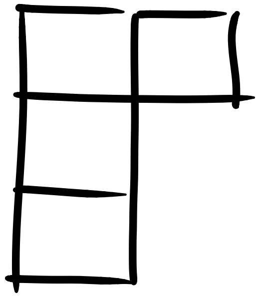
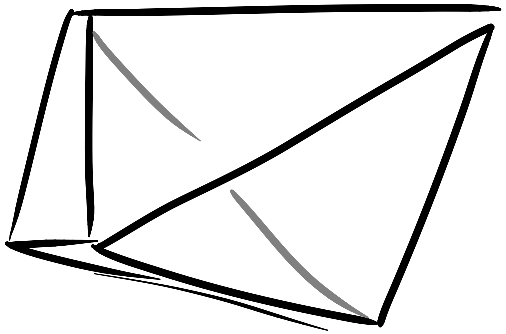
Simons Reunion Workshop, Berkeley (December 2015)





On the computational complexity of the membership problem for moment polytopes

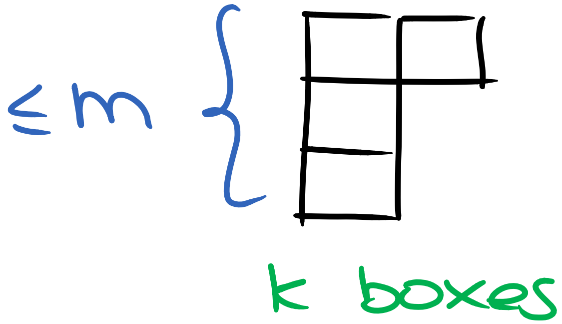
Michael Walter, Stanford University

joint work with P. Bürgisser, M. Christandl, K. Mulmuley; M. Vergne

Notation

Young diagram λ:

- row lengths $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$
- partition of k into $\leq m$ parts



They parametrize the irreducible representations of:

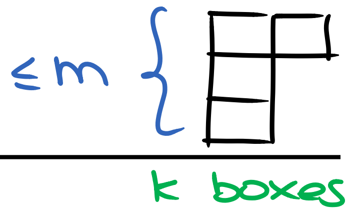
Symmetric group S_k

Specht module $[\lambda]$

General linear group $GL(m)$

Weyl module V_λ^m

Kronecker coefficients



$$[\lambda] \otimes [\mu] = \bigoplus_{\nu} g_{\lambda\mu\nu} [\nu]$$

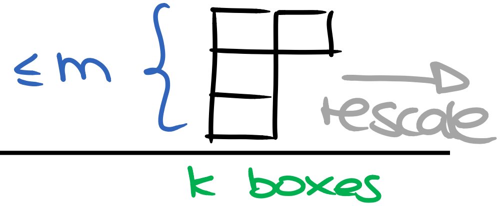
Many interesting connection to other areas of mathematics as well as applications (quantum physics, geometric complexity theory), in part via

$$\mathrm{Sym}^k(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \bigoplus_{\lambda\mu\nu} g_{\lambda\mu\nu} V_\lambda^m \otimes V_\mu^m \otimes V_\nu^m$$

Despite 75+ years of history, many properties remain mysterious!

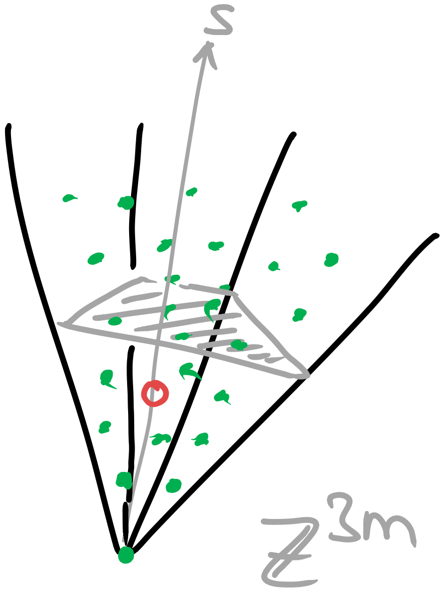
Littlewood-Richardson coefficients are special Kronecker coefficients [Murnaghan].

