The holographic entropy cone

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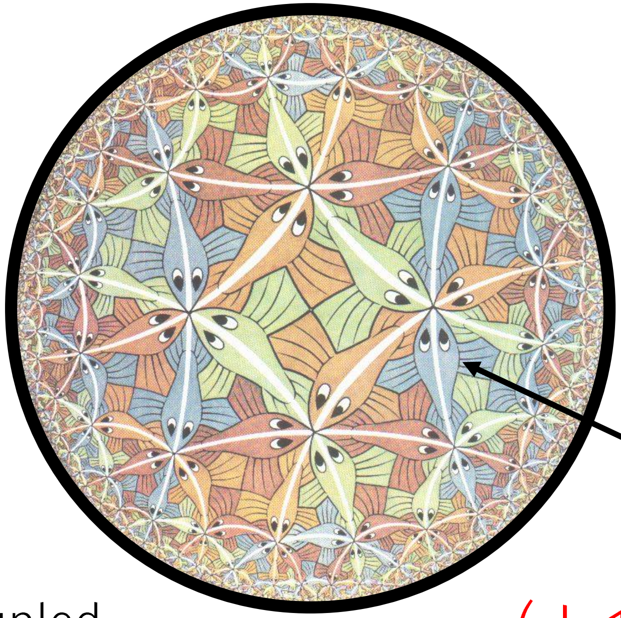
QIP '16

joint work with S. Nezami, J. Sully (Stanford), N. Bao, H. Ooguri, B. Stoica (Caltech)

Holography: the gauge/gravity correspondence

[Maldacena]



boundary

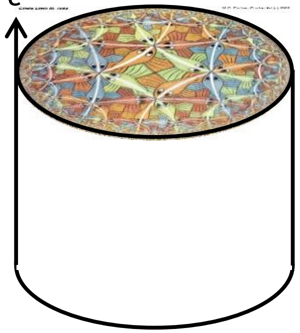
d-dim QFT

large N, strongly coupled

bulk

(d+1)-dim q. gravity

semiclassical



Why care? (As a quantum information theorist...)

→ Patrick's plenary talk

Entanglement is "explained" via geometry, and vice versa. [Swingle]

