Tensor Networks, Fundamental Theorems, and Computational Complexity

Michael Walter (Uni Bochum)

PCTS, Princeton University, March 2023

joint work with Arturo Acuaviva, Visu Makam, Harold Nieuwboer, David Pérez-García, Friedrich Sittner, Freek Witteveen (QIP 2013, arXiv:2209.14358)
Complexity of many-body quantum physics

Many-body quantum states have **exponentially large** description:

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

In practice, entanglement **local** $\sim$ compact description:

Start with local entangled pairs...

$\ldots$ and “glue” by applying **local** transformations:
Complexity of many-body quantum physics

Many-body quantum states have exponentially large description:

$$|\Psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle$$

In practice, entanglement local $\sim$ compact description:

Start with local entangled pairs...

...and “glue” by applying local transformations:
What is a tensor network?

Given a tensor

\[ T = \sum_{i,j,k} T_{ijk} \langle i \rangle \langle j \rangle \langle k \rangle \]

we represent it graphically as

Contraction of tensors is shown graphically as
The tensor network tool box

**Tensor network**: define many-body state by contracting “local” tensors

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

e.g.

MPS [White, Fannes-Nachtergaele-Werner, …]  

PEPS [Verstraete-Cirac]

Numerical tool on classical and quantum computers

Analytical tool that provides “dual descriptions” of complex phenomena: symmetries, topological phases, renormalization, …
The tensor network tool box

**Tensor network:** define many-body state by contracting “local” tensors

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

e.g.

**MPS** [White, Fannes-Nachtergaele-Werner, …]

**PEPS** [Verstraete-Cirac]

**Numerical tool** on classical and quantum computers

**Analytical tool** that provides “dual descriptions” of complex phenomena: symmetries, topological phases, renormalization, …
Innocent or not? First examples

▸ **Long-range entanglement:** If \( \Psi = |000\rangle + |111\rangle \) then

\[
\cdots \cdots = |00 \cdots 00\rangle + |11 \cdots 11\rangle
\]

▸ **Quantum cellular automata:**

Unitary, but *not* a quantum circuit!

▸ **An example from TCS:**

\[
= \sum_{i,j,k} A_{ij} B_{jk} C_{ki} = \text{tr}(ABC)
\]

and hence is the matrix multiplication tensor.
Innocent or not? First examples

- **Long-range entanglement:** If $|\psi\rangle = |000\rangle + |111\rangle$ then

  $\cdots \cdots = |00 \cdots 00\rangle + |11 \cdots 11\rangle$

- **Quantum cellular automata:**

  Unitary, but *not* a quantum circuit!

- **An example from TCS:**

  $= \sum_{i,j,k} A_{ij} B_{jk} C_{ki} = \text{tr}(ABC)$

  and hence is the matrix multiplication tensor.
Innocent or not? First examples

- **Long-range entanglement:** If $|\psi\rangle = |000\rangle + |111\rangle$ then
  \[
  \cdots \quad = |00 \cdots 00\rangle + |11 \cdots 11\rangle
  \]

- **Quantum cellular automata:**
  \[
  \cdots
  \]
  Unitary, but *not* a quantum circuit!

- **An example from TCS:**
  \[
  = \sum_{i,j,k} A_{ij} B_{jk} C_{ki} = \text{tr}(ABC)
  \]
  and hence is the matrix multiplication tensor.
Innocent or not? First examples

▶ Long-range entanglement: If $\psi = |000\rangle + |111\rangle$ then

$$\cdots \cdots = |00 \cdots 00\rangle + |11 \cdots 11\rangle$$

▶ Quantum cellular automata:

Unitary, but *not* a quantum circuit!

▶ An example from TCS:

$$= \sum_{i,j,k} A_{ij} B_{jk} C_{ki} = \text{tr}(ABC)$$

and hence is the matrix multiplication tensor.
Consider 3-leg tensor $M$ with bond dimension $D$ and physical dimension $d$:

This determines a quantum state on any $n$ sites:

Its coefficients are $\langle i_1, \ldots, i_n | M_n \rangle = \text{tr}(M^{(i_1)} M^{(i_2)} \cdots M^{(i_n)})$, hence the name.
1D: Matrix product states (MPS)

Consider 3-leg tensor $M$ with bond dimension $D$ and physical dimension $d$:

This determines a quantum state on any $n$ sites:

Its coefficients are $\langle i_1, \ldots, i_n | M_n \rangle = \text{tr}(M^{(i_1)} M^{(i_2)} \cdots M^{(i_n)})$, hence the name.
1D: Matrix product states (MPS)

Consider 3-leg tensor $M$ with bond dimension $D$ and physical dimension $d$:

This determines a quantum state on any $n$ sites:

Its coefficients are $\langle i_1, \ldots, i_n|M_n \rangle = \text{tr}(M^{(i_1)}M^{(i_2)}\cdots M^{(i_n)})$, hence the name.
Computing with tensor networks

Essentially all computations amount to tensor contractions:

\[ \langle \psi | \chi \rangle = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \]

**Computational complexity:** Easy in 1D. In principle, hard in 2D and higher.
2D: Projected entangled pair states (PEPS)

Consider 5-leg tensor $T$ with bond dimension $D$ and physical dimension $d$:

![Diagram of 5-leg tensor T with bond dimension D and physical dimension d]

This determines a quantum state on any $n \times m$ sites:
2D: Projected entangled pair states (PEPS)

Consider 5-leg tensor $T$ with bond dimension $D$ and physical dimension $d$:

This determines a quantum state on any $n \times m$ sites:
Geometry vs. entanglement

\[ S(A) = - \text{tr} \rho_A \log \rho_A \]

Area law: 
\[ S(A) \leq |\partial A| \log D. \]

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy area law and have accurate MPS representations [Hastings].
- These can be found in polynomial time [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, ...]
Geometry vs. entanglement

\[ S(A) = -\text{tr} \rho_A \log \rho_A \]

Area law: \[ S(A) \leq |\partial A| \log D. \]

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy area law and have accurate MPS representations [Hastings].
- These can be found in polynomial time [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, …]
Geometry vs. entanglement

\[ S(A) = - \text{tr} \rho_A \log \rho_A \]

\textbf{Area law:} \[ S(A) \leq |\partial A| \log D. \]

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy \textit{area law} and have accurate MPS representations [Hastings].
- These can be found in \textit{polynomial time} [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, ...]
S(A) = −tr \( \rho_A \log \rho_A \)

**Area law:** \( S(A) \leq |\partial A| \log D \).

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy *area law* and have *accurate MPS representations* [Hastings].
- These can be found in polynomial time [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, ...]
**Geometry vs. entanglement**

\[ S(A) = -\text{tr} \rho_A \log \rho_A \]

**Area law:** \[ S(A) \leq |\partial A| \log D. \]

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy area law and have **accurate MPS representations** [Hastings].
- These can be found in **polynomial time** [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, ...]
Geometry vs. entanglement

\[ S(A) = - \text{tr} \rho_A \log \rho_A \]

**Area law:** \[ S(A) \leq |\partial A| \log D. \]

- Any tensor network satisfies an area law, determined by its geometry.
- In 1D, low-energy states of local gapped quantum systems satisfy *area law* and have accurate MPS representations [Hastings].
- These can be found in *polynomial time* [Landau-Vazirani-Vidick, Arad et al].
- In 2D and higher conjecture, many examples, proof only in special cases [Anshu-Arad-Gosset, ...]
Fundamental theorems
A fundamental question

Given two tensors, when do they generate the *same* tensor network state?

Easy to find such situations! MPS:

\[ \text{\#1} = \text{\#2} \Rightarrow \text{\#3} = \text{\#4} \]

PEPS:

\[ \Rightarrow \]

These *gauge symmetries* preserve the quantum state for any system size! Moreover, can take limits of such symmetries...
A fundamental question

Given two tensors, when do they generate the same tensor network state?

Easy to find such situations! MPS:

PEPS:

These gauge symmetries preserve the quantum state for any system size! Moreover, can take limits of such symmetries...
A fundamental question

Given two tensors, when do they generate the same tensor network state?

Easy to find such situations! MPS:

\[ \Rightarrow \]

PEPS:

These gauge symmetries preserve the quantum state for any system size!
Moreover, can take limits of such symmetries…
A fundamental question

Given two tensors, when do they generate the same tensor network state?