Kronecker coefficients: asymptotics



$$G(m) = \{(\lambda,\mu,\nu) : g_{\lambda\mu\nu} > 0\}$$

Asymptotic support is convex cone: [Mumford], [Kirwan]

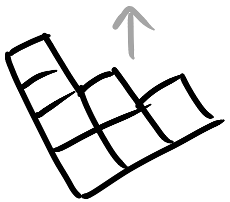
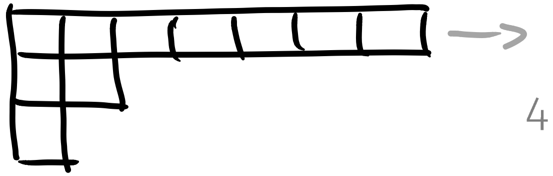


- outside: $g \equiv 0$ inside: $\exists s : g_{s\lambda, s\mu, s\nu} > 0$

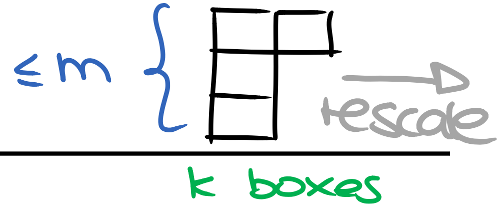
in general, $s > 1$: failure of saturation, "holes"!

- piecewise quasi-polynomiality

[Guillemin-Sternberg], [Meinrenken-Sjamaar]

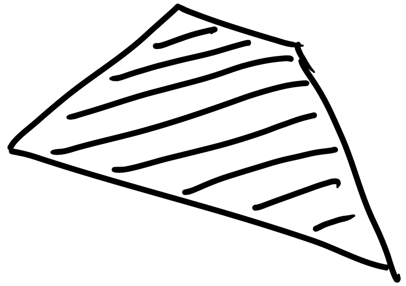
Various other asymptotics have been studied:  , 

The Kronecker polytopes



$$\Delta(m) = \left\{ \frac{(\lambda,\mu,\nu)}{k} : g_{\lambda\mu\nu} > 0 \right\}$$

...is a convex polytope: the Kronecker polytope.



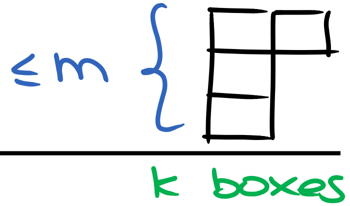
More generally: Define moment polytope

$$\Delta_G(M) = \left\{ \frac{\lambda}{k} : V_\lambda \subseteq \mathrm{Sym}^k(M) \right\}$$

where G compact connected Lie group, M unitary representation.

E.g., Littlewood-Richardson coefficients give rise to Horn polytopes.

This talk



$$\Delta(m) = \left\{ \frac{(\lambda,\mu,\nu)}{k} : g_{\lambda\mu\nu} > 0 \right\}$$

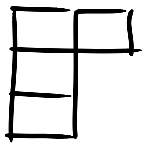
1. Effective “combinatorial” description of moment polytopes
   [Vergne-W., 2014]

2. Computational complexity: NP ∩ coNP.
   [Bürgisser-Christandl-Mulmuley-W., 2015]

Motivation: Interest in computing moment polytopes in practice, quantum marginal problem; understand hardness vs. failure of saturation (cf. [BIH])

1. Inequalities for moment polytopes

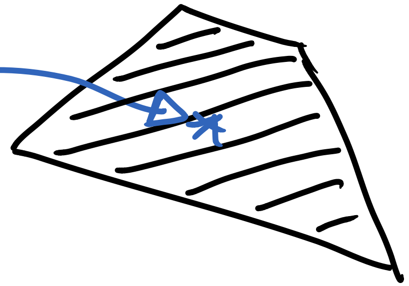
Geometric description



$$\Delta(m) = \left\{ \frac{(\lambda,\mu,\nu)}{k} : g_{\lambda\mu\nu} > 0 \right\}$$

can also be described in geometric terms via "moment map"

$\mathbb{P}((\mathbb{C}^m)^{\otimes 3}) \ni \psi$



$(\operatorname{spec} \rho_A, \ldots, \operatorname{spec} \rho_C)$

where $\operatorname{tr} \rho_A X_A = \langle \psi | X_A \otimes I \otimes I | \psi \rangle$ etc. ("reduced density matrices")

