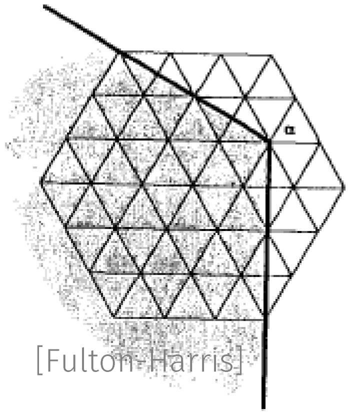
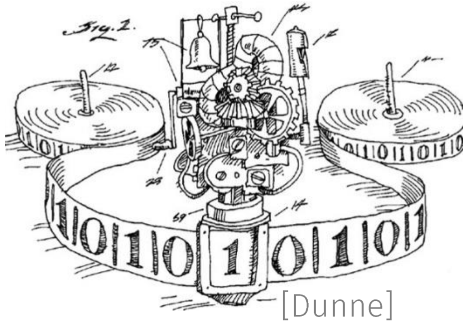


[Fulton-Harris]



[Dunne]

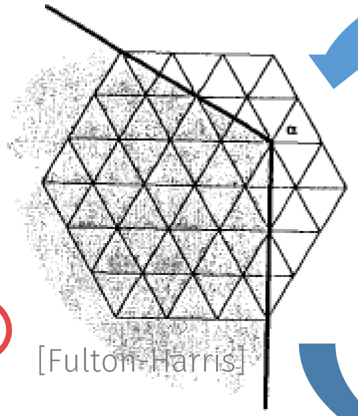
Kronecker coefficients and complexity theory

Michael Walter, Stanford University

University of Rome Tor Vergata, March 2016

conceptual framework:
language and results

4



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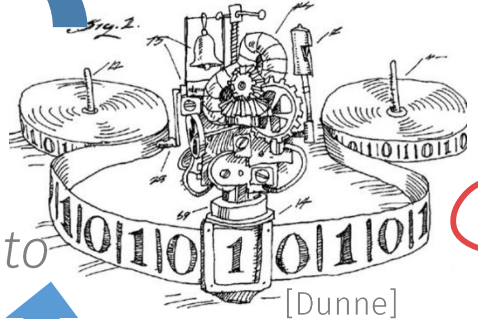
[Fulton-Harris]

geometric approach to
P≠NP conjecture



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[Dunne]

2

quantum marginal problem

Kronecker coefficients and complexity theory

Michael Walter, Stanford University

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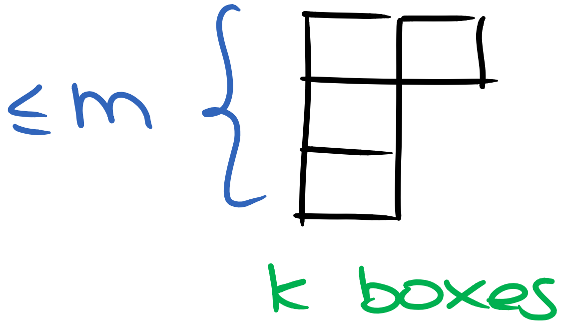
Young diagrams

Young diagram λ :

- row lengths $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$
- partition of k into $\leq m$ parts



They parametrize the irreducible representations of:

Symmetric group S_k :

Specht module $[\lambda]$

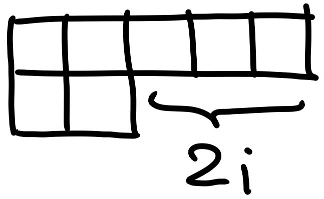
General linear group $GL(m)$:

Weyl module V_λ^m

Well-known decompositions

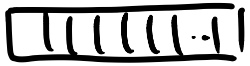
Clebsch-Gordan rule for SU(2):

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{i+j} V_k$$



Schur-Weyl duality:

$$(\mathbb{C}^m)^{\otimes k} = \bigoplus_{\lambda} V_\lambda^m \otimes [\lambda]$$