Easy to find such situations! MPS:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a};
\node (B) at (1,0) [shape=cube, fill=yellow!30] {b};
\end{tikzpicture}} = \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=orange!30] {a'};
\node (B) at (1,0) [shape=cube, fill=pink!30] {b'};
\end{tikzpicture} \Rightarrow \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a''};
\node (B) at (1,0) [shape=cube, fill=yellow!30] {b''};
\node (C) at (2,0) [shape=cube, fill=orange!30] {a'''};
\end{tikzpicture} = \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a'''};
\node (B) at (1,0) [shape=cube, fill=yellow!30] {b'''};
\end{tikzpicture}
\end{array}
\]

PEPS:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a};
\end{tikzpicture}} = \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=yellow!30] {a'};
\node (B) at (1,0) [shape=cube, fill=pink!30] {b'};
\end{tikzpicture} \Rightarrow \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a''};
\node (B) at (1,0) [shape=cube, fill=yellow!30] {b''};
\node (C) at (2,0) [shape=cube, fill=orange!30] {a'''};
\end{tikzpicture} = \begin{tikzpicture}
\node (A) at (0,0) [shape=cube, fill=blue!30] {a'''};
\node (B) at (1,0) [shape=cube, fill=yellow!30] {b'''};
\end{tikzpicture}
\end{array}
\]

These gauge symmetries preserve the quantum state for any system size!
Moreover, can take limits of such symmetries...
In 1D, gauge symmetry and taking limits is the only redundancy. We can efficiently pick \textbf{canonical form}. It is unique up to \textit{unitaries} & satisfies:

\textbf{Fundamental Theorem of MPS (Cirac–PG–Schuch, de las Cuevas)}

Two tensors $M$ and $N$ give rise to the same quantum states $|M_n\rangle = |N_n\rangle$ for all system sizes $n$ if and only if they have \textbf{same canonical forms}.

Many applications!

▶ Classification of symmetries and topological phases
▶ Classification of quantum cellular automata
▶ Better-behaved numerics

\textit{No such result known in higher dimensions (before our work)!}
Fundamental theorem for MPS

In 1D, gauge symmetry and taking limits is the only redundancy. We can efficiently pick canonical form. It is unique up to unitaries & satisfies:

Fundamental Theorem of MPS (Cirac–PG–Schuch, de las Cuevas)

Two tensors $M$ and $N$ give rise to the same quantum states $|M_n\rangle = |N_n\rangle$ for all system sizes $n$ if and only if they have same canonical forms.

Many applications!
- Classification of symmetries and topological phases
- Classification of quantum cellular automata
- Better-behaved numerics

No such result known in higher dimensions (before our work)!
Suppose $M$ is tensor with **global (on-site) symmetry** for any system size $n$:

$$u^\otimes n |M_n\rangle = |M_n\rangle$$

Fundamental theorem implies that there is unitary $U$ such that

$$= $$

In this way, classification of SPT phases $\sim$ classification of projective representations. [Chen-Gu-Wen, Schuch-Perez-Garcia-Cirac, Pollman et al].
Illustration: Classification of symmetries

Suppose $M$ is tensor with **global (on-site) symmetry** for any system size $n$:

$$u^\otimes n |M_n\rangle = |M_n\rangle$$

Fundamental theorem implies that there is unitary $U$ such that

$$\begin{array}{c}
\end{array}$$

In this way, classification of SPT phases $\sim$ classification of projective representations.  [Chen-Gu-Wen, Schuch-Perez-Garcia-Cirac, Pollman et al].
Computational complexity

In particular, \( |M_n⟩ = |N_n⟩ \) for all \( n \)? can be decided in polynomial time.

Bad news: For PEPS, \( |T_{m,n}⟩ = |S_{m,n}⟩ \) for all \( m, n \)? is undecidable!

Suggests in 2D and higher, no useful fundamental theorem should exist. However, this is not so – need to change the perspective...
Computational complexity

In particular, “$|M_n⟩ = |N_n⟩$ for all $n$?” can be decided \textbf{in polynomial time}.

\textbf{Bad news:} For PEPS, “$|T_{m,n}⟩ = |S_{m,n}⟩$ for all $m, n$?” is \textbf{undecidable}!

Suggests in 2D and higher, \textit{no} useful fundamental theorem should exist.

However, this is not so – need to change the perspective...
Computational complexity

In particular, “$|M_n\rangle = |N_n\rangle$ for all $n$?” can be decided in polynomial time.

Bad news: For PEPS, “$|T_{m,n}\rangle = |S_{m,n}\rangle$ for all $m, n$?” is undecidable!

