The Holographic Entropy Cone

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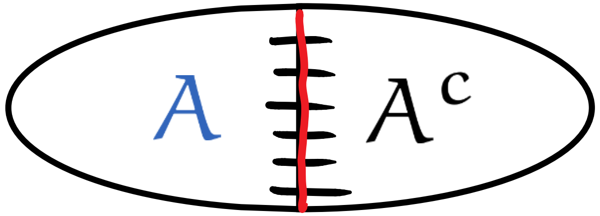
joint work with S. Nezami, J. Sully (Stanford), N. Bao, H. Ooguri, B. Stoica (Caltech)

Entanglement Entropy

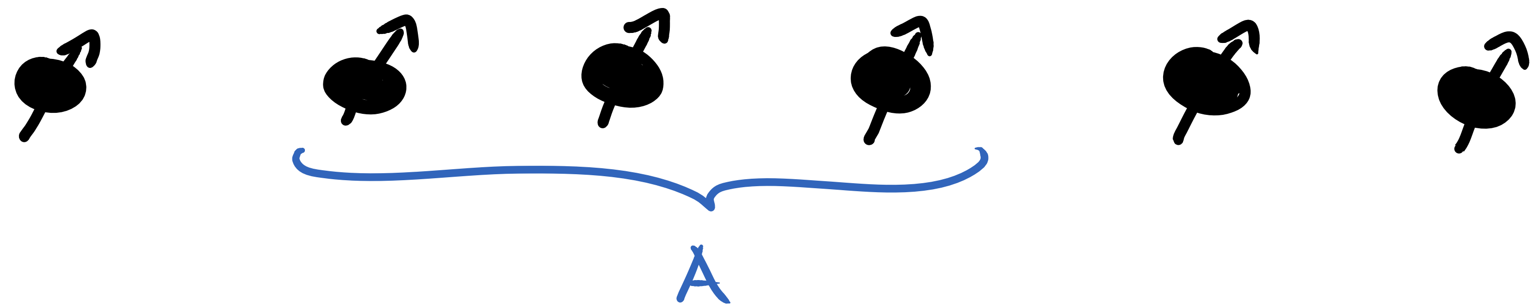


$$S(A) = -\operatorname{tr} \rho_A \log \rho_A$$

Measure of quantum information, entanglement, ...

Entanglement Entropy in Spin Systems

[Hastings]



Area law for gapped phases: $S(A) \leqslant c\,|\partial A|$

Exponential decay of correlations: $I(A:B) \leqslant e^{-d(A,B)/\xi}$

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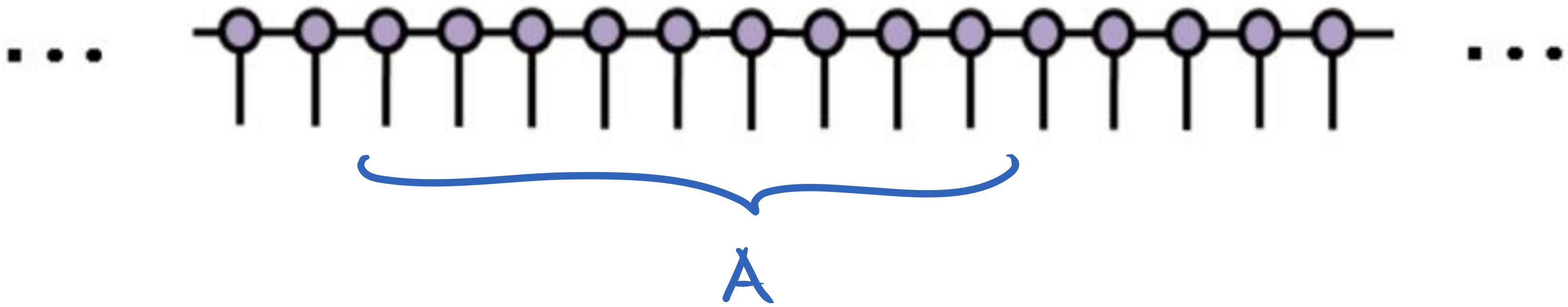

matrix product state

Area law for gapped phases: $S(A) \leqslant c\,|\partial A|$

Exponential decay of correlations: $I(A:B) \leqslant e^{-d(A,B)/\xi}$

Figure from [Evenbly—Vidal]

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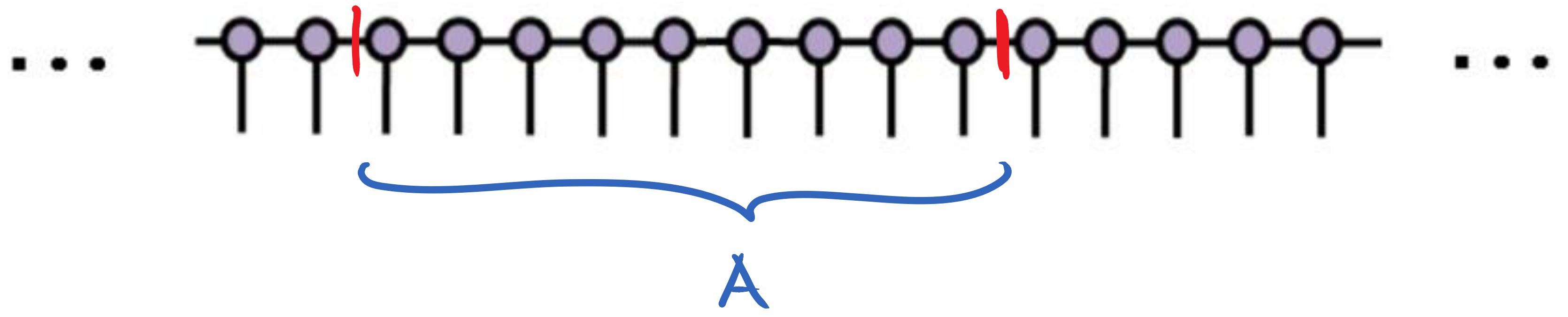


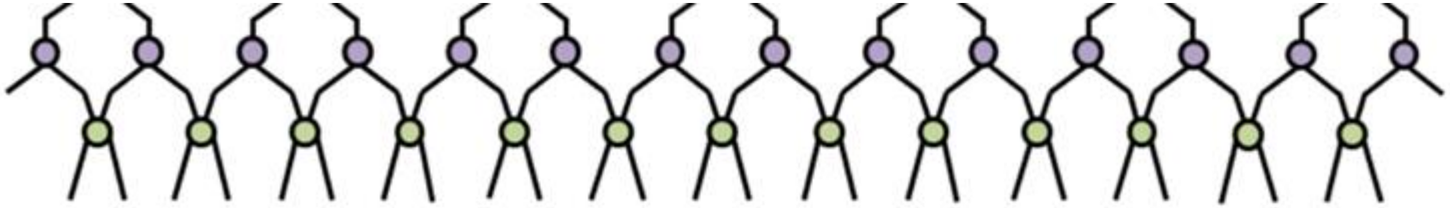
matrix product state

Area law for gapped phases: $S(A) \leqslant c\,|\partial A|$ in 1d

Exponential decay of correlations: $I(A:B) \leqslant e^{-d(A,B)/\xi}$

Figure from [Evenbly—Vidal]