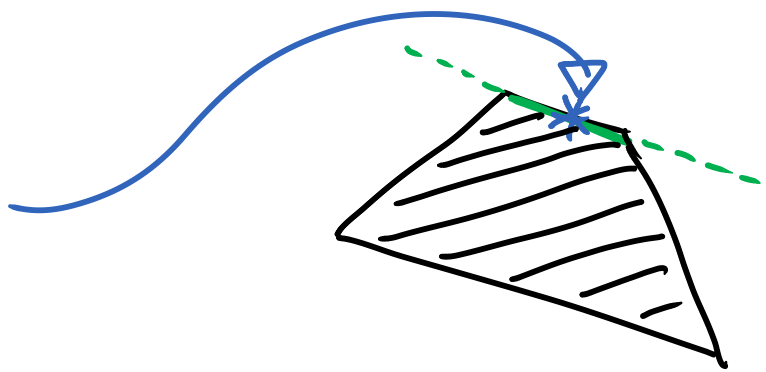
Proof via basic GIT. $\rightsquigarrow$ Quantum marginal problem, q. version of discrete tomography!
Similar: Horn polytopes vs. spectra of Hermitian matrices with A + B + C = 0.

Basic idea

$$H \cdot r = (H_A, H_B, H_C) \cdot (r_A, r_B, r_C) \overset{!}{\geq} z$$

We derive necessary conditions on H and z to be a valid inequality by studying the moment map in first and second order (geometric picture):

$$\mathbb{P}((\mathbb{C}^m)^{\otimes 3}) \ni \psi$$



Result can be expressed purely in terms of representation-theoretic data!

cf. [Berenstein-Sjamaar], [Klyachko], [Ressayre].

Roots and weights

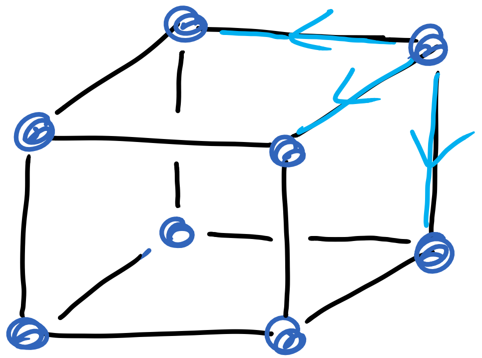
$G = SU(m)^3 \curvearrowright M = (\mathbb{C}^m)^{\otimes 3}$

Negative roots: $N = \{ (e_i - e_j, 0, 0) : i > j \} \cup \dots$

Weights: $\Omega = \{ (e_i, e_j, e_k) \}$

$e_i \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

E.g., m=2:



negative roots of $SU(2)^3$

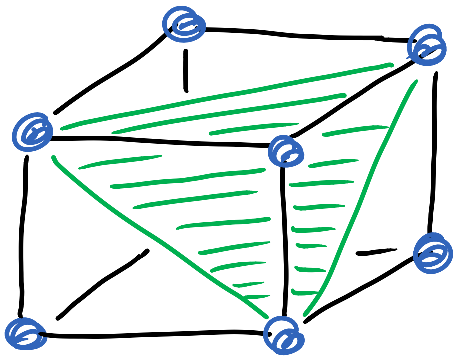
weights of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

Inequalities for the Kronecker polytopes

[Vergne-W., 2014]

Theorem: The Kronecker polytope $\Delta(m)$ is cut out by inequalities $H \cdot r \geq z$ satisfying the following conditions ("Ressayre elements"):

- $\Omega(H=z) = \{\varphi \in \Omega : \quad H \cdot \varphi = z\}$ span hyperplane



- $N(H<0) = \{\alpha \in N : \quad H \cdot \alpha < 0\}$

  $\Omega(H<z) = \{\omega \in \Omega : \quad H \cdot \omega < z\}$

  have same cardinality, and

$$\det \Big[\sum_{\varphi \in \Omega(H=z)} X_\varphi \, \delta_{\alpha+\varphi=\omega} \Big]_{\omega,\alpha} \neq 0$$

Sufficiency: Characterization of $\Delta(m)$ via highest weight polynomials in the coordinate ring.

Numerical results

[Vergne-W., 2014]

Conditions are <u>concrete</u> → <u>effective</u> in low dimensions (can go beyond what had been computed before).

#	H_A	H_B	H_C	z
1	$(-5,-1,3,3)$	$(-5,3,3,-1)$	$(5,1,-3,-3)$	5
2	$(-5,-1,3,3)$	$(1,-3,-3,5)$	$(3,3,-1,-5)$	5
3	$(-5,3,-1,3)$	$(-5,3,-1,3)$	$(5,1,-3,-3)$	5

...... etc. ......