Suggests in 2D and higher, no useful fundamental theorem should exist. However, this is not so – need to change the perspective...
Gauge symmetry in higher dimension

When two PEPS tensors are related by gauge symmetry, they determine not only the same state on square grids... 

\[ \Rightarrow \]

Group action on tensors

These gauge symmetries preserve the many-body state for any system size!

For MPS:

and for PEPS:

on a square lattice with periodic boundaries.

Intuition: There are many inequivalent surfaces in 2D!

Summary of results: If one allows for arbitrary graphs, gauge symmetry and taking limits is the only redundancy. Can again find canonical form that satisfies all the same properties as before! Let’s see how this works...
Gauge symmetry in higher dimension

When two PEPS tensors are related by gauge symmetry, they determine not only the same state on square grids... but on any graph:

\[ \begin{array}{c}
\text{Intuition:} \quad \text{There are many inequivalent surfaces in 2D!}
\end{array} \]

Summary of results: If one allows for arbitrary graphs, gauge symmetry and taking limits is the only redundancy. Can again find canonical form that satisfies all the same properties as before! Let’s see how this works...
Gauge symmetry in higher dimension

When two PEPS tensors are related by gauge symmetry, they determine not only the same state on square grids... but on any graph:

\[
\begin{align*}
\text{Intuition: } & \text{ There are many inequivalent surfaces in 2D!} \\
\end{align*}
\]

**Summary of results:** *If one allows for arbitrary graphs, gauge symmetry and taking limits is the only redundancy. Can again find canonical form that satisfies all the same properties as before!* *Let’s see how this works...*
Gauge symmetry in higher dimension

When two PEPS tensors are related by gauge symmetry, they determine not only the same state on square grids... but on any graph:

\[
\begin{align*}
\begin{array}{c}
\text{Intuition: There are many inequivalent surfaces in 2D!}
\end{array}
\end{align*}
\]

**Summary of results:** *If one allows for arbitrary graphs*, gauge symmetry and taking limits is the only redundancy. Can again find **canonical form** that satisfies all the same properties as before! *Let’s see how this works...*
Our proposal: The minimal canonical form

Group action of $G = \text{GL}(D) \times \text{GL}(D)$:

$$S = (g, h) \cdot T$$

Define minimal canonical form of PEPS tensor $T$ by minimizing $\ell^2$-norm:

$$T_{\text{min}} := \text{argmin} \left\{ \|S\|_2 : S \in G \cdot T \right\}.$$  

- Closure is needed so that minimum is attained, but also natural since taking limits is allowed.
- First rigorous canonical form in 2D (+ similarly works in higher dim)!
Our proposal: The minimal canonical form

Group action of $G = \text{GL}(D) \times \text{GL}(D)$:

$$S = (g, h) \cdot T$$

Define **minimal canonical form** of PEPS tensor $T$ by minimizing $\ell^2$-norm:

$$T_{\text{min}} := \arg\min \{ \|S\|_2 : S \in G \cdot T \}.$$  

- Closure is **needed** so that minimum is attained, but also **natural** since taking limits is allowed.
- First rigorous canonical form in 2D (+ similarly works in higher dim)!
What does it characterize?

Recall the group action of $G = \text{GL}(D) \times \text{GL}(D)$:

$S = (g, h) \cdot T$

We say $T$, $T'$ are gauge equivalent if $G \cdot T$ and $G \cdot T'$ intersect. This implies that they give the same many-body states!

Result (Canonical form)

1. The minimal canonical form exists and is unique up to $U(D) \times U(D)$.
2. Two tensors have the same minimal canonical forms if and only if they are gauge equivalent.
What does it characterize?

Recall the group action of $G = \text{GL}(D) \times \text{GL}(D)$:

$$S = (g, h) \cdot T$$

We say $T$, $T'$ are gauge equivalent if $G \cdot T$ and $G \cdot T'$ intersect. This implies that they give the same many-body states!

**Result (Canonical form)**

1. The minimal canonical form exists and is unique up to $U(D) \times U(D)$.
2. Two tensors have the same minimal canonical forms if and only if they are gauge equivalent.
When are two tensors gauge equivalent?

**Fundamental Theorem of PEPS**

Two tensors $T$ and $T'$ give rise to the same tensor network state on any admissible graph $\Gamma$ if and only if they are gauge equivalent, so if and only if they have the same minimal canonical forms.

- In fact, $e^{O(D^2)}$ vertices suffice to distinguish two PEPS tensors, and $e^{\Omega(D)}$ vertices are necessary. For MPS, we note $\tilde{O}(D)$ vertices suffice.
Decidability

In particular, “$|T_\Gamma\rangle = |T'_\Gamma\rangle$ for all graphs $\Gamma$?” is decidable.