cf. [Czech et al] at QIP '15

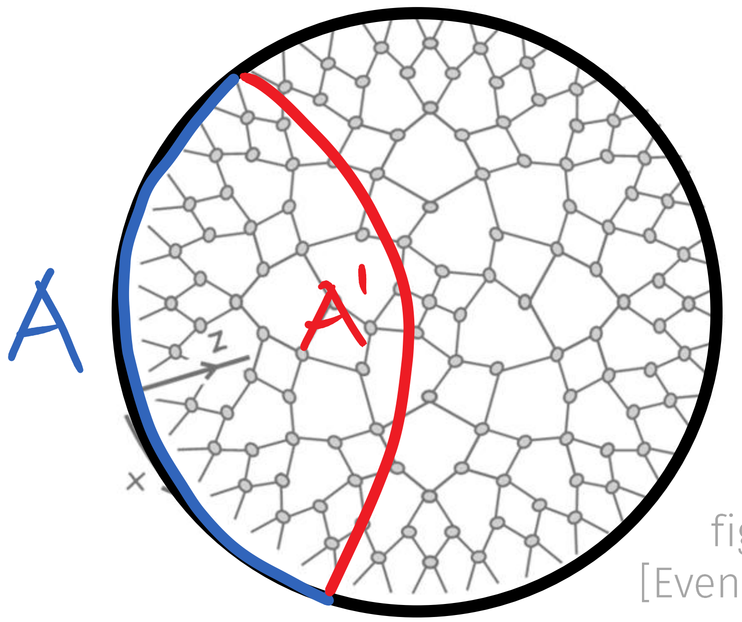


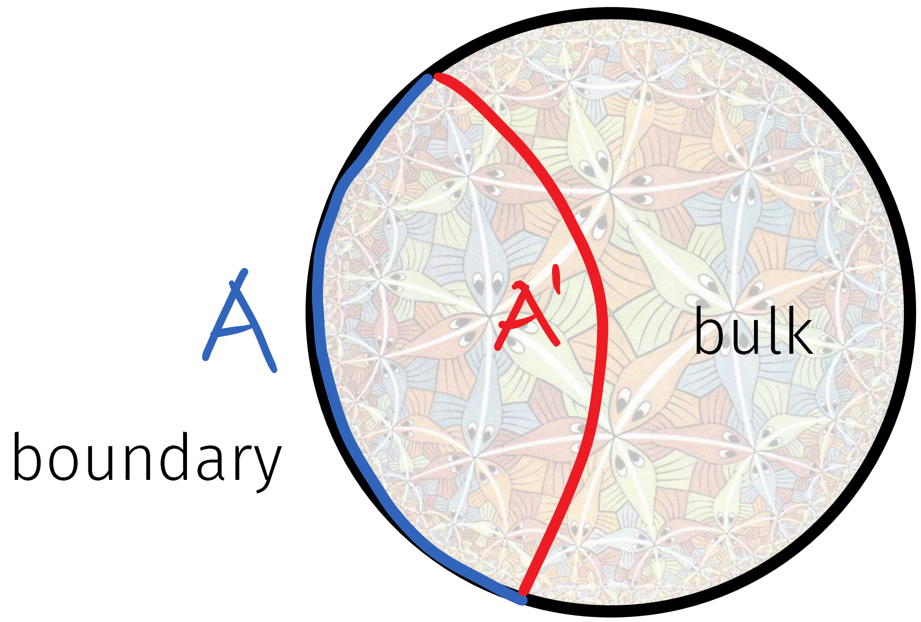
figure from [Evenbly-Vidal]

Holographic systems behave like q. error correcting codes. [Almheiri et al]
Further curious QIT properties (...)

QIT ideas useful to construct rigorous toy models of holography! → Fernando

The holographic entropy formula

[Ryu-Takayanagi]

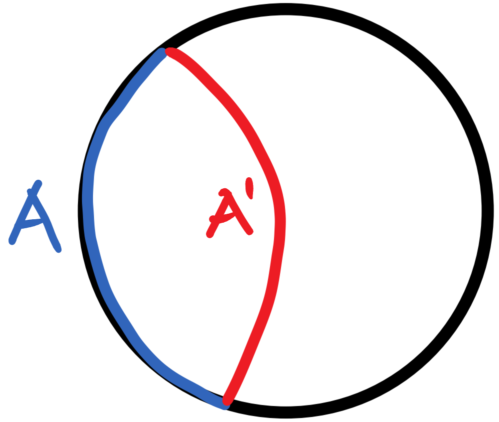


$$S(A) = \min_{A' \sim A} |A'|$$

entangled entropy in boundary — length of minimal geodesic in bulk

This talk

$$S(A) = \min_{A' \sim A} |A'|$$



"Deconstruction" of the holographic entropy formula:

- Entropy cone
- Combinatorics
- Applications

The holographic entropy cone

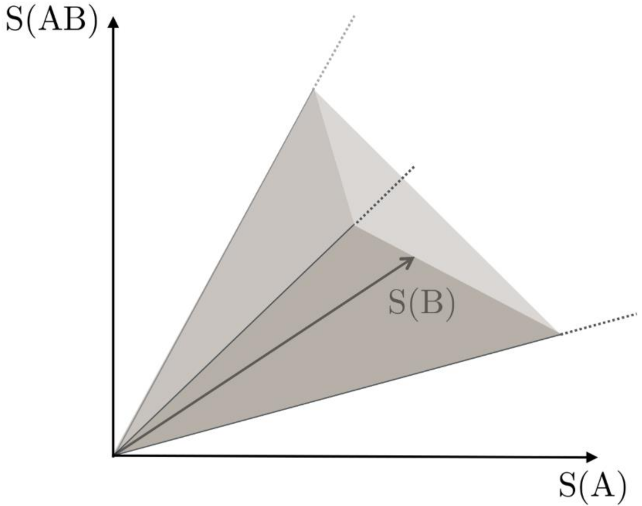
$S(A) = \min|A'|$

$$\mathcal{C}_n = \{(S(A_1), \ldots, S(A_1A_2), \ldots, S(A_1 \ldots A_n)) \in \mathbb{R}^{2^n - 1}\}$$

where we allow for *arbitrary* geometries and boundary regions.

