Free fermion entanglement renormalization from wavelets

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joint work with Haegeman, Swingle, Cotler, Evenbly, Scholz (arXiv:1707.06243)
& with Scholz, Swingle, Witteveen (forthcoming)

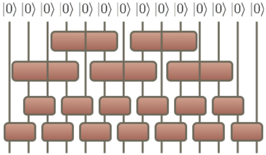
Tensor network states

Efficient variational classes for many-body quantum systems:

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle$$

e.g.

matrix product states



MERA

- can have interpretation as quantum circuit

Useful theoretical formalism:

- geometrize entanglement structure: *generalized area law*
- bulk-boundary dualities: *lift physics to the virtual level*
- quantum phases, topological order, RG, holography, ...

Tensor networks and quantum field theories

- tensor networks are discrete and finite representations
- quantum field theories are infinite and defined in the continuum

Two successful approaches:

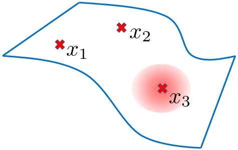
- *Lattice:* MPS, PEPS, MERA
- *Continuum:* cMPS, cMERA

How to measure goodness of approximation?
What do tensor networks really capture?
Why do tensor networks work so well?

Tensor networks for correlation functions

Given many-body system in state ρ and choice of operators $\{O_\alpha\}$, define correlation function:

$$C(\alpha_1, \cdots, \alpha_n) = \mathrm{tr}[\rho O_{\alpha_1} \cdots O_{\alpha_n}]$$



Goal: Design tensor network for correlation functions!

- unified perspective: system can be continuous, discreteness imposed by how we probe it
- tensor network for state sufficient — but not optimal
- in lattice models can recover state, but *only* for complete set of $\{O_\alpha\}$

Examples: Zaletel-Mong (MPS/q. Hall states), König-Scholz (MPS/CFTs), cf. quantum marginal problem

Our results

We construct tensor networks for free fermion systems:

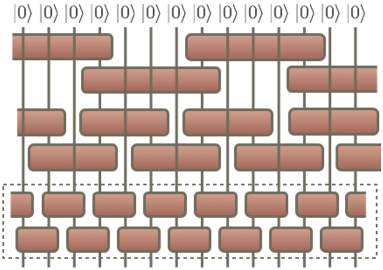
- 1D Dirac fermions on lattice & continuum
- non-relativistic 2D fermions on lattice (Fermi surface)

Key features:

- tensor networks that target correlation functions
- rigorous approximation guarantees
- quantum circuits that 'renormalize entanglement': (branching) MERA
- explicit circuit construction, no variational optimization required

We achieve this using insight from signal processing: wavelet theory.

MERA: multi-scale entanglement renormalization ansatz (Vidal)



↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

↑ entanglement renormalization

↕ organize q. information by scale

- layers are short-depth quantum circuits (disentangle & coarse-grain)
- variational class for critical systems in 1D
- any MERA can be extended to a 'holographic' mapping
- reminiscent of holography (Swingle), starting point for tensor network models (HaPPY; Hayden-…-W.)

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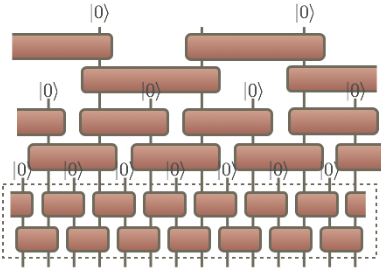

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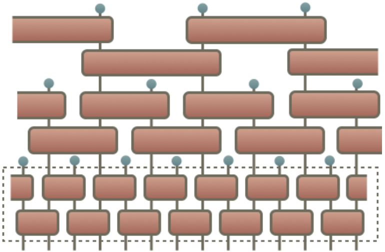

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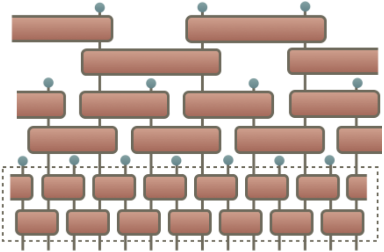

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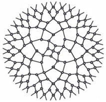
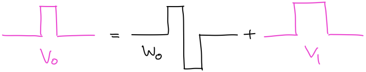


Wavelet transforms

Wavelet transforms locally resolve classical signal into different scales:



'multi-resolution analysis'

Formally: $L^2(\mathbb{R}) \supseteq \cdots \supseteq V_{j-1} \supseteq V_j \supseteq V_{j+1} \supseteq \dots$ $\quad V_j = W_j \oplus V_{j+1}$

- V_j = signal at scale up to j, $\quad W_j$ = signal at scale j
- spanned by bases $\phi(2^{-j}x - n)$, $\quad \psi(2^{-j}x - n)$