**Intuition**: Undecidability for $|T_{n,m}\rangle$ reduces to periodic tiling problem. Its undecidability in turn relies on existence of aperiodic tile sets such as:

There exist $T \neq 0$ such that all $|T_{n,m}\rangle = 0$, but no algorithm can recognize them! [Scarpa et al]

However, if allow arbitrary graphs (topologies) this distinction collapses!
Decidability

In particular, “\(|T \Gamma\rangle = |T' \Gamma\rangle\) for all graphs \(\Gamma\)” is decidable.

**Intuition:** Undecidability for \(|T_{n,m}\rangle\) reduces to periodic tiling problem. Its undecidability in turn relies on existence of aperiodic tile sets such as:

There exist \(T \neq 0\) such that all \(|T_{n,m}\rangle = 0\), but no algorithm can recognize them! [Scarpa et al]

However, if allow arbitrary graphs (topologies) this distinction collapses!
Decidability

In particular, “$|T_\Gamma\rangle = |T'_\Gamma\rangle$ for all graphs $\Gamma$?” is decidable. 😊

Intuition: Undecidability for $|T_{n,m}\rangle$ reduces to periodic tiling problem. Its undecidability in turn relies on existence of aperiodic tile sets such as:

There exist $T \neq 0$ such that all $|T_{n,m}\rangle = 0$, but no algorithm can recognize them! [Scarpa et al]

However, if allow arbitrary graphs (topologies) this distinction collapses!
How to characterize the canonical form?

We show that a tensor $T$ is in minimal canonical form if and only if

$$
\rho = |T\rangle \langle T|
$$

satisfies

$$
\rho_{\text{right}} = \overline{\rho_{\text{left}}} \quad \text{and} \quad \rho_{\text{bottom}} = \overline{\rho_{\text{top}}}.
$$

This means the state $\rho = |T\rangle \langle T|$ satisfies $\rho_{\text{right}} = \overline{\rho_{\text{left}}}$ & $\rho_{\text{bottom}} = \overline{\rho_{\text{top}}}$. Physical interpretation?
Why does it work? Geometric invariant theory (GIT)

A field of mathematics that studies equivalence for “nice” actions of group $G$ on vector space $V$. In our case:

$$G = \text{GL}(D) \text{ and } V = \text{MPS tensors of fixed format,}$$

$$G = \text{GL}(D) \times \text{GL}(D) \text{ and } V = \text{PEPS tensors of fixed format.}$$

Notion of equivalence: $G \cdot v$ and $G \cdot v'$ intersect. GIT tells us:

- Minimum norm vectors unique up to unitaries. [Kempf-Ness]
- Equivalent iff $P(v) = P(v')$ for all $G$-invariant polynomials $P$. [Mumford]

What are the invariant polynomials in our case? Quantum states!

$$P(M) = \langle i_1, \ldots, i_n | M_n \rangle \text{ for } n = \tilde{O}(D).$$

New perspective gives stronger results!

$P(T) = \langle i_1, \ldots, i_n | T_{\Gamma} \rangle$ for arbitrary graphs $\Gamma$
Why does it work? Geometric invariant theory (GIT)

A field of mathematics that studies equivalence for “nice” actions of group $G$ on vector space $V$. In our case:

$G = \text{GL}(D)$ and $V = \text{MPS} \text{ tensors of fixed format}$,

$G = \text{GL}(D) \times \text{GL}(D)$ and $V = \text{PEPS} \text{ tensors of fixed format}$.

**Notion of equivalence**: $G \cdot v$ and $G \cdot v'$ intersect. 😊 GIT tells us:

- Minimum norm vectors unique up to unitaries. [Kempf-Ness]
- Equivalent iff $P(v) = P(v')$ for all $G$-invariant polynomials $P$. [Mumford]

What are the invariant polynomials in our case? Quantum states!

$P(M) = \langle i_1, \ldots, i_n|M_n \rangle$ for $n = \tilde{O}(D)$.

New perspective gives stronger results! [Procesi-Razmyslov-Formanek, Derksen-Makam]

$P(T) = \langle i_1, \ldots, i_n|T_\Gamma \rangle$ for arbitrary graphs $\Gamma$. 