This is a convex cone, the holographic entropy cone.

rescaling
disjoint union

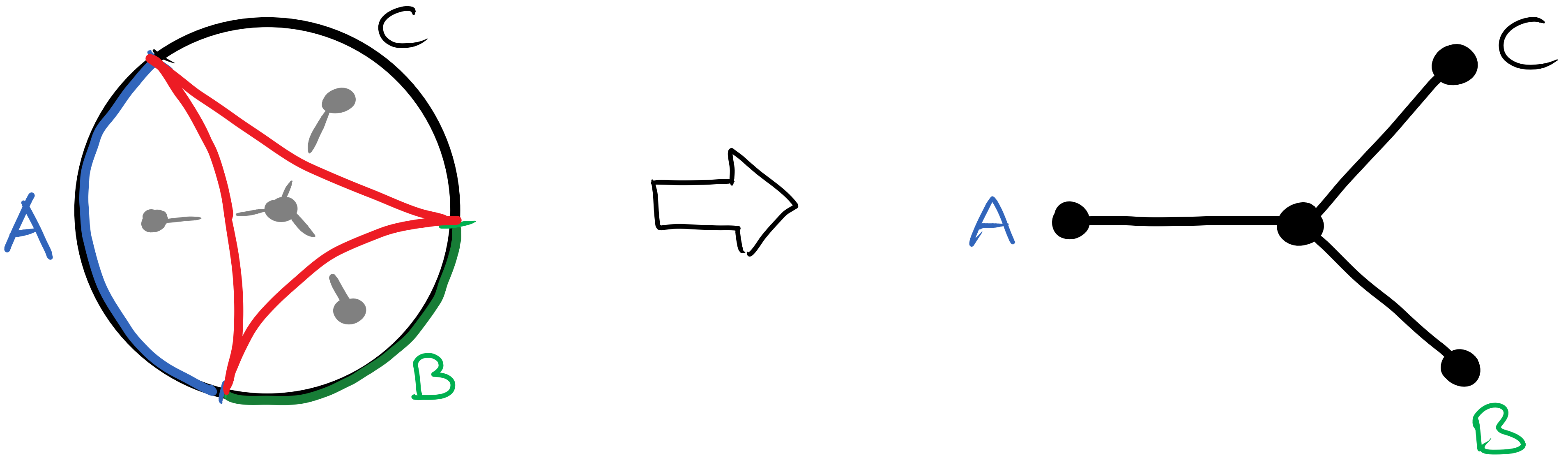


facets: entropy inequalities
extreme rays: most extreme entropy vectors

"Phase space" of holographic entropies.

A combinatorial version of holographic entropy

Instead of studying all possible smooth geometries, we may reduce to graphs:



Holographic entropy of subsystem = weight of min-cut in graph

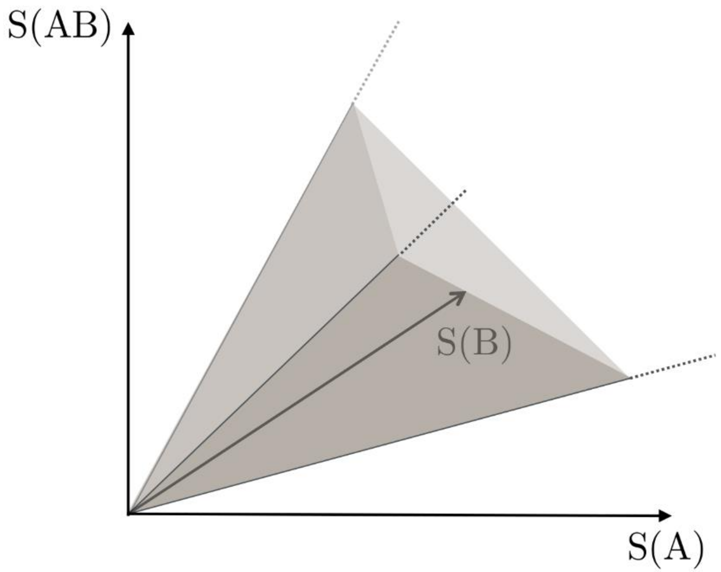
Graphs to geometries via pair-of-pants decomposition → time slices of Lorentzian wormholes.

Application 1: Polyhedrality

Theorem: Each holographic entropy cone is polyhedral.

That is, there are:

- Finitely many entropy inequalities
- Finitely many extreme rays



Proof by reduction to graphs with a *fixed* number of vertices.

This is *not* known for general quantum states (and *false* for the Shannon entropy).