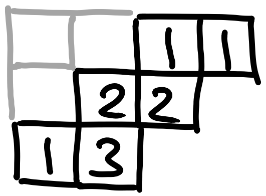
e.g.,  *is the symmetric subspace*

Littlewood-Richardson coefficients

$$V^m_\lambda \otimes V^m_\mu = \bigoplus_\nu c^{\lambda\mu}_\nu V^m_\nu$$

Littlewood-Richardson rule:

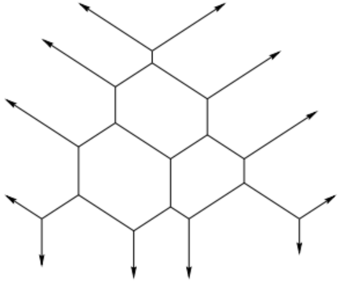
$c^{\lambda\mu}_\nu$ = # of *LR tableaux* of shape ν/λ with *weight* μ

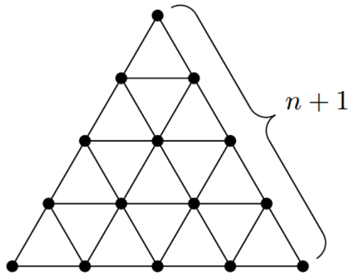


Honeycomb and hive models: [Knutson-Tao]

$c^{\lambda\mu}_\nu$ = # of *honeycombs* with *boundary conditions*

= # *of integral hives with boundary conditions*





Both formulas count combinatorial gadgets – they are evidently positive!

Moreover, we can efficiently determine if nonzero. [Mulmuley-Sohoni]

Littlewood-Richardson coefficients

$$V_\lambda^m \otimes V_\mu^m = \bigoplus_\nu c_\nu^{\lambda\mu} V_\nu^m$$

Saturation property: [Knutson-Tao]

$$c_{s\nu}^{s\lambda, s\mu} > 0 \implies c_\nu^{\lambda\mu} > 0$$

Symplectic geometry: directly related to eigenvalues of Hermitian matrices with

$$A + B = C$$

→ Horn's inequalities

Kronecker coefficients

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu} g_{\lambda\mu\nu} [\nu]$$

Many interesting connection to other areas of mathematics & applications (→later).

In part via:

$$\mathrm{Sym}^k(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \bigoplus_{\lambda\mu\nu} g_{\lambda\mu\nu} V_\lambda^m \otimes V_\mu^m \otimes V_\nu^m$$

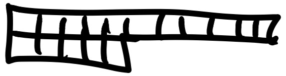
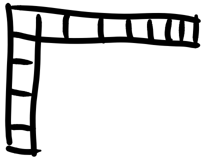
Despite 75+ years of history, many properties remain poorly understood!

Littlewood-Richardson coefficients are special Kronecker coefficients.

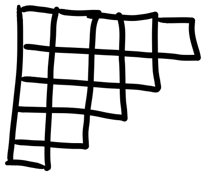
Kronecker coefficients: formulas

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu} g_{\lambda\mu\nu} [\nu]$$

Explicit formulas in various special cases:

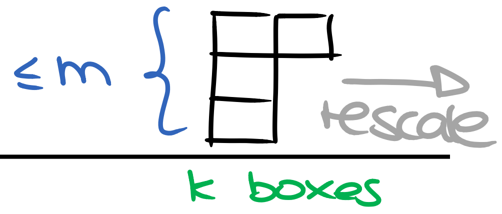
- Two rows  [Orellana et al], [Blasiak-Mulmuley-Sohoni]
- Hooks  [Remmel], [Blasiak]

Recent progress on the Saxl conjecture: [Ikenmeyer], [Pak-Panova-Vallejo]