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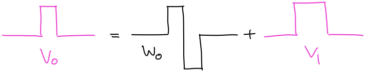

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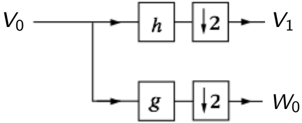
Discrete wavelet transform: corresponding basis transformation



- defined by convolution with low-pass filter h and high-pass filter g

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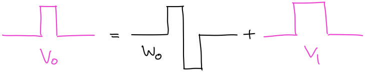

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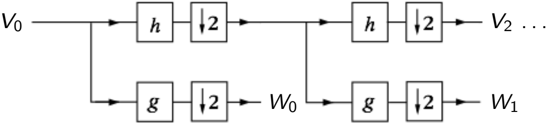
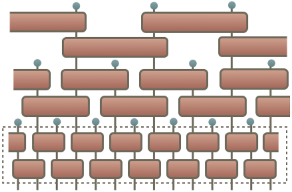


- self-similar, resolves discrete input signal in $\ell^2(\mathbb{Z})$ into different scales

MERA and wavelets

Key insight: Second quantizing 1D wavelet transform $\rightsquigarrow$ MERA circuit!



- in fact, obtain 'holographic' mapping (Qi)
- length of classical filter $\widehat{=}$ depth of quantum circuit (Evenbly-White)

Ansatz: To construct free-fermion ground states, design wavelet transforms that target positive/negative energy modes.

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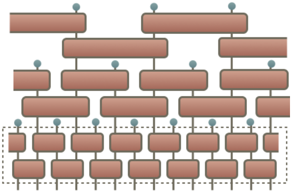

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1D Dirac fermions – Lattice model and result

Massless ‘staggered’ Dirac fermions on 1D lattice (Kogut-Susskind):

$$H_{1\mathrm{D}} = -\sum_n b^\dagger_{1,n} b_{2,n} - b^\dagger_{2,n} b_{1,n+1} + b^\dagger_{2,n} b_{1,n} - b^\dagger_{1,n+1} b_{2,n}$$

- equivalent to nearest-neighbor hopping Hamiltonian
- easily solved using Fourier transform – but *not* using local q. circuit!

We construct MERA networks that target correlation functions:

$$C(\{f_i\}) := \langle b^\dagger_{j_1}(f_1) \cdots b_{j_{2N}}(f_{2N}) \rangle$$

Result (simplified)

$$C(\{f_i\})_{\mathrm{exact}} \approx C(\{f_i\})_{\mathrm{MERA}}$$

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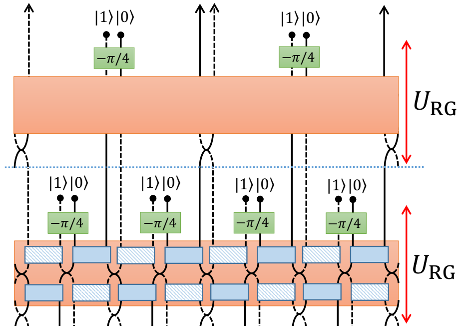

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1D Dirac fermions – Ground state

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$$= \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \begin{bmatrix} b_1(k) \\ b_2(k) \end{bmatrix}^\dagger \begin{bmatrix} 0 & \mathrm{e}^{-\mathrm{i}k} - 1 \\ \mathrm{e}^{\mathrm{i}k} - 1 & 0 \end{bmatrix} \begin{bmatrix} b_1(k) \\ b_2(k) \end{bmatrix}.$$

Diagonalize:

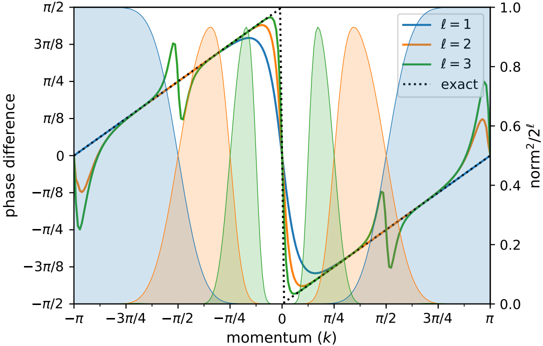
$$u(k) = \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{i}\,\mathrm{sign}(k)\mathrm{e}^{\mathrm{i}k/2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad u^\dagger h u = \begin{bmatrix} E_-(k) & 0 \\ 0 & E_+(k) \end{bmatrix}$$

- Fourier trafo highly *nonlocal*. But can choose *any* basis of Fermi sea!
- want pairs of modes related by $-\mathrm{i}\,\mathrm{sign}(k)\mathrm{e}^{\mathrm{i}k/2}$.