However: Number of inequalities grows rapidly!

(a,b,c)	$(2,2,2)$ [$\boxed{43}$]	$(3,3,3)$ [$\boxed{45}$]	$(4,4,4)$
Inequalities	9 (3)	114 (25)	1749 (323)
Facets	6 (2)	45 (10)	270 (50)
Extreme Rays	5 (3)	33 (11)	328 (65)

Aside: Determinant & Horn condition

[Vergne-W., 2014]

Recall the last condition:

$$\det\left[\sum_{\varphi\in\Omega(H=z)} X_\varphi\, \delta_{\alpha+\varphi=\omega}\right]_{\omega,\alpha} \neq 0$$

This ”determinant polynomial” is highest weight vector (w.r.t. subgroup).

Thus determines point in (lower-dimensional) moment polytope:

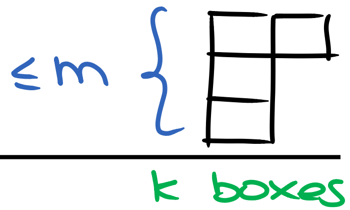
$$\kappa_{H,z} \in \Delta_{G'}(M')$$

necessary condition!

In the case of the Horn polytopes, this reduces precisely to the recursive definition of Horn’s inequalities (i.e., also sufficient).

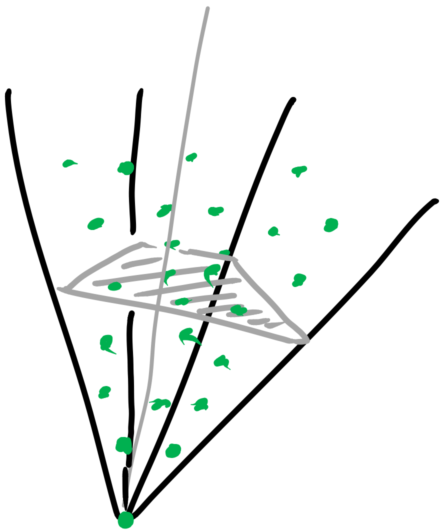
2. Computational complexity

Kronecker polytopes as a decision problem



KronPolytopes: Given three Young diagrams as input, decide if

$$\frac{(\lambda,\mu,\nu)}{k} \in \Delta(m) \ ?$$



Equivalently, decide if $\exists s: \ g_{s\lambda,s\mu,s\nu} > 0$?

Importantly, the height m is not bounded.

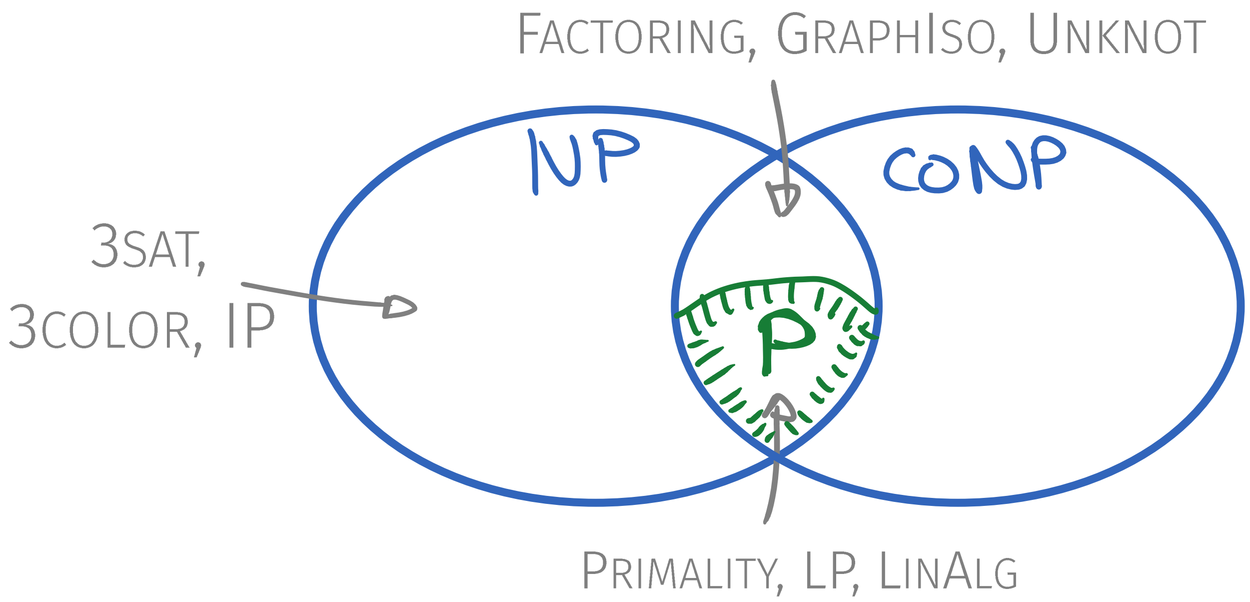
Challenges: No useful bounds on stretching factor *s*. Quadratically constrained program (NP-hard in general). Large # of Ressayre elements.