Application 2: Holographic entropy inequalities

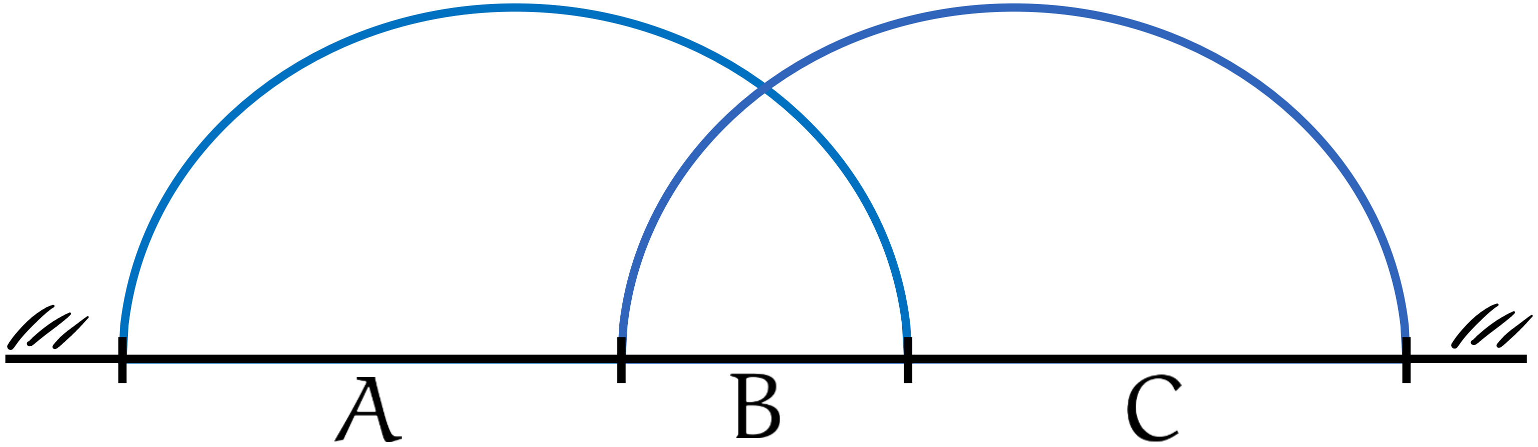
Strong subadditivity in holography

[Headrick–Takayanagi]

$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$

bulk

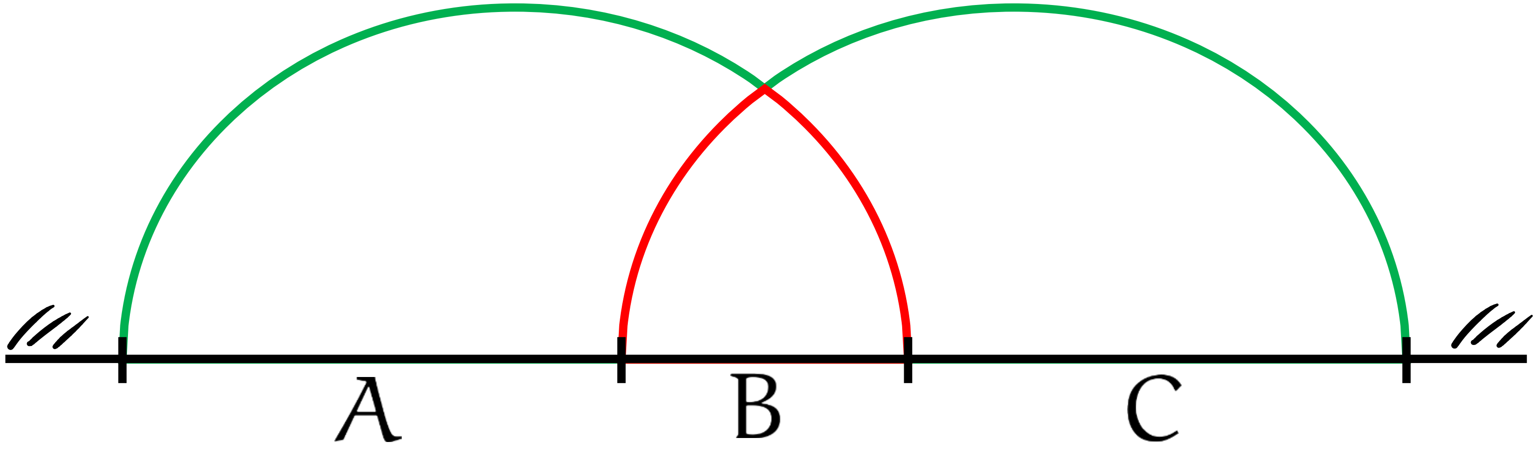
boundary



Strong subadditivity in holography

[Headrick–Takayanagi]

