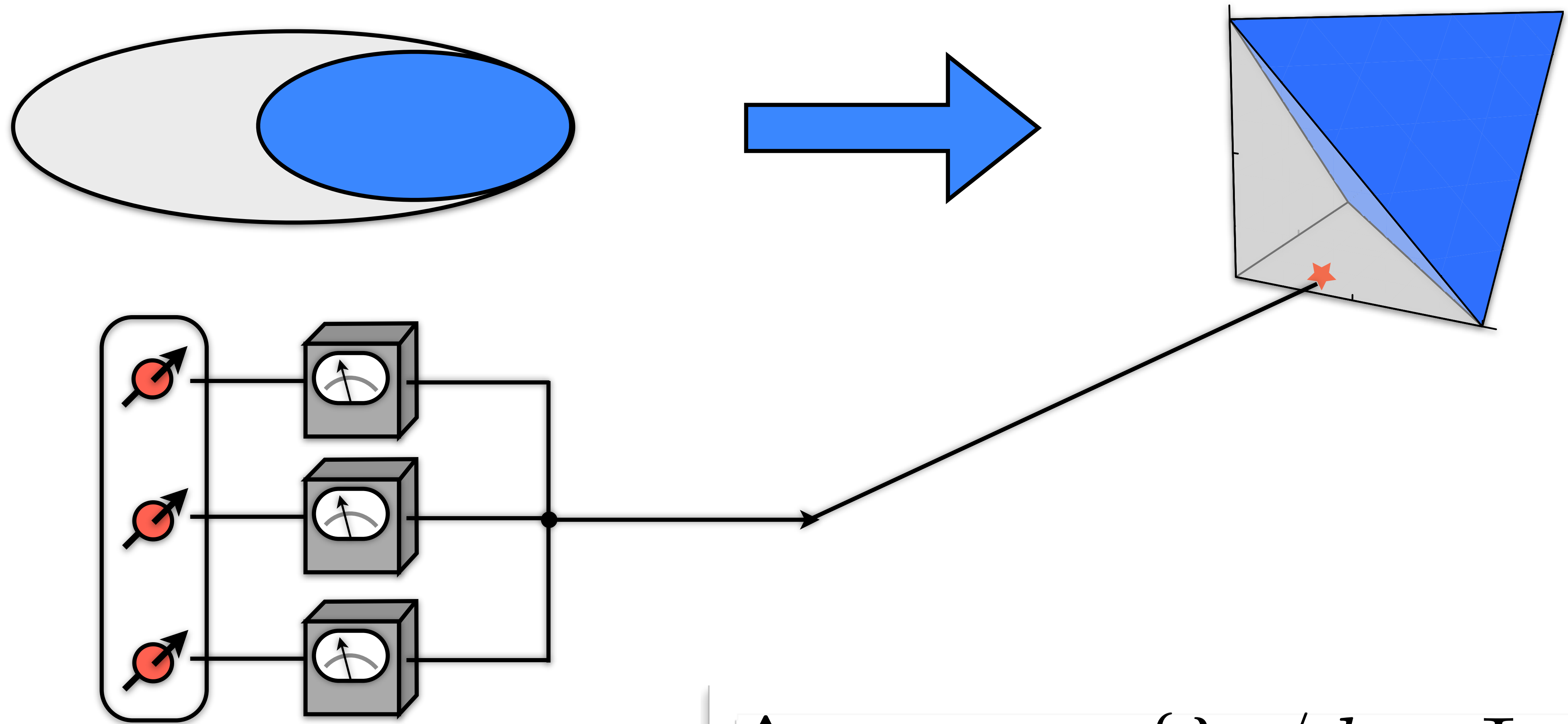

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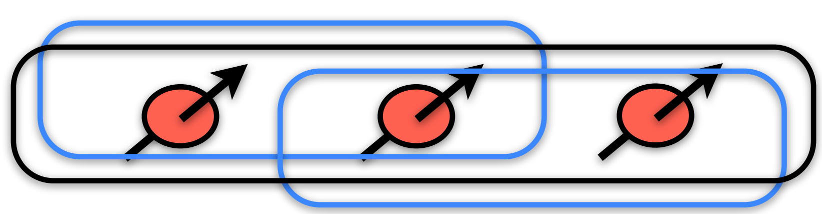
See also Burak's talk on "Recoupling Coefficients and Quantum Entropies"!



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