$g_{\delta\delta\nu} \overset{?}{>} 0$ whenever $\delta =$ 

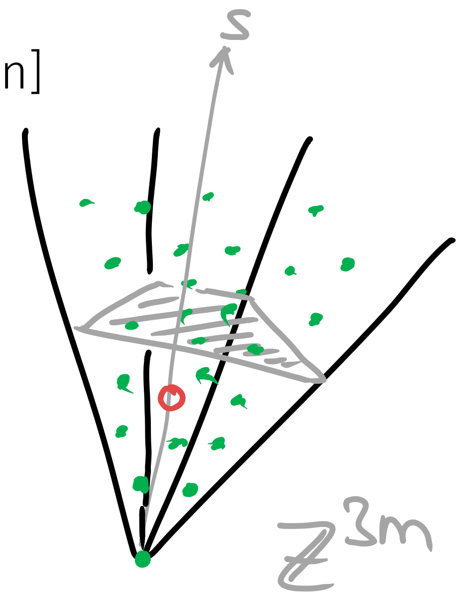
Open problem: Find combinatorial interpretation!

Kronecker coefficients: asymptotics



$$C(m) = \{(\lambda,\mu,\nu) : g_{\lambda\mu\nu} > 0\}$$

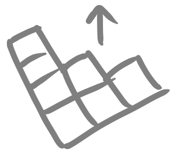
Asymptotic support is convex cone: symplectic geometry [Mumford], [Kirwan]



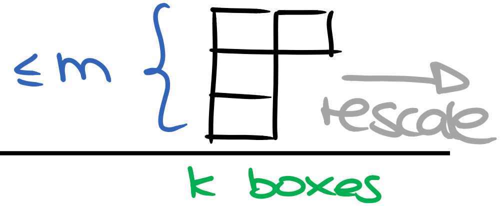
outside: $g \equiv 0$ inside: $\exists s: g_{s\lambda, s\mu, s\nu} > 0$

in general, $s > 1$: failure of saturation, "holes"!

$g_{\lambda\mu\nu}$ is piecewise quasi-polynomial. [Meinrenken-Sjamaar]

Various other asymptotics have been studied:  

Motivation I: The Kronecker polytopes



$$\Delta(m) = \left\{ \frac{(\lambda,\mu,\nu)}{k} : g_{\lambda\mu\nu} > 0 \right\}$$

...is a convex polytope: the Kronecker polytope.

More generally: moment polytope associated with arbitrary representation of a compact connected Lie group.

symplectic geometry

- explicit inequalities known

[Klyachko], [Berenstein-Sjamaar], [Ressayre], [Vergne-W.]

- efficient algorithms of high interest in quantum physics: quantum marginal problem

$$\mathbb{P}((\mathbb{C}^m)^{\otimes 3}) \ni \psi$$

Another example: Littlewood-Richardson coefficients give rise to Horn polytopes.

Motivation II: Geometric complexity theory

How many multiplications are required to multiply 2 x 2 matrices?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_2 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

In fact, 7 < 8 are enough! → $O(n^{2.807\ldots})$ elementary multiplications for n x n [Strassen]

Best known algorithm: $O(n^{2.3729\ldots})$ [Stothers], [Vassilevska-Williams], [Winograd]

What is the minimal exponent of matrix multiplications?

Motivation II: Geometric complexity theory

[Mulmuley-Sohoni]

Idea: Rephrase in terms of tensor varieties, study using algebraic geometry!

$$v_{\text{hard}} \in M_n \otimes M_n \otimes M_n^* \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$$

$$v_{\text{easy}} = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \qquad G = GL(m)^3$$

The goal is to show that: $v_{\text{hard}} \notin \overline{G \cdot v_{\text{easy}}}$

This would imply that we need $> r$ elementary multiplications for $n \times n$ matrices.

[Burgisser-Ikenmeyer]

Landsberg: r=7 is optimal for n=2. Similarly: Permanent vs. determinant (Valiant's conjecture).

Representation-theoretic obstructions

[Mulmuley-Sohoni]

Instead of determining equations for the varieties, we seek to find "representation-theoretic obstructions":

$$V_\lambda \subseteq R(\overline{G \cdot v_{\text{hard}}}) \quad \text{but} \quad V_\lambda \not\subseteq R(\overline{G \cdot v_{\text{easy}}})$$

This naturally leads to certain Kronecker coefficients and related multiplicities (symmetric Kronecker coefficients, plethysms, ...). E.g.:

$$g_{\square\square\square\square,\,\square\square\square\square,\,\nu}$$

[Buergisser-Landsberg-Manivel-Weyman]

Much recent work on Kronecker coefficients has been motivated by this connection to geometric complexity theory.