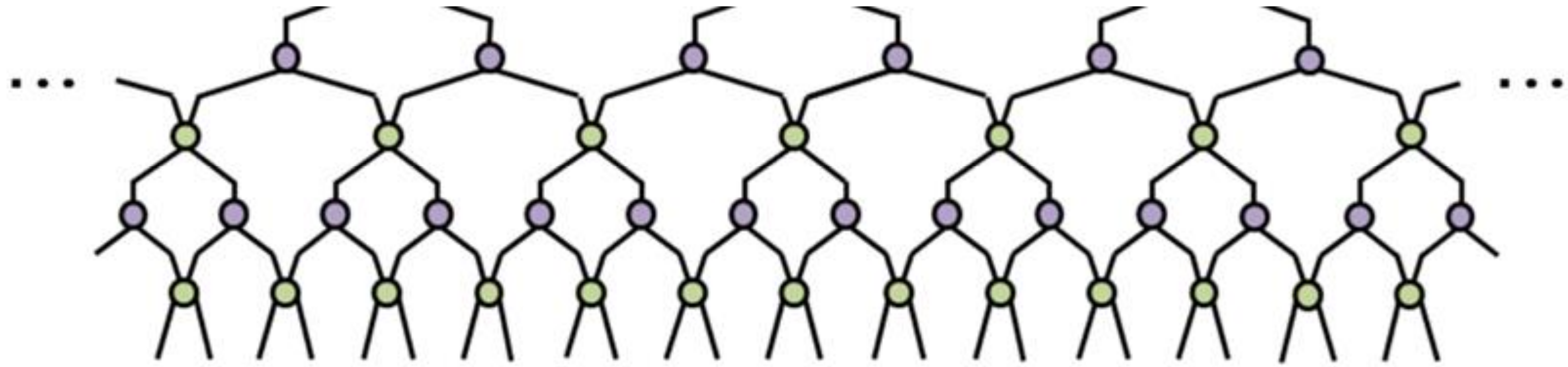
Critical Systems



Entanglement at each scale (MERA)

Figure from [Evenbly—Vidal]

Critical Systems



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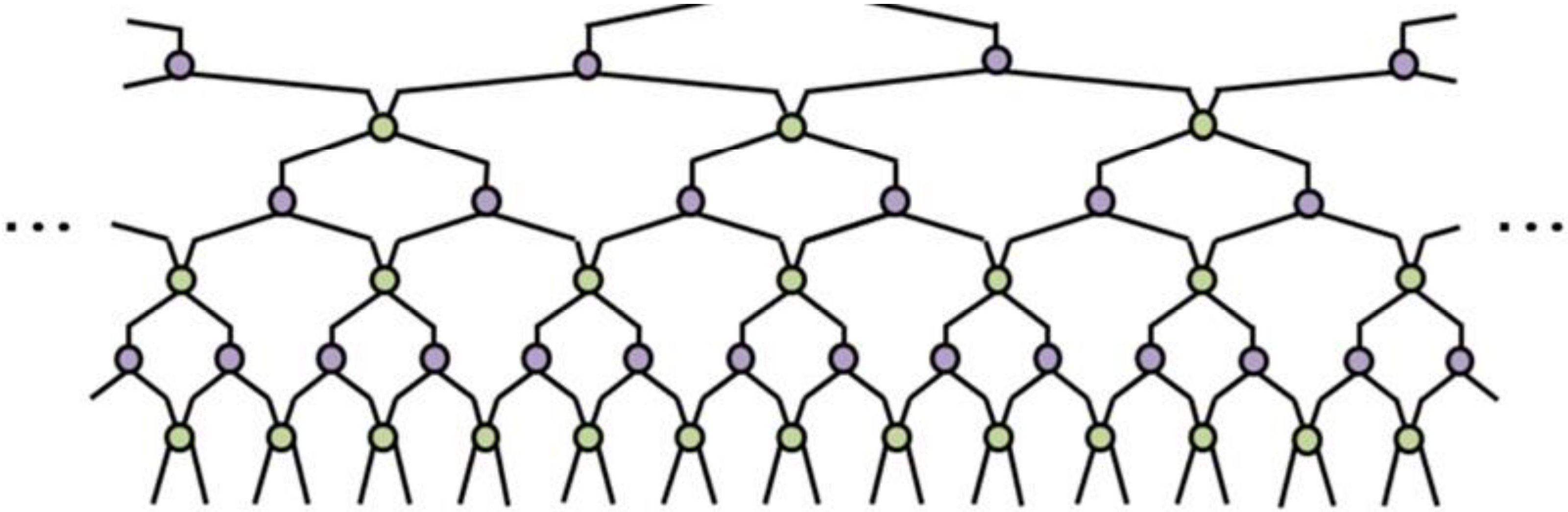

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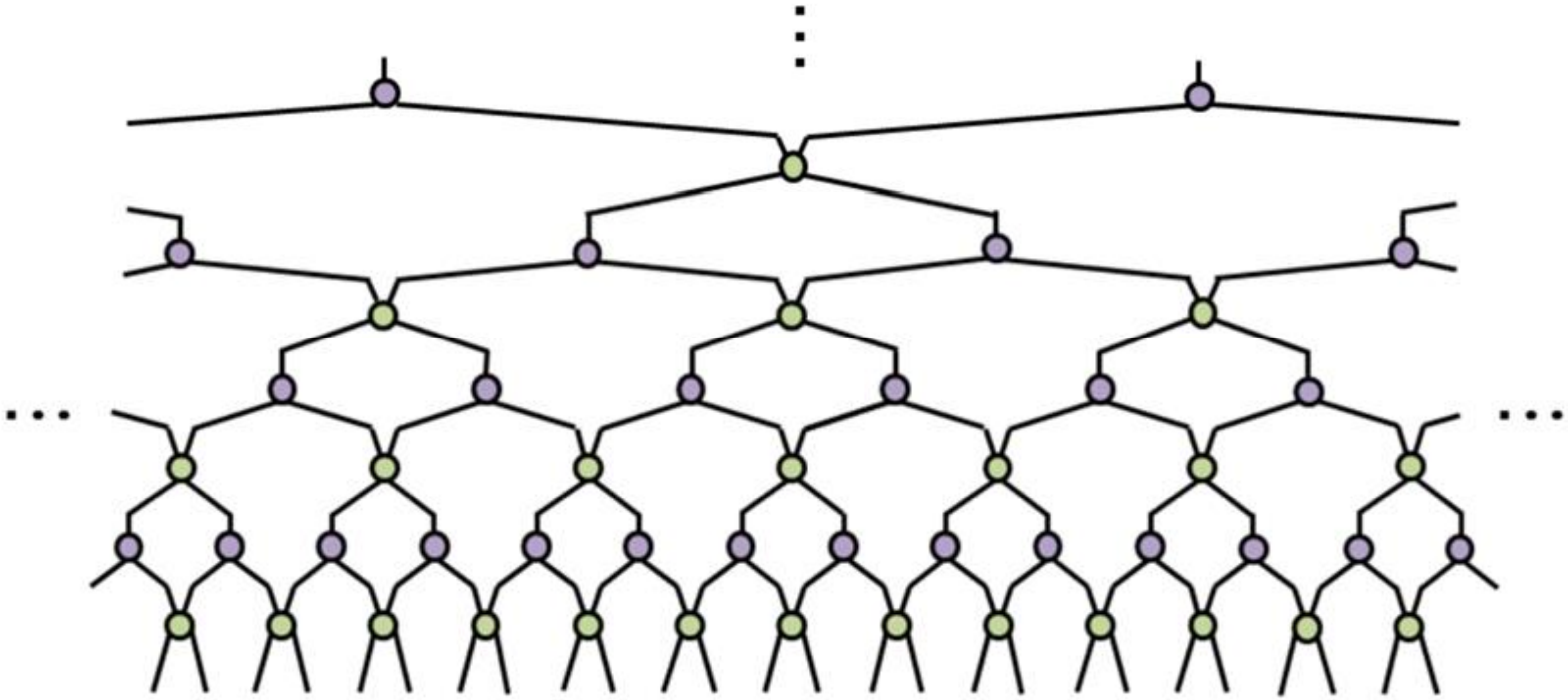