$$S(AB) + S(BC) =$$



Strong subadditivity in holography

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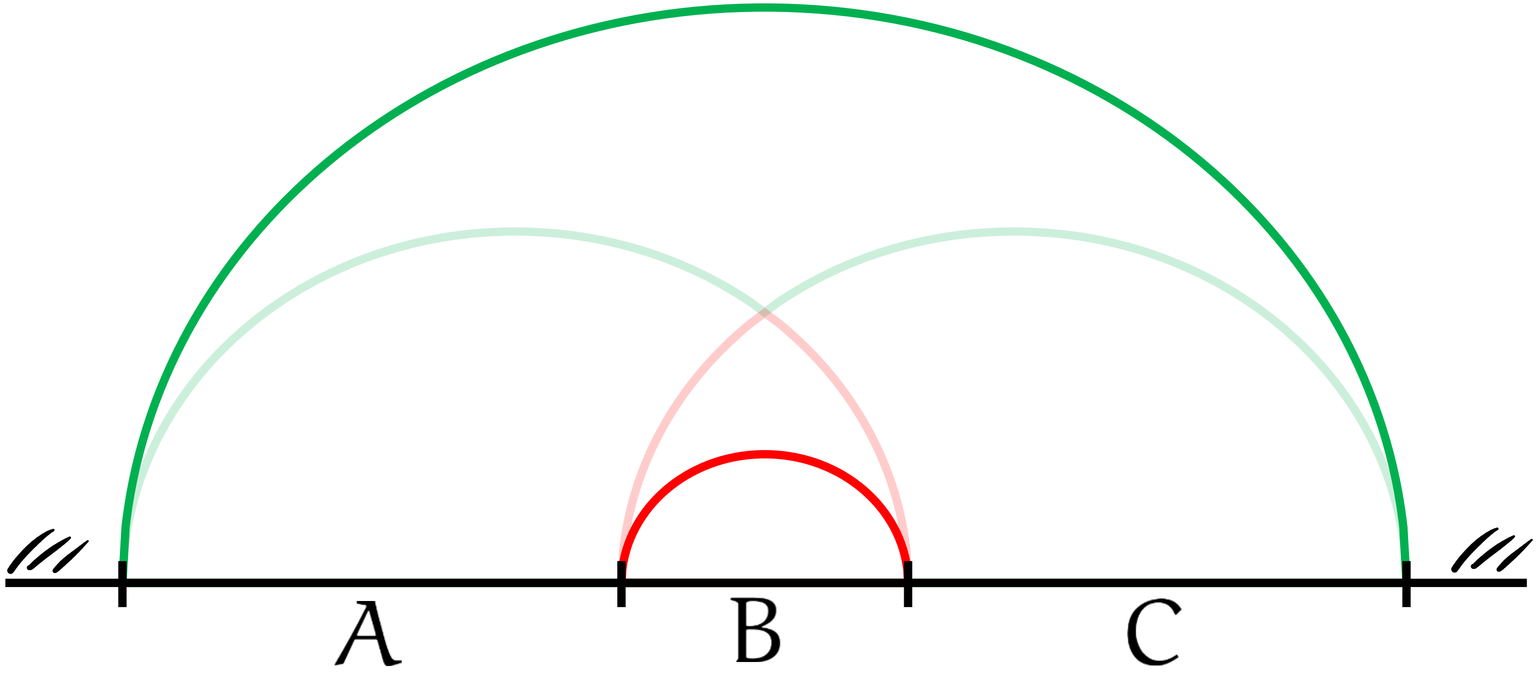
$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$



Making this precise requires finding decompositions of the minimal surfaces that work in *all* geometric configurations.

Of course, strong subadditivity for general quantum states is much more difficult [Lieb-Ruskai].

Proofs by contraction

$$S(AB) + S(BC) + S(AC) \geqslant S(A) + S(B) + S(C) + S(ABC)$$

A	1	0	1	1	0	0	1
B	1	1	0	0	1	0	1
C	0	1	1	0	0	1	1
O	0	0	0	0	0	0	0

"occurrence vectors"

Theorem: If table can be extended to Hamming contraction then the entropy inequality is valid.

Encodes cutting/pasting prescription. Easy to prove from graph model. Greedy algorithm.

Monogamy of mutual information

[Hayden-Headrick-Maloney]

$$S(AB) + S(BC) + S(AC) \geqslant S(A) + S(B) + S(C) + S(ABC)$$

	$S(AB)$	$S(BC)$	$S(AC)$	$S(A)$	$S(B)$	$S(C)$	$S(ABC)$
A	1	0	1	1	0	0	1
B	1	1	0	0	1	0	1
C	0	1	1	0	0	1	1
O	0	0	0	0	0	0	0

$$I(A:B) + I(A:C) \leqslant I(A:BC)$$

Does *not* hold for general quantum systems. It “excludes” non-trivial quantum Markov states, GHZ states, ...