Kronecker coefficients: mathematical challenges

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu} g_{\lambda\mu\nu} [\nu]$$

1. Decide when a Kronecker coefficient is non-zero!
   Asymptotic polytopes well-understood, but failure of saturation makes it "difficult".

2. Find a positive, combinatorial formula!
   Like the Littlewood-Richardson rule.

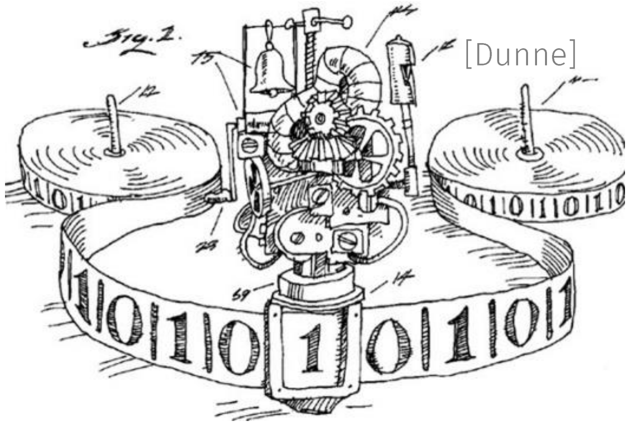
3. Understand the failure of saturation!
   Minimal stretching factor? How to find holes?

This talk: Explicitly study the <u>complexity</u> of these problems!

Computational complexity primer

Computational complexity theory

Study of computational problems: decision problems (*“is n a prime?”*) and counting problems (*“how many prime factors does n have?”*).



[Dunne]

problem instance → **Algorithm** → answer

input, encoded in bits | *in a formal computational model, e.g., Turing machine* | *output, encoded in bits*

<u>Central question:</u> What is the difficulty of a computational problem?

I.e., can we hope for an efficient solution? Or will all algorithms take a long time? Contrast with computability theory (“does there exist *any* algorithm”) & algorithm engineering (“find a *fast* algorithm”).

The complexity class P

P: Computational problems that admit an efficient algorithm.

i.e., runtime polynomial in the input size

Intuition: Those are the computationally feasible problems.

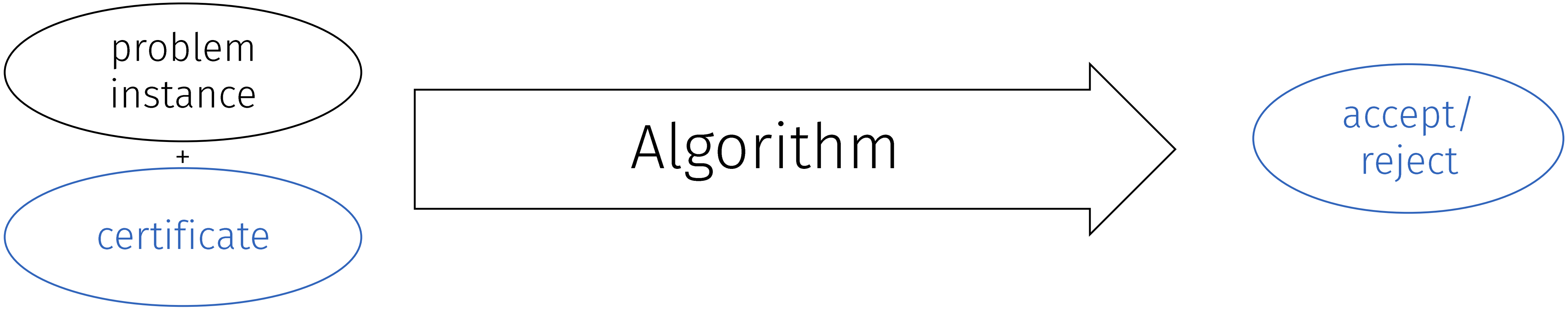
Examples: Linear algebra; linear optimization; min-cut; Fourier transforms; …
Often due to mathematical structure, dualities, …

We may then zoom in and ask for the most efficient algorithm & matching lower bounds. E.g., know how to multiply two n by n matrices in time $O(n^{2.372\ldots})$ [Le Gall], but best lower bound is $3n^2 - o(n^2)$ [Landsberg]!