1D Dirac fermions – Wavelets

Task: Engineer pair of wavelet transforms such that high-pass filters (and hence modes) are related by $-\mathrm{i}\,\mathrm{sign}(k)\mathrm{e}^{\mathrm{i}k/2}$.

- studied in signal processing, motivated by *translation-invariance* (!)
- impossible with finite filters, but possible to arbitrary accuracy (Selesnick)



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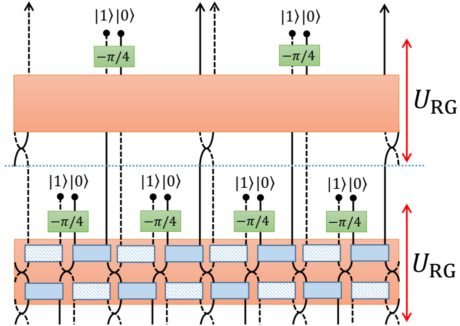


Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of phase relation of filters
- W – 'size' of filters/ 'depth' per layer

Consider correlation function of N creation and annihilation operators

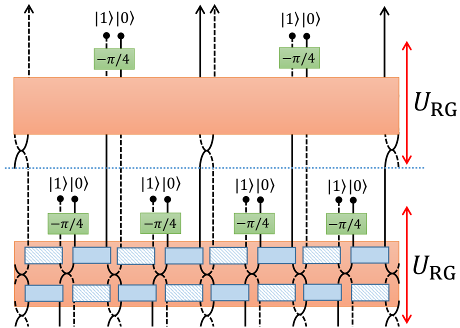
$$C(\{f_i\}) := \left\langle b^\dagger_{j_1}(f_1) \cdots b^\dagger_{j_N}(f_N)\, b_{j_{N+1}}(f_{N+1}) \cdots b_{j_{2N}}(f_{2N}) \right\rangle$$

supported on S lattice sites.

Result (simplified)

$$\left| C(\{f_i\})_{\mathrm{exact}} - C(\{f_i\})_{\mathrm{MERA}} \right| \lesssim \sqrt{SN}\, W \max\{2^{-\mathcal{L}/4}, \varepsilon\}$$

1D Dirac fermions – MERA



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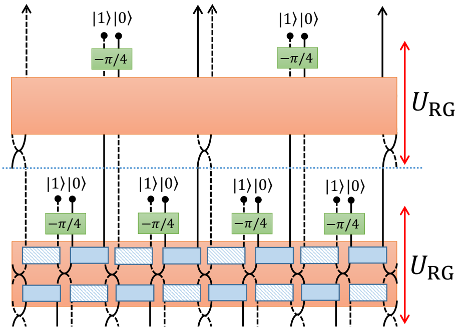

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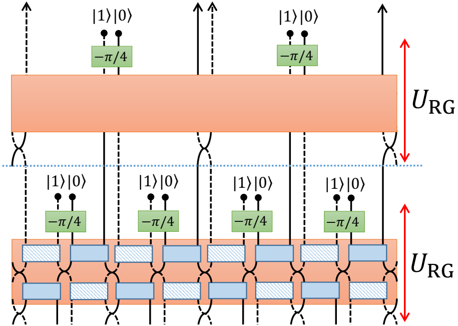

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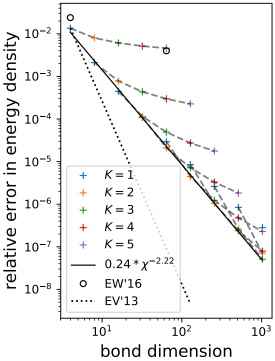
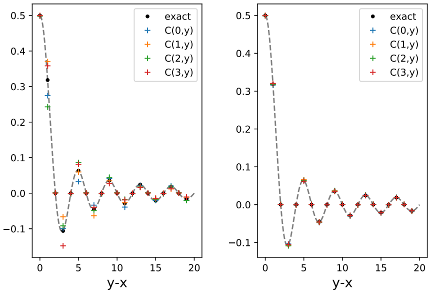
1D Dirac fermions – Numerics

Energy error



Green function $C(x, y) = \langle a_x^\dagger a_y \rangle$



bond dimension 2^2 bond dimension 2^6

1D Dirac fermions – Continuum

Massless Dirac fermions in (1+1)d:

$$\begin{bmatrix} \mathrm{i}(\partial_t + \partial_x) & 0 \\ 0 & \mathrm{i}(\partial_t - \partial_x) \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0$$

- need to produce modes $\psi_\pm(x)$ supported in $k < 0$ / $k > 0$

Natural construction: 'Continuum limit' of inverse wavelet transform!