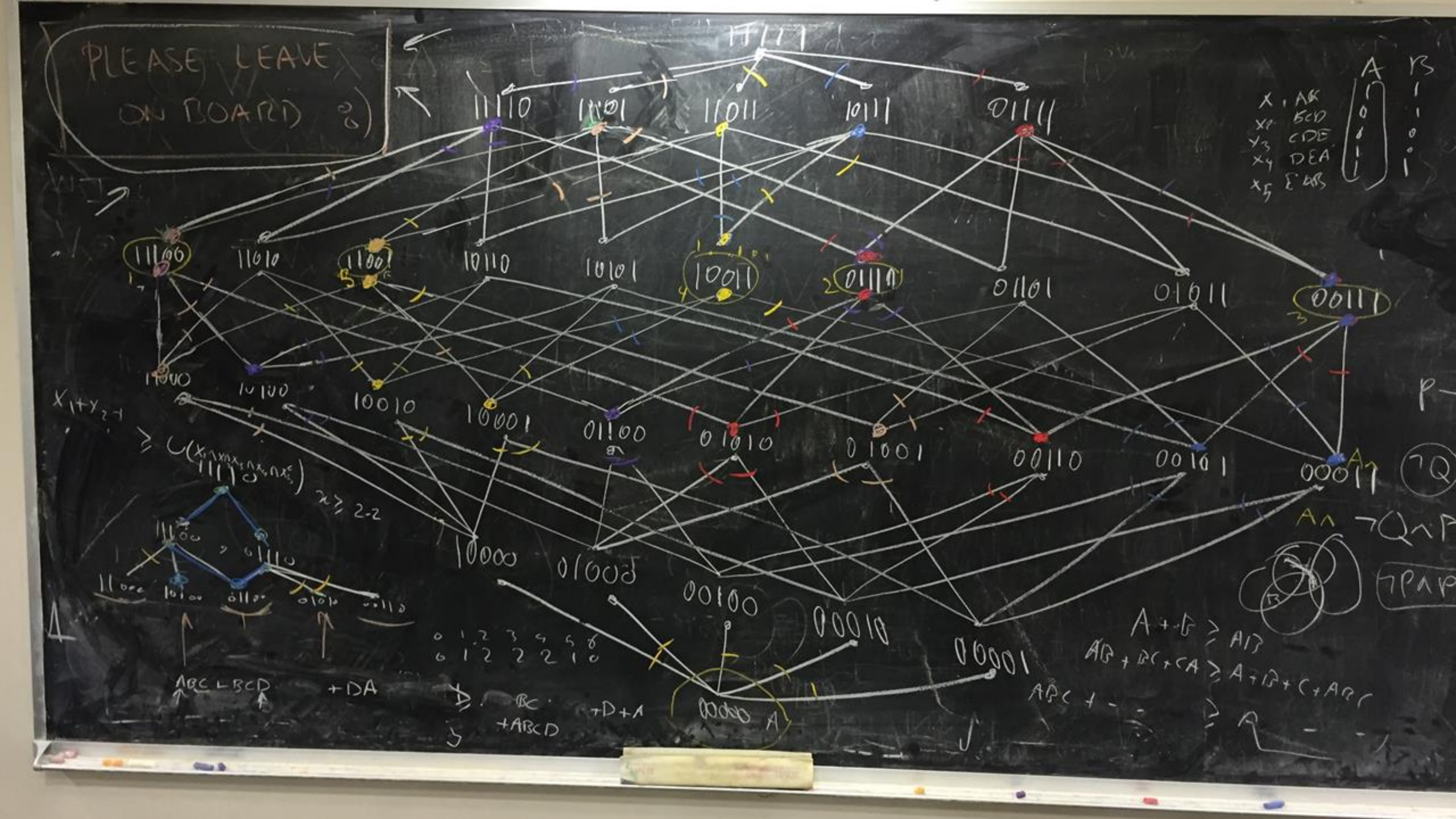
A cyclic family of entropy inequalities

$$\sum_{i=1}^{2k+1} S(A_i | A_{i+1} \dots A_{i+k}) \geqslant S(A_1 \dots A_{2k+1})$$

Part of an infinite new family; unifies all previously known holographic entropy inequalities.

We could also verify entropy inequalities conjectured for the von Neumann entropy.