21 / 23
Why does it work? Geometric invariant theory (GIT)

A field of mathematics that studies equivalence for “nice” actions of group $G$ on vector space $V$. In our case:

$$G = \text{GL}(D) \quad \text{and} \quad V = \text{MPS tensors of fixed format},
$$

$$G = \text{GL}(D) \times \text{GL}(D) \quad \text{and} \quad V = \text{PEPS tensors of fixed format}.$$

**Notion of equivalence:** $G \cdot v$ and $G \cdot v'$ intersect. 😊 GIT tells us:

- **Minimum norm** vectors unique up to unitaries. [Kempf-Ness]
- Equivalent iff $P(v) = P(v')$ for all $G$-invariant polynomials $P$. [Mumford]

What are the invariant polynomials in our case? Quantum states!

$$P(M) = \langle i_1, \ldots, i_n | M_n \rangle \quad \text{for} \quad n = \tilde{O}(D).$$

New perspective gives stronger results!

[P Procesi-Razmyslov-Formanek, Derksen-Makam]

$$P(T) = \langle i_1, \ldots, i_n | T_\Gamma \rangle \quad \text{for arbitrary graphs } \Gamma.$$
Why does it work? Geometric invariant theory (GIT)

A field of mathematics that studies equivalence for “nice” actions of group $G$ on vector space $V$. In our case:

\[
G = \text{GL}(D) \quad \text{and} \quad V = \text{MPS tensors of fixed format},
\]
\[
G = \text{GL}(D) \times \text{GL}(D) \quad \text{and} \quad V = \text{PEPS tensors of fixed format}.
\]

Notion of equivalence: $G \cdot v$ and $G \cdot v'$ intersect. GIT tells us:

- Minimum norm vectors unique up to unitaries. [Kempf-Ness]
- Equivalent iff $P(v) = P(v')$ for all $G$-invariant polynomials $P$. [Mumford]

What are the invariant polynomials in our case? Quantum states!

\[
P(M) = \langle i_1, \ldots, i_n | M_n \rangle \text{ for } n = \tilde{O}(D).
\]

New perspective gives stronger results!

[Procesi-Razmyslov-Formanek, Derksen-Makam]

\[
P(T) = \langle i_1, \ldots, i_n | T_\Gamma \rangle \text{ for arbitrary graphs } \Gamma.
\]
Algorithms for the minimal canonical form

Given a tensor $T$, how to compute $T_{\text{min}}$? Nontrivial, even for MPS!

Result (Algorithms)

For fixed bond dimension $D$, can approximate $T_{\text{min}}$ in polynomial time.

Combines computer science ideas from the solution of Paulsen’s problem with recent results on optimization on groups (operator, tensor scaling).

Key idea: $g \mapsto \|g \cdot v\|$ is convex along geodesics of curved space arising from noncommutative group. Accordingly, local algorithms can find canonical form. Versatile framework, many applications. [Bürgisser-…-W-Wigderson]
Algorithms for the minimal canonical form

Given a tensor $T$, how to compute $T_{\text{min}}$? Nontrivial, even for MPS!

Result (Algorithms)

For fixed bond dimension $D$, can approximate $T_{\text{min}}$ in polynomial time.

Combines computer science ideas from the solution of Paulsen’s problem
with recent results on optimization on groups (operator, tensor scaling).

Key idea: $g \mapsto \|g \cdot v\|$ is convex along geodesics of curved space arising
from noncommutative group. Accordingly, local algorithms can find
canonical form. Versatile framework, many applications. [Bürgisser-…-W-Wigderson]
Given a tensor $T$, how to compute $T_{\text{min}}$? Nontrivial, even for MPS!

**Result (Algorithms)**

For fixed bond dimension $D$, can approximate $T_{\text{min}}$ in polynomial time.

Combines computer science ideas from the solution of Paulsen’s problem with recent results on optimization on groups (operator, tensor scaling).

**Key idea:** $g \mapsto \|g \cdot v\|$ is convex along geodesics of curved space arising from noncommutative group. Accordingly, local algorithms can find canonical form. Versatile framework, many applications. [Bürgisser-...-W-Wigderson]
Summary and outlook

Tensor networks describe high-dimensional data succinctly. Applications from physics to numerics to computer science.

Fundamental theorems and canonical forms are key tools. We propose first rigorous general such tools beyond 1D.

To achieve this, we connect tensor networks to powerful tools from geometric invariant theory and recent progress on optimization algorithms in theoretical computer science.