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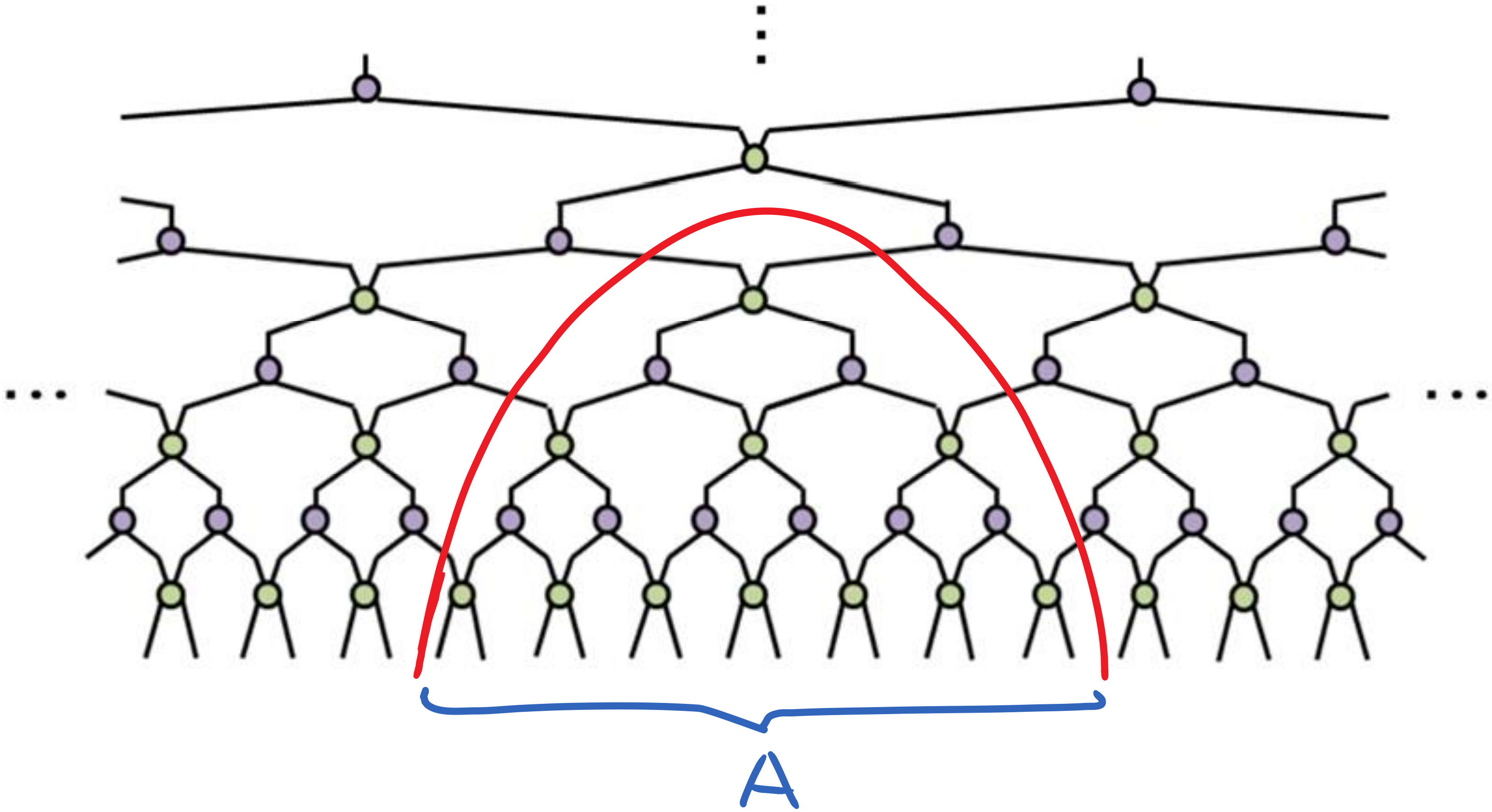

$$S(A) \sim \log \frac{|A|}{\mu}$$

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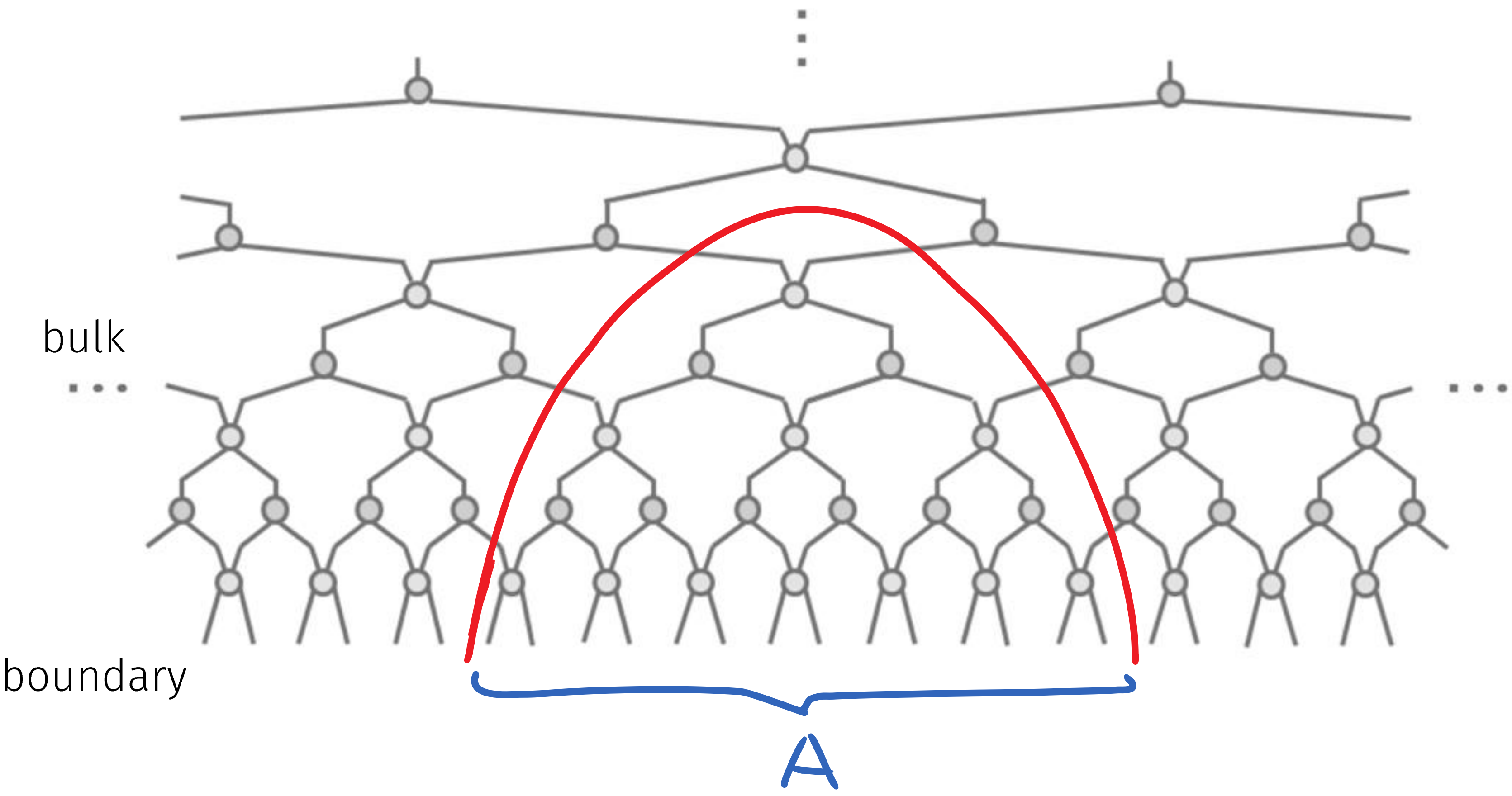
Figure from [Evenbly—Vidal]

Critical Systems & Hyperbolic Geometry

[Swingle]



$$S(A) \sim \log \frac{|A|}{\mu}$$

Hyperbolic half-plane

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Figure from [Evenbly—Vidal]