Proof by construction of explicit family of contractions.
Main application: Entropic constraints on the existence of smooth dual geometries.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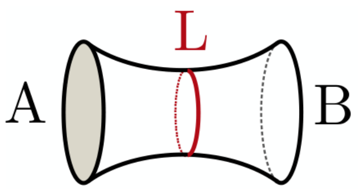
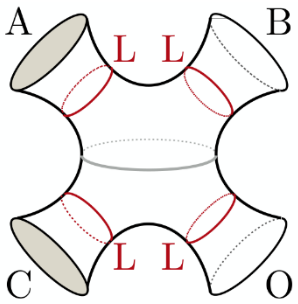
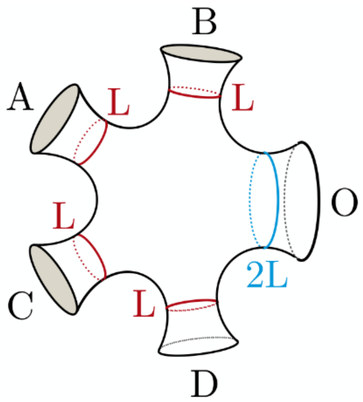
A B
x_1 AB
x_2 BCD
x_3 CDE
x_4 DEA
x_5 EAB
$x_1 + x_2 - 1$
0 1 2 3 4 5 6
0 1 2 2 2 1 0
ABC + BCD
+DA
+ABCD
+D+A
A + B ≥ AB
AB + BC + CA ≥ A + B + C + ABC

The holographic entropy cones

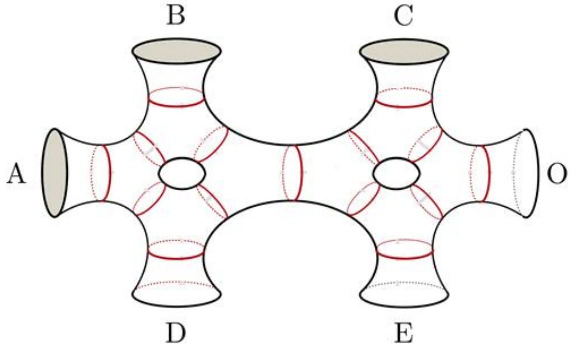
n ≤ 4 subsystems: subadditivity and monogamy are complete







n ≥ 5 subsystems: several new inequalities



extreme rays have higher genus, nontrivial interior cycles

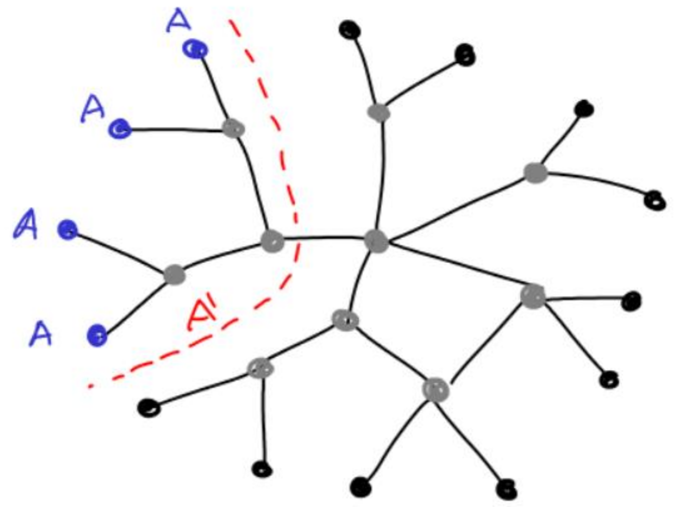
"Application" 3: Random tensor networks

→ Patrick (cf. Fernando)

Theorem: Holographic entropies are entropies of stabilizer states.

Probabilistic method: random tensor network states satisfy RT formula with high probability.

In particular, always quantum mechanical.
Strong consistency check!