The complexity class NP

Not all decision problems are known to admit an efficient algorithm. But often the answer can be efficiently verified!

e.g., factoring a number vs. *verifying a factorization; coloring a graph* vs. *checking a coloring*



NP: If answer "YES" then there exists small certificate that can be efficiently verified.

Can be rather nontrivial to prove that a problem is in NP (e.g., UNKNOT). Many problems not in NP.

P vs. NP

P: There exists an efficient algorithm.

NP: If answer YES then there exists small certificate that can be efficiently verified.

Conjecture: P ≠ NP.

Widely believed to be true, for empirical as well as philosophical reasons:

"Surely, finding a proof must be harder than verifying it..."

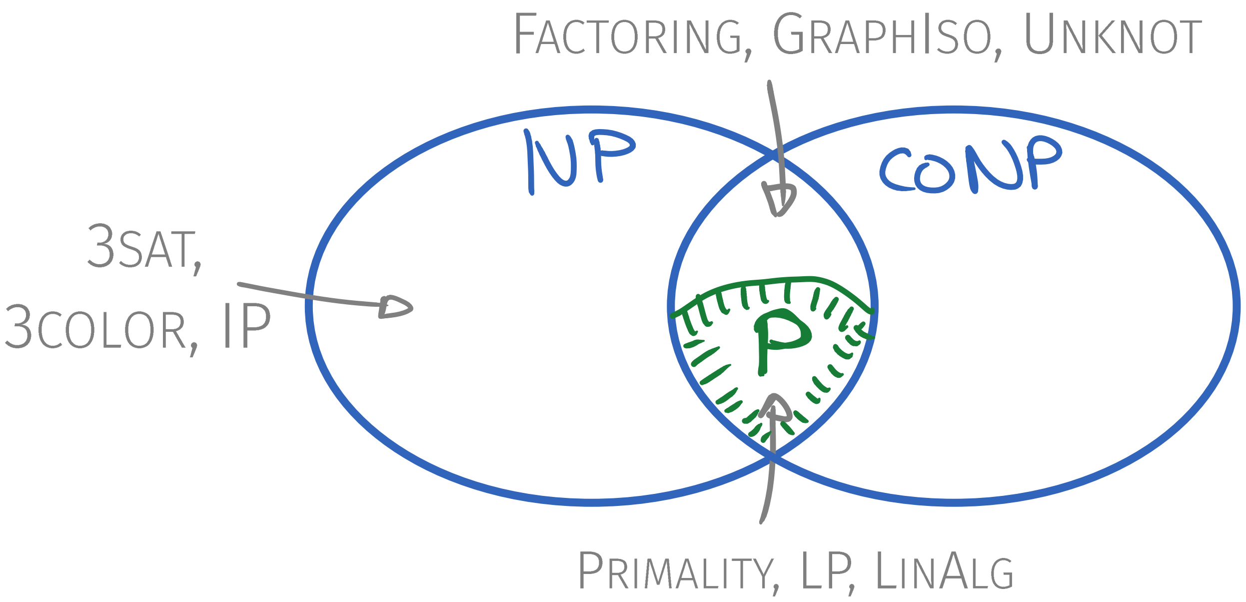
Interestingly, there are proofs that *exclude* entire proof strategies of P ≠ NP!

A glimpse at the complexity landscape

P: There exists an efficient algorithm.

NP: If answer YES then there exists small certificate that can be efficiently verified.

CoNP: If answer NO then there exists small certificate that can be efficiently verified.