The complexity classes P, NP, and coNP

P: There exists an efficient algorithm.

NP: If answer YES then there exists small certificate that can be efficiently verified.

CoNP: If answer NO then there exists small certificate that can be efficiently verified.

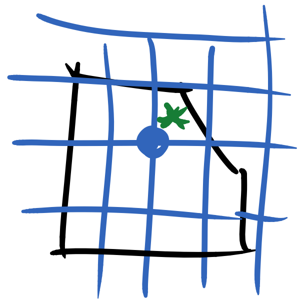


"efficient" = polynomial time; "small" = polynomial bitsize (in the bitsize of the input)

Complexity of Kronecker polytopes [Bürgisser-Christandl-Mulmuley-W., 2015]

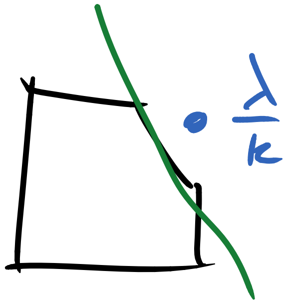
Theorem: The problem KRONPOLYTOPES is in NP and coNP.

NP: Certificate is vector in $(\mathbb{C}^m)^{\otimes 3}$



Point in polytope can be computed efficiently.
We prove that finite precision is not an issue (walls of polytope are not too steep).

CoNP: Certificate is Ressayre element (H, z) for separating hyperplane.



Ressayre condition can be checked efficiently (if also given point at which to evaluate determinant polynomial).

The bigger picture: classical & quantum complexity

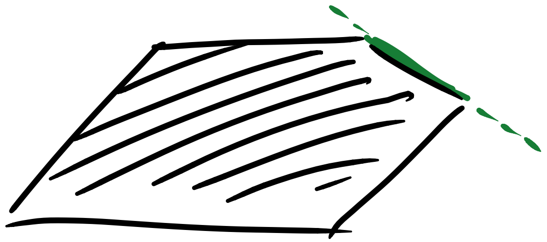
	Kronecker coefficients		Littlewood-R.	
COUNTING $g(\lambda,\mu,\nu) = ?$	#P-hard, GapP #BQP	[Bürgisser-Ikenmeyer, Narayanan] [Harrow-Christandl-W.]	#P-complete	
POSITIVITY $g(\lambda,\mu,\nu) > 0$	NP-hard QMA	[Christian's talk] [Harrow-Christandl-W.]	P	[Knutsen-Tao], [Blasiak-Mulmuley-Sohoni]
MOMENTPOLYTOPES $\exists l: g(l\lambda, l\mu, l\nu) > 0$	NP $\cap$ coNP	[this talk]	P	

NB: For bounded height (fixed *m*), all these problems are in P [Christandl-Doran-W.].

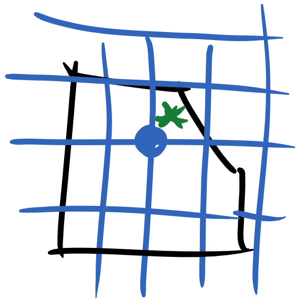
Summary

Moment polytopes describe the asymptotic support of representation-theoretic multiplicities. They have been studied in many different contexts (including GCT, quantum physics, ...).

1. Effective "combinatorial" description
   [Vergne-W., 2014]



2. Computational complexity: NP ∩ coNP.
   [Bürgisser-Christandl-Mulmuley-W., 2015]



Results generalize to unitary representations of compact connected Lie groups.

Is the membership problem in fact in P? Positivity in NP? The coefficient in #P? Quantum gap?

Thank you for your attention