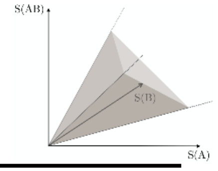


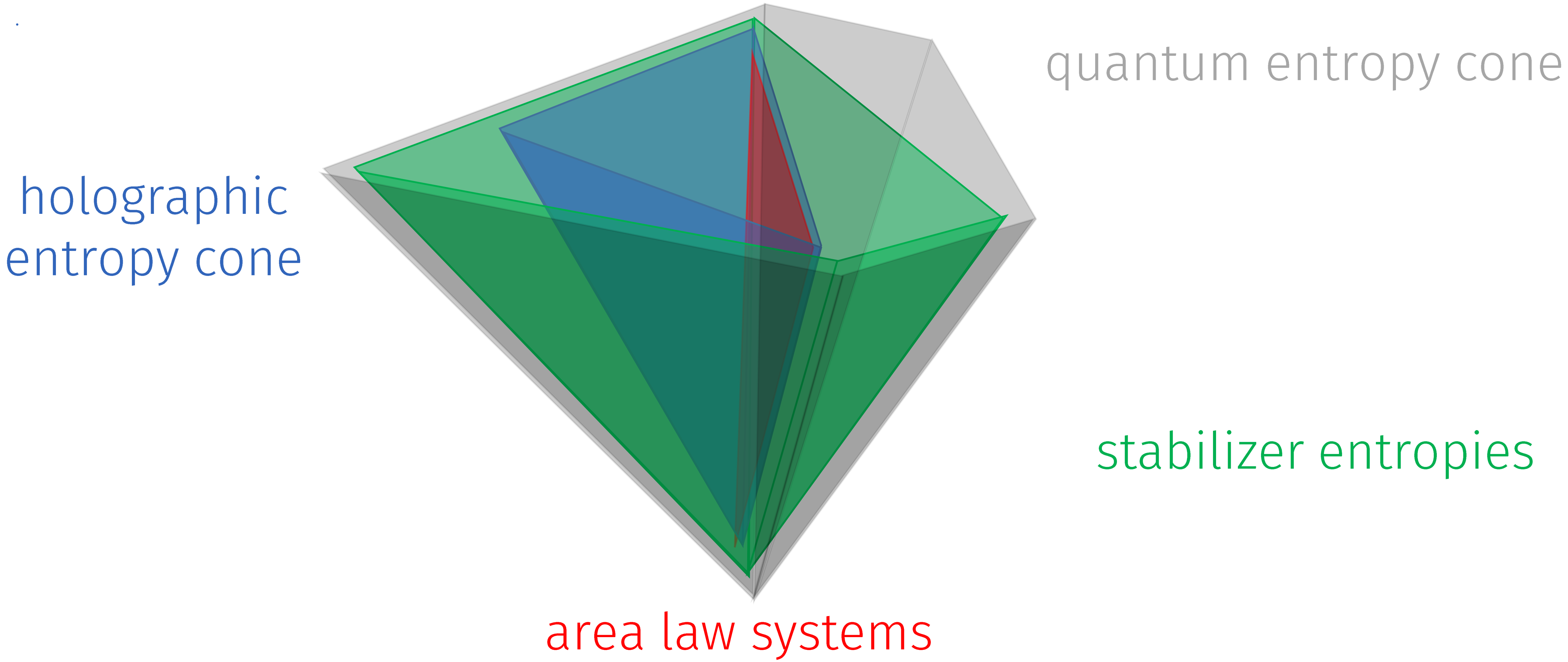
$$S(A) = \min_{A' \sim A} |A'|$$

An extension of the method can be used to construct a toy model for the holographic correspondence. [Hayden-Nezami-Qi-Thomas-W.-Zhao]

These are *not* graph states! Interestingly, stabilizers also have classical dual. Cf. [Ruskai et al], [Gross-W.].

Entropy cones – the big picture

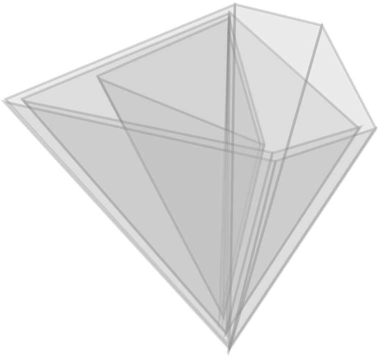




In particular, the RT formula really does behave like a von Neumann entropy. Conversely, can check conjectures about the latter via holography. Amusing aside: Kitaev-Preskill topological entropy (I_3) emerges as unique measure of area law violation.

Summary

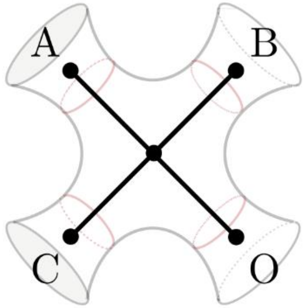
arXiv: 1505.07839, 1507.05650, 1601.01694



Holographic entropy cone as organizing principle

Min-cuts in graphs as combinatorial model

- “proofs by contraction” of new entropy inequalities
- back to geometry: Lorentzian wormholes
- back to quantum states: random tensor networks



Thank you for your attention!