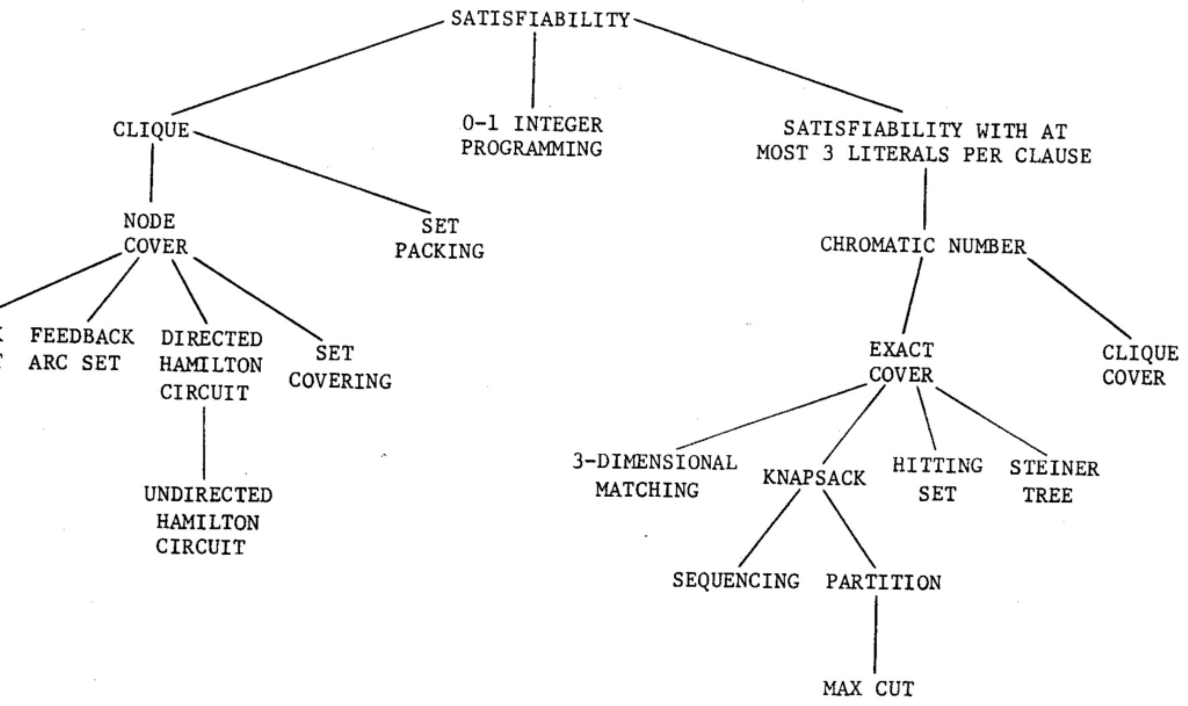
Only a small part of the complexity landscape (time, space, random, quantum, ...).

Comparing complexity

X can be reduced to Y if X can be solved efficiently using an efficient algorithm for Y.

Y is NP-hard if any problem in NP can be reduced to Y.

Y is NP-complete if NP-hard and contained in NP.



NP-complete problems exist! [Cook], [Levin]

In fact, many natural combinatorial problems are NP-complete. [Karp]

If any NP-complete problem has an efficient solution, then P=NP.

Various possible definitions of reduction (reuse, post-processing, ...).

Complexity of *counting* problems

P: There exists an efficient algorithm.

NP: If answer YES then there exists small certificate that can be efficiently verified.

#P: Answer = number of certificates accepted by an NP-algorithm. [Valiant]

Natural complexity class for counting gadgets that are easily verified.

e.g., counting 3-colorings of a graph, integral hives, ...

Arguably what we would call a ”positive, combinatorial formula”! [Mulmuley]

Complexity & representation theory

Branching problems as computational problems

$$V = \bigoplus_{\lambda} m_\lambda V_\lambda$$

- <u>Decision problem:</u> Decide if multiplicity > 0.
- <u>Counting problem:</u> Compute the multiplicity.

We may thus use computational complexity theory to study their difficulty!

We always need to consider a family of representations (complexity is about asymptotics).