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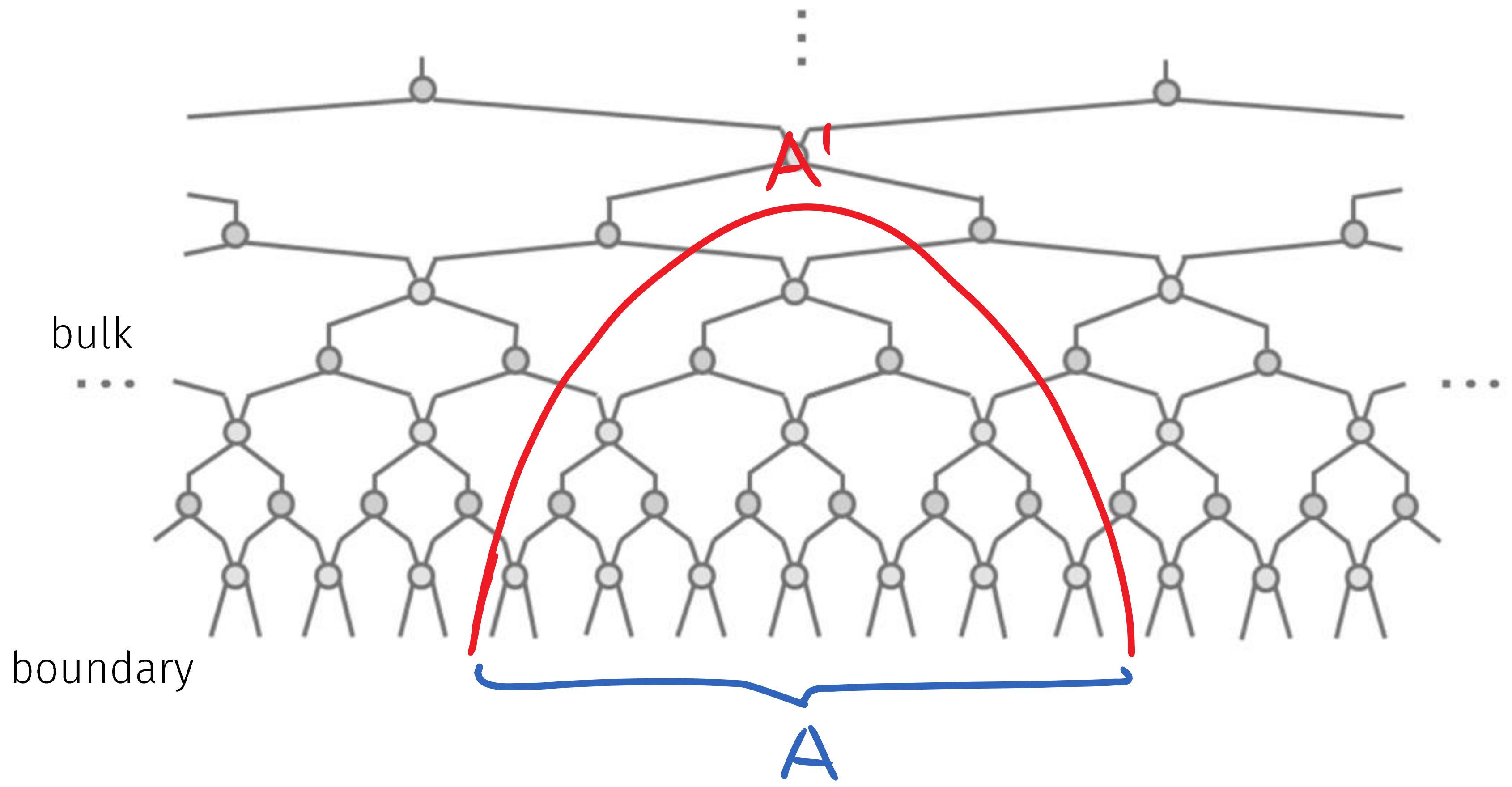

Hyperbolic half-plane

$$S(A) \sim \min_{A' \sim A} |A'|$$

entanglement entropy
= *length of minimal geodesic in bulk geometry*

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Figure from [Evenbly—Vidal]

Critical Systems & Hyperbolic Geometry

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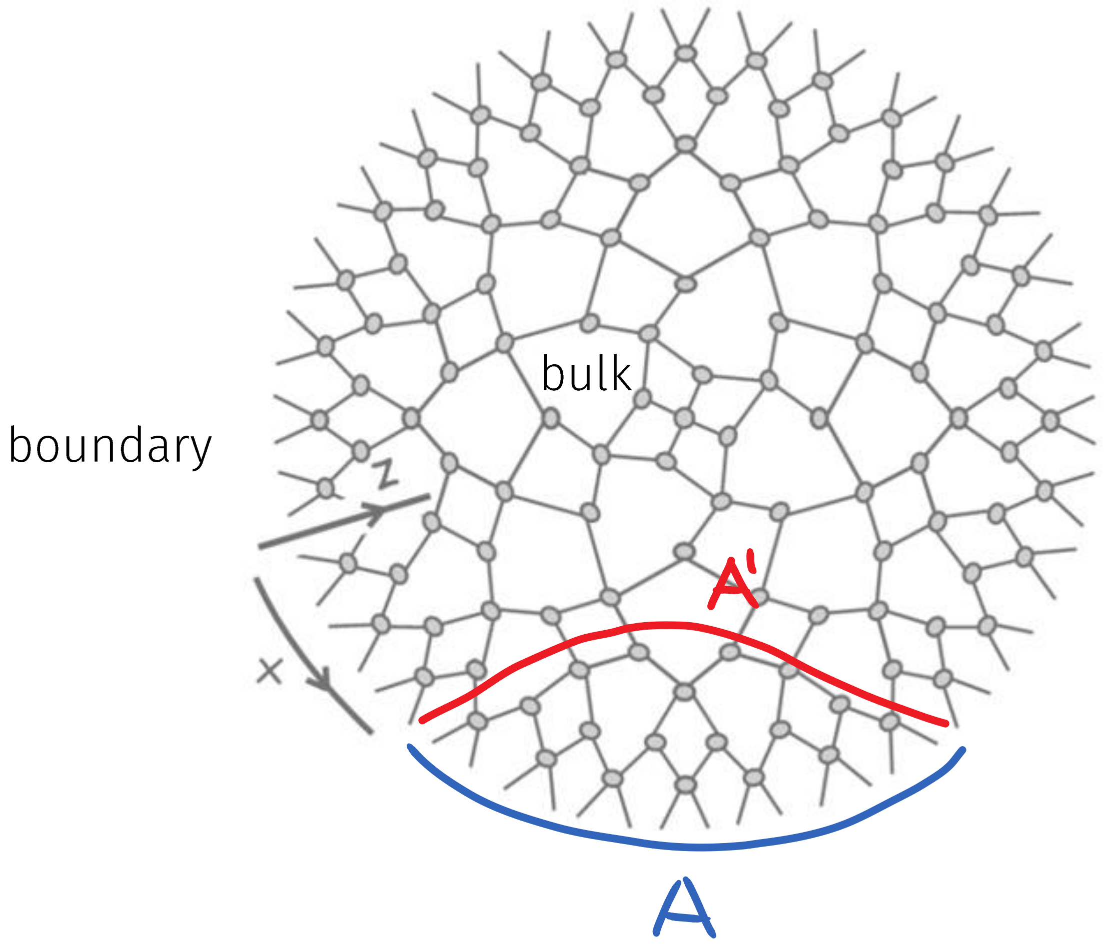


Hyperbolic disk

$$S(A) \sim \min_{A' \sim A} |A'|$$

entanglement entropy
= *length of minimal geodesic*
in bulk geometry

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

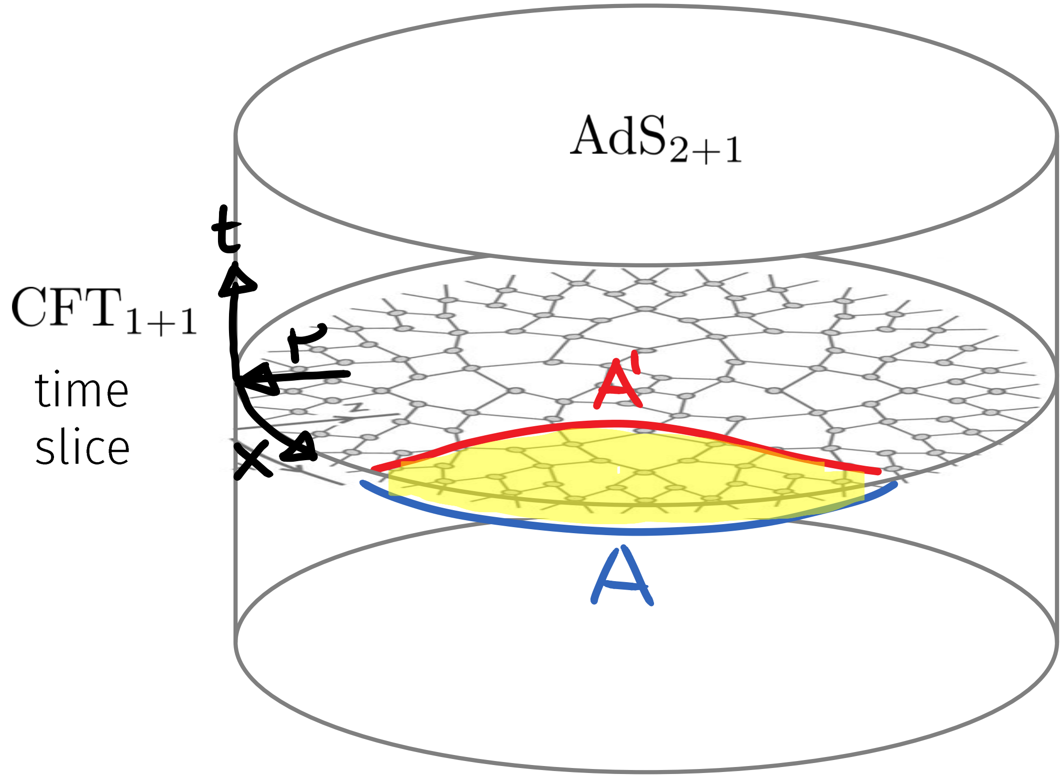
Figure from [Evenbly—Vidal]

Entanglement Entropy in Holography

[Maldacena]

Gauge/gravity correspondence ("holography"):

d-dim QFT ↔ d+1-dim gravity theories
boundary bulk



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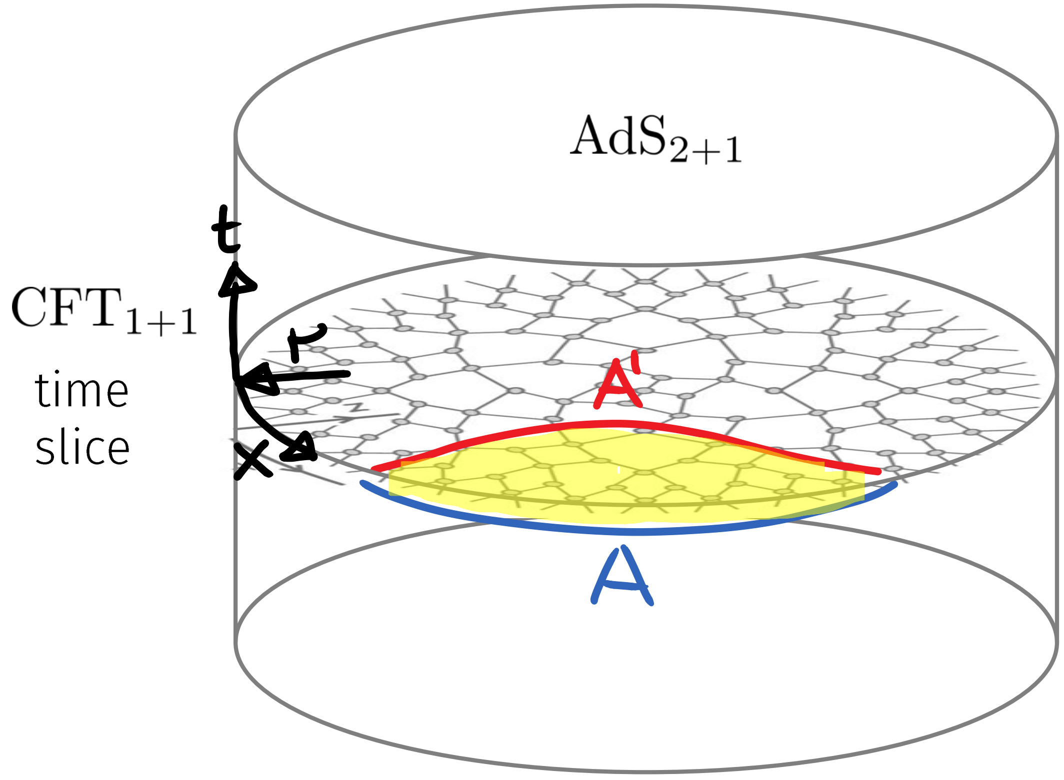
Gauge/gravity correspondence ("holography"):

large N, strongly coupled — semiclassical

d-dim QFT ↔ d+1-dim gravity theories

boundary — bulk



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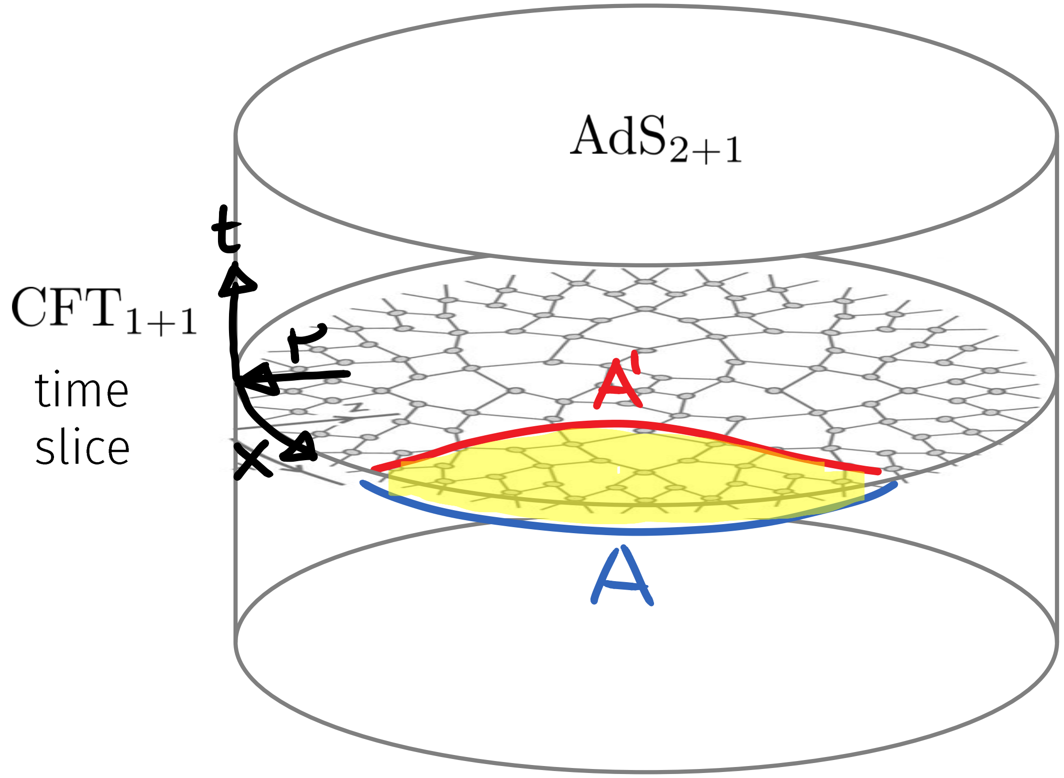


Ryu-Takayanagi formula:

$$S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$$

entanglement entropy of boundary region
= "area" of minimal (homologous) "surface" in bulk

- for time-independent states (static space-times)
- typically infinite → UV cut-off (IR cut-off)
- proved to various degrees

Entanglement Entropy in Holography

AdS_{2+1} / CFT_{1+1}

$$S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$$

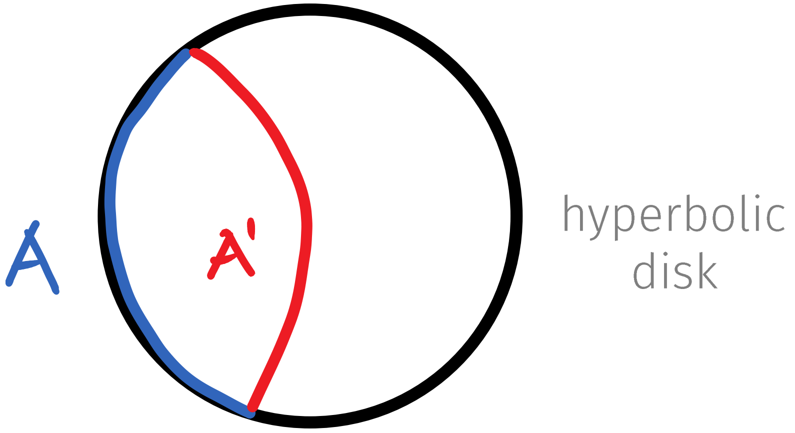
$$\frac{3R}{2G} = c$$

Vacuum state:

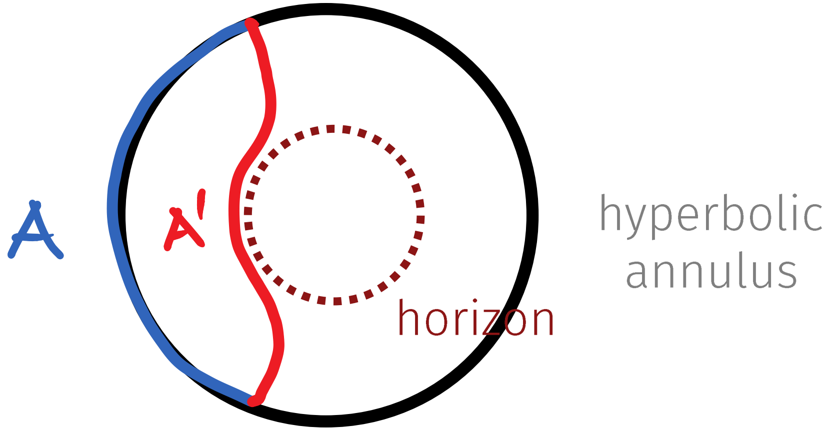
$$|\Omega\rangle_{CFT}$$

Thermal state:

$$\rho_{CFT}(\beta) = \sum_n e^{-\beta E_n} |n\rangle\langle n|$$



Empty AdS



BTZ black hole

(T large enough)

Entanglement Entropy in Holography

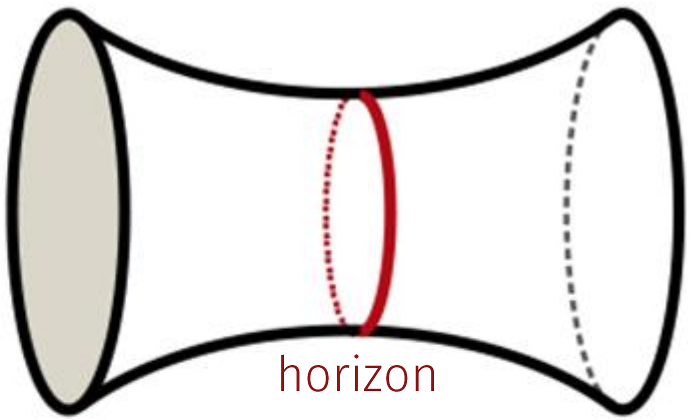
[Van Raamsdonck], [Maldacena], [M.—Susskind]

$$S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$$

"Thermofield double" state:

$$|\Psi_\beta\rangle_{CFT_L \otimes CFT_R} = \sum_n e^{-\beta E_n/2} |n\rangle |n\rangle$$

entangled state of two CFTs



no causality violation

not traversable

Einstein-Rosen bridge

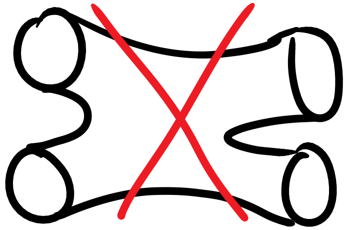
geometry from entanglement; "ER=EPR"

Properties of Holographic Entanglement Entropy

$$S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$$

- $S(A) = S(A^c)$ for pure states
- Strong subadditivity: $S(AB) + S(BC) \geqslant S(B) + S(ABC)$ [Headrick–Takayanagi]
- Monogamy: $I(A:B) + I(A:C) \leqslant I(A:BC)$ [Hayden et al]

$$\sum_n e^{-\beta E_n/2} |n\rangle |n\rangle |n\rangle |n\rangle \neq$$



This Talk

What are the holographic constraints on entropy?

When can an entangled state have a smooth dual geometry?

What are the extremal states/geometries?

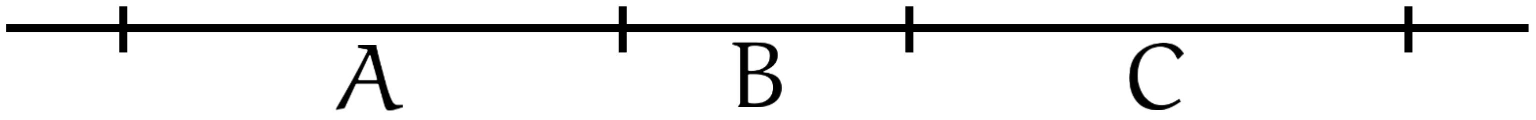
I.e., those that are on the brink of violating an entropy inequality

Constraining Holographic Entropy

Proof of strong subadditivity

[Headrick—Takayanagi]

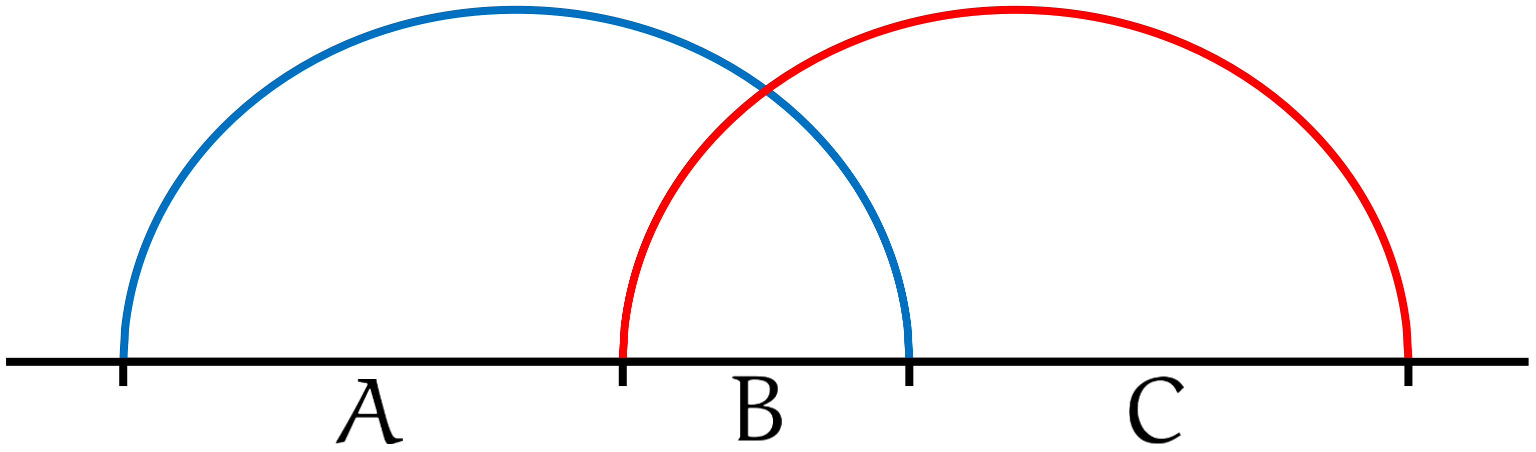
$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$



Proof of strong subadditivity

[Headrick–Takayanagi]

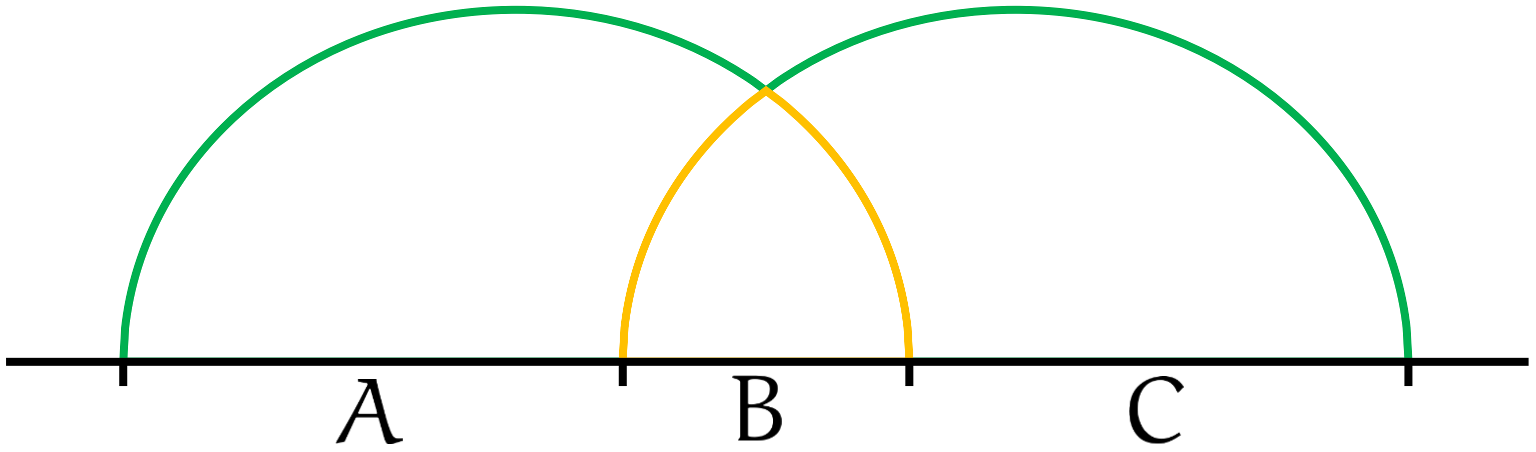
$$S(AB) + S(BC)$$



Proof of strong subadditivity

[Headrick—Takayanagi]

$$S(AB) + S(BC) =$$



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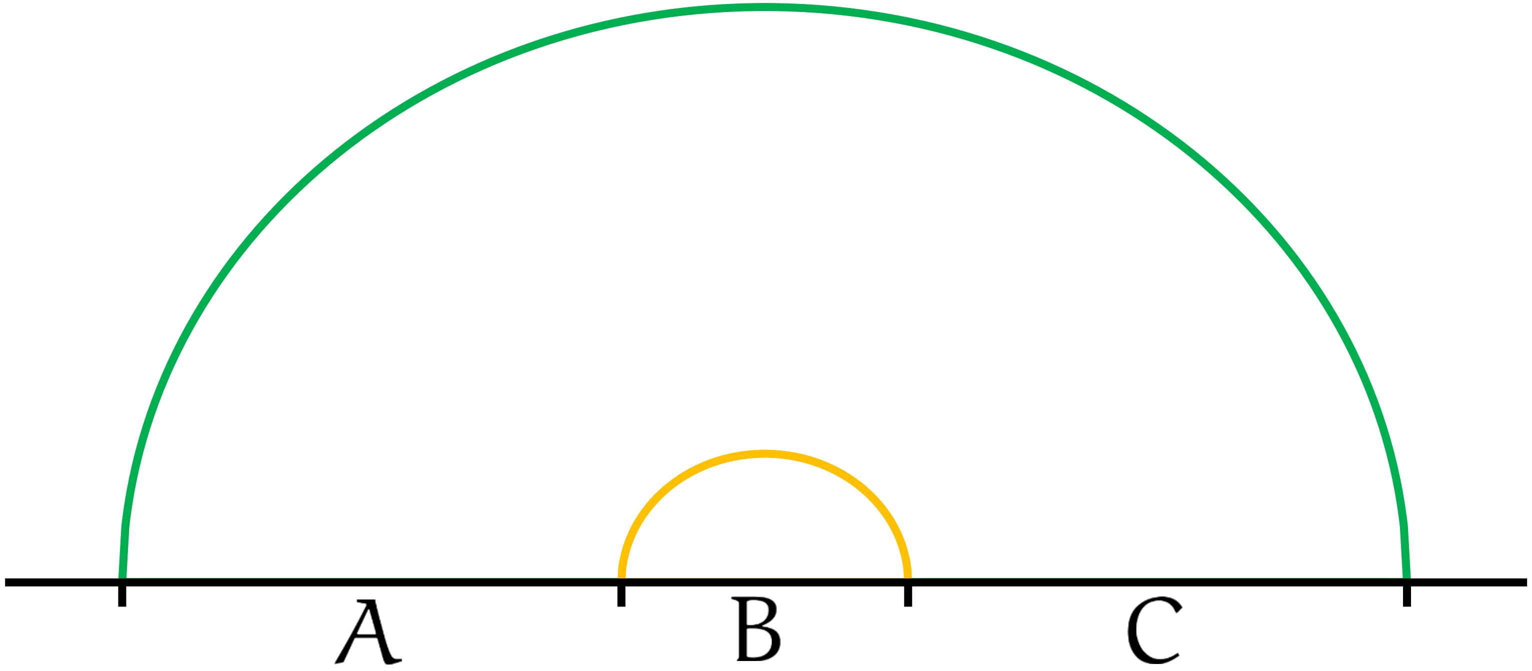


Proving holographic entropy inequalities

A similar cartoon proves monogamy inequality.

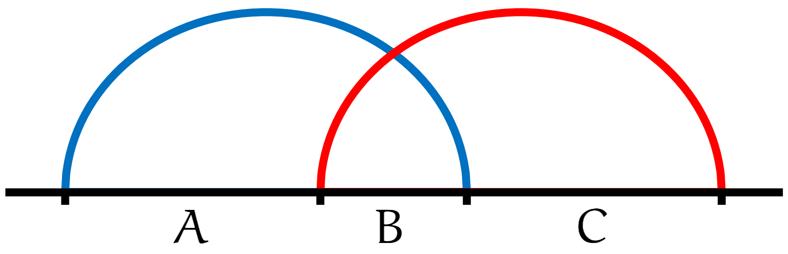
Making this precise, however, requires finding decompositions of the minimal surfaces that work in *all* geometric configurations.

We now show that there is a general combinatorial method to achieve this.

Inclusion/exclusion and the hypercube

$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$

Each minimal surface comes with a bulk region:



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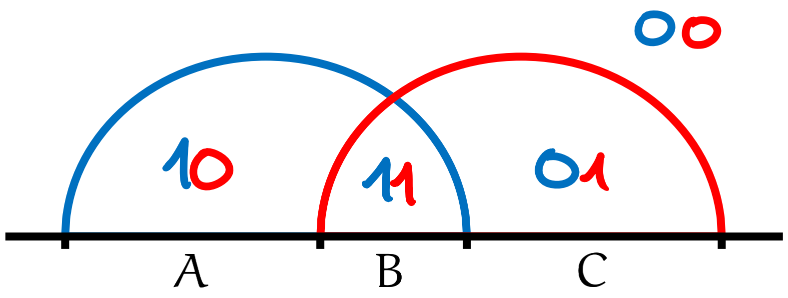

Inclusion-exclusion → bulk is cut into 2^L pieces

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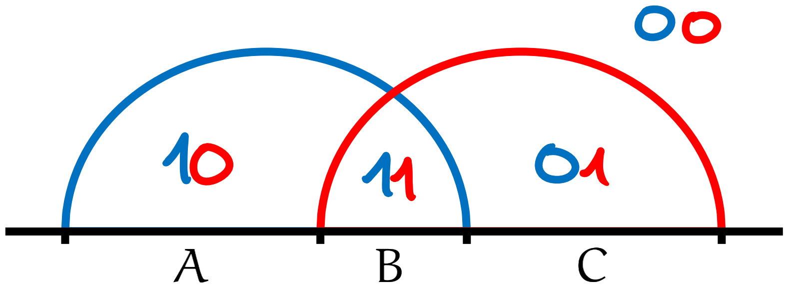
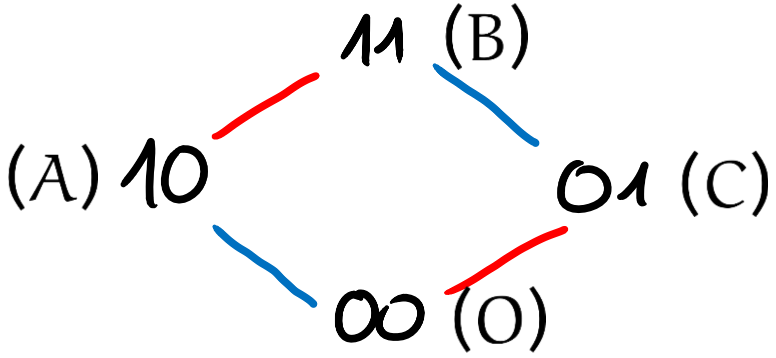


Inclusion-exclusion → bulk is cut into 2^L pieces → hypercube graph:



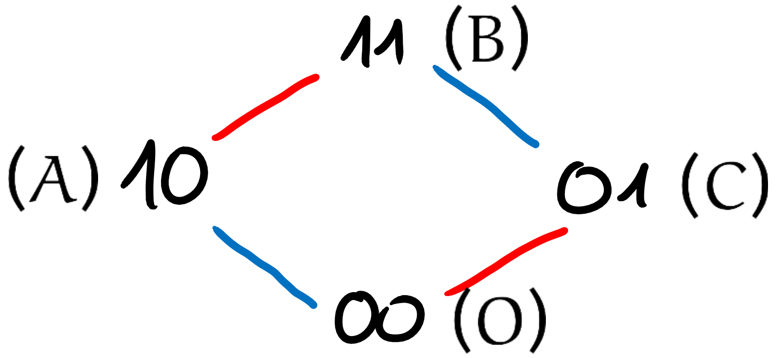
vertices → bulk pieces

edges → boundary pieces

Hypercube proofs of holographic entropy inequalities

$$S(AB) + S(BC) \geqslant S(B) + S(ABC)$$

1. Construct hypercube graph:



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Hypercube proofs of holographic entropy inequalities

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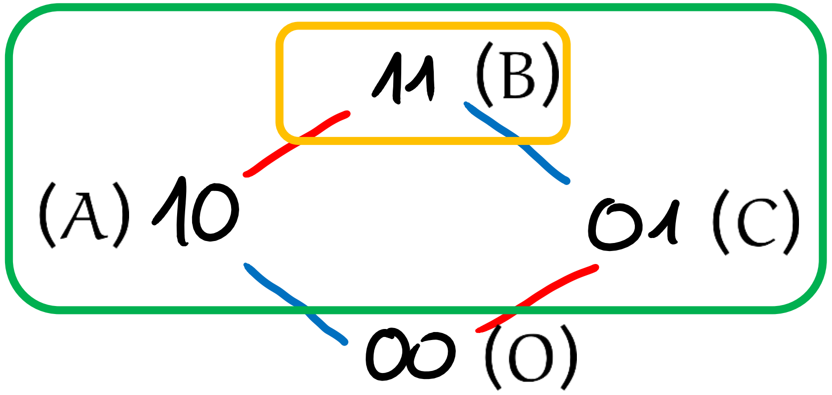


vertices → bulk pieces

edges → boundary pieces

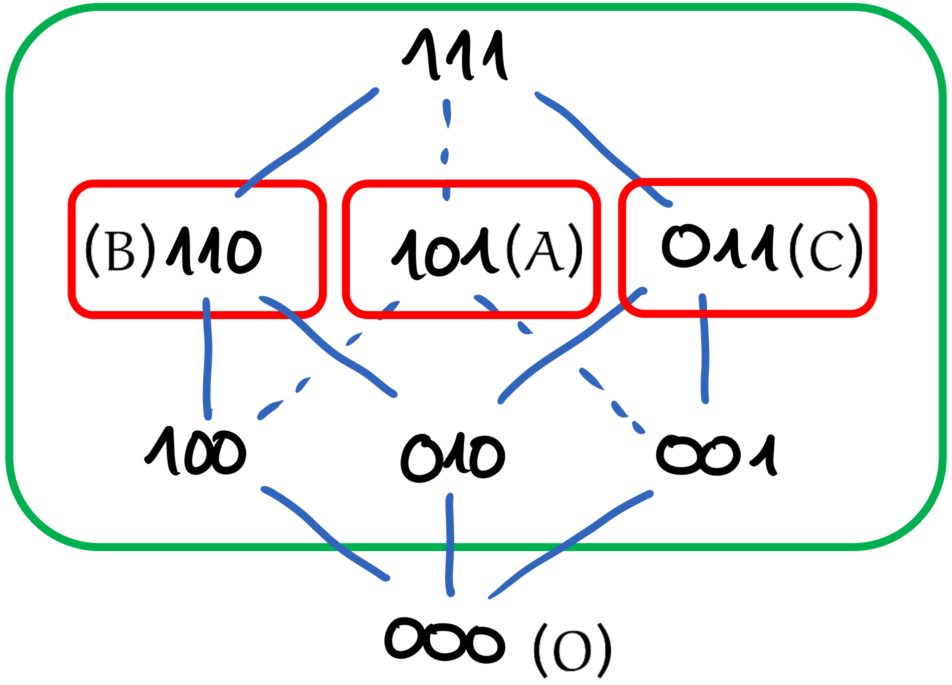
2. Choose subset for each right-hand side term s.th. each edge cut at most once.

Then the holographic entropy inequality is correct.

Surprisingly powerful: Can be fully algebraized ("proofs by contraction"); greedy algorithm; many new entropy inequalities; equality conditions; …

Hypercube proof of monogamy

$$S(AB) + S(BC) + S(AC) \geqslant S(A) + S(B) + S(C) + S(ABC)$$



The cyclic inequalities