- for pair of transforms as before: outputs ψ_1, ψ_2 (wavelet functions) related by $\mathrm{i}\,\mathrm{sign}(k) \rightsquigarrow \psi_\pm = \psi_1 \pm \mathrm{i}\psi_2$

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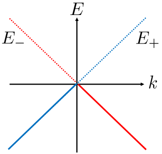

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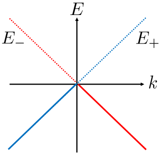

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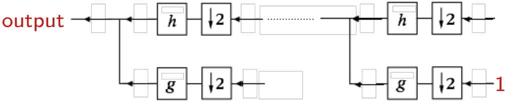

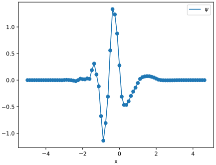

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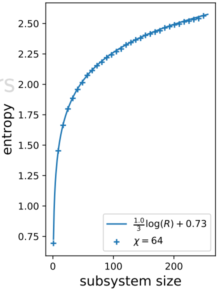
1D Dirac fermions – Verifying conformal data

- Central charge: $S(R) = \frac{c}{3} \log R + c'$
- Scaling dimensions: usual procedure: find operators that coarse-grain to themselves (Evenbly-Vidal)



$\rightsquigarrow$ diagonalize 'scaling superoperator' (eigenvalues $2^{-\Delta_\alpha}$)

Here can also simply verify scaling of two-point functions.

Similarly: OPE coefficients, q. error correction properties (Kim-Kastoryano)

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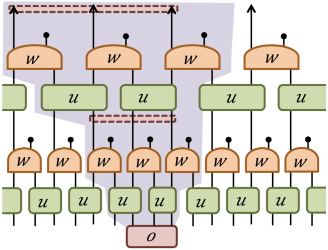

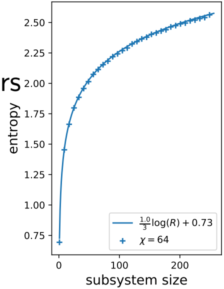

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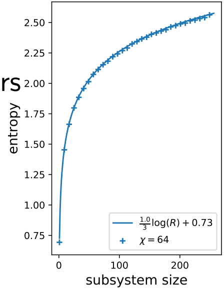

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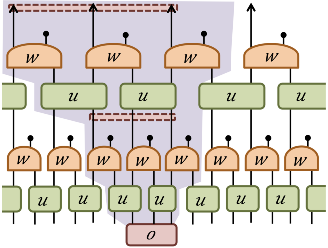

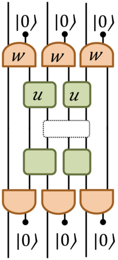

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Non-relativistic 2D fermions – Lattice model

$$H_{1\mathrm{D}} \cong -\sum_n a_n^\dagger a_{n+1} + h.c.$$

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{2\mathrm{D}} = -\sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c.$$

Fermi surface:

- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
- Green's function factorizes w.r.t. rotated axes

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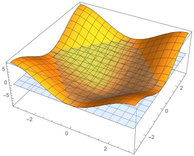
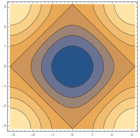
$$H_{1\text{D}} \cong -\sum_n a_n^\dagger a_{n+1} + h.c.$$

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{2\text{D}} = -\sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c.$$

Fermi surface:





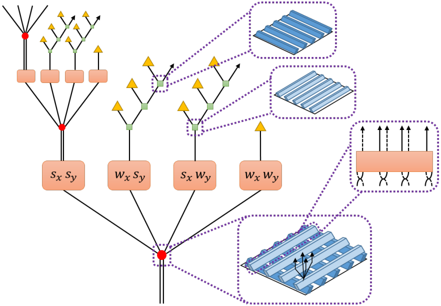
- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
- Green's function factorizes w.r.t. rotated axes

Non-relativistic 2D fermions – Branching MERA

Natural construction: Tensor product of wavelet transforms!

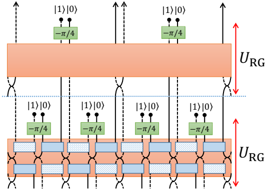
$$W\psi = \psi_s \oplus \psi_w \quad \rightsquigarrow \quad (W \otimes W)\psi = \psi_{ss} \oplus \psi_{ws} \oplus \psi_{sw} \oplus \psi_{ww}$$

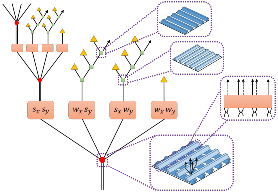
After second quantization, obtain variant of branching MERA (Evenbly-Vidal):



Similar approximation theorem holds.

Summary and outlook





Entanglement renormalization for free fermions (lattice and continuum):

- Rigorous approximation of correlation functions
- Explicit quantum circuits from wavelet transforms
- Similarly: TFD state

Outlook:

- Massive theories, Dirac cones, . . .
- WZW CFTs (w/ Scholz, Swingle, Witteveen); building block?
- Interacting theories? Starting point for variational optimization?

Thank you for your attention!