Complexity of Littlewood-Richardson coefficients

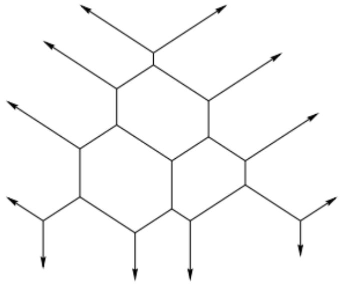
$$V^m_\lambda \otimes V^m_\mu = \bigoplus_\nu c^{\lambda\mu}_\nu V^m_\nu$$

Input: Three Young diagrams such that $|\lambda| + |\mu| = |\nu|$

- Decision problem: P

  [Mulmuley-Sohoni]

  Proof relies on honeycombs & LP results.



- Counting problem: #P-complete

  [Narayanan]

  Combinatorial formula shows that in #P. Hardness by reduction from contingency tables.

Thus *any* other #P problem can be solved by computing LR coefficients!
E.g., exists mapping {graphs} → {Young diagrams} s.th. # of 3-colorings = f(LR coeff).

Consequences largely unexplored...

Complexity of Kronecker coefficients

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu} g_{\lambda\mu\nu} [\nu]$$

Input: Three Young diagrams such that $|\lambda| = |\mu| = |\nu|$

- <u>Decision problem:</u> NP-hard   Is it in NP?   [Ikenmeyer-Mulmuley-W.]

  This was previously conjectured to be in P!

  "Hopeless" to look for efficient algorithm (i.e., to find a simple characterization).

- <u>Counting problem:</u> #P-hard   Is there a #P formula?

  ...since LR coefficients are special Kronecker coefficients.

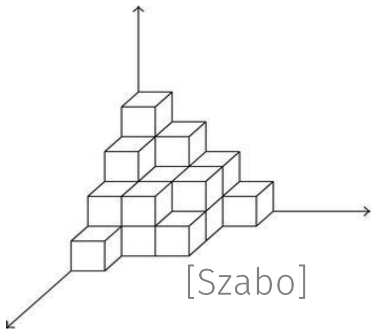
For Young diagrams of <u>bounded height</u>, both problems in P! [Christandl-Doran-W.]

Sketch of proof

[Ikenmeyer-Mulmuley-W.]

Theorem: Deciding positivity of Kronecker coefficients is NP-hard.

Alternative characterization: $\# V^m_{\lambda^T} \otimes V^m_{\mu^T} \otimes V^m_{\nu^T} \subseteq \Lambda^N((\mathbb{C}^m)^{\otimes 3})$



Weight vectors = point sets; weight = slice sums

Deciding if there exists a point set with given slice sums is NP-hard. [Brunetti et al]

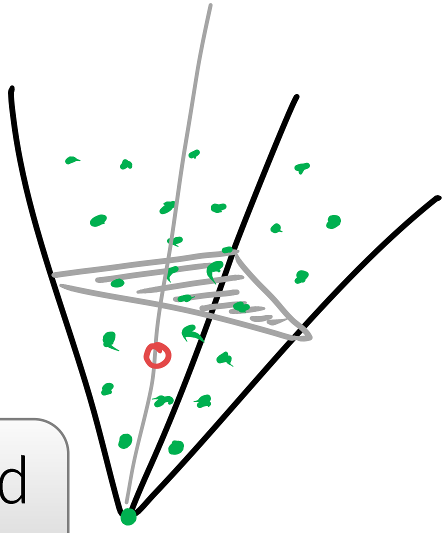
Relevant point sets are always “pyramids” → correspond to highest weight vectors.

The failure of saturation

[Ikenmeyer-Mulmuley-W.]

We are interested in finding examples of “holes”:

$g_{\lambda\mu\nu} = 0$ but $g_{s\lambda, s\mu, s\nu} > 0$ for some $s > 1$



Corollary: There exist “many” such holes and they can be constructed explicitly and efficiently.

Proof: We have a sequence of injective reductions

3D MATCHING → 4D PARTITION → … → 3D CONSISTENCY → KRONECKER>0

& 3D MATCHING has many “NO” instances.

The resulting holes are significantly beyond current methods – cannot even verify!

Asymptotic positivity

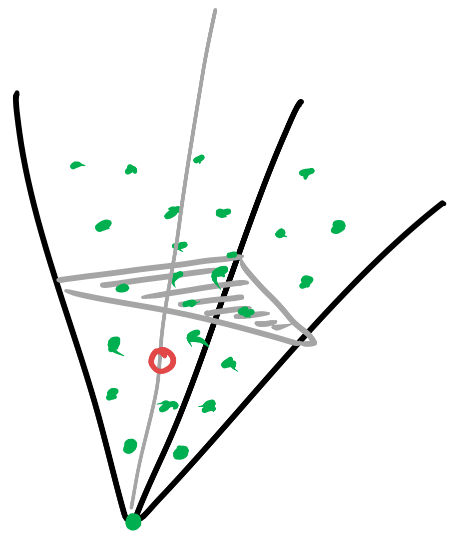
[Buergisser-Christandl-Mulmuley-W.]

We may also consider the asymptotic positivity problem:

Given three Young diagrams,

$$\exists s : g_{s\lambda, s\mu, s\nu} > 0 \ ?$$

That is, is the triple contained in the cone $C(m)$?



Theorem: Deciding asymptotic positivity is in NP and CoNP.

"not" NP-hard!

efficient algorithm?

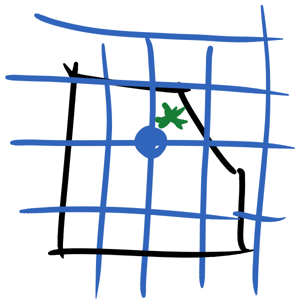
- Motivation: Computing moment polytopes in practice, quantum marginal problem.
- Suggests hardness of positivity problem is in part due to failure of saturation.

Sketch of proof

[Buergisser-Christandl-Mulmuley-W.]

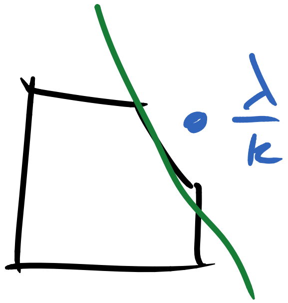
Theorem: Deciding asymptotic positivity is in NP and CoNP.

NP: Certificate is vector in $(\mathbb{C}^m)^{\otimes 3}$



Point in polytope can be computed efficiently. We prove that finite precision is not an issue (walls of polytope are not too steep).

CoNP: Certificate is separating hyperplane (H, z)



Inequality can be verified efficiently (if also given point at which to evaluate determinant polynomial). [Vergne-W.]

Generalization to arbitrary groups, representations requires efficient algos for Lie algebra representation.

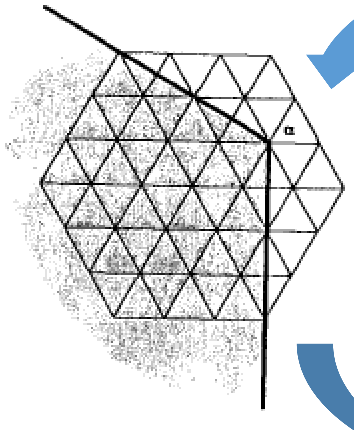
Summary

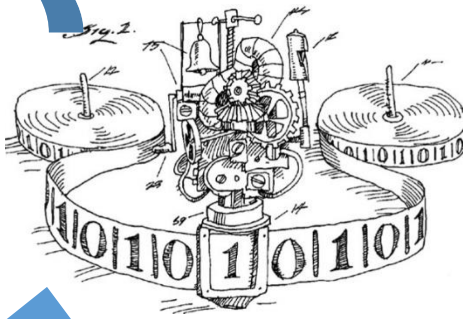
#P = "combinatorial formula"

NP-hardness of the positivity problem

explicit "holes"

Complexity theory: conceptual framework for studying the difficulty of mathematical problems; a theory that can yield new mathematical results





New challenges in representation theory motivated by applications in geometric complexity theory, theoretical quantum physics

Thank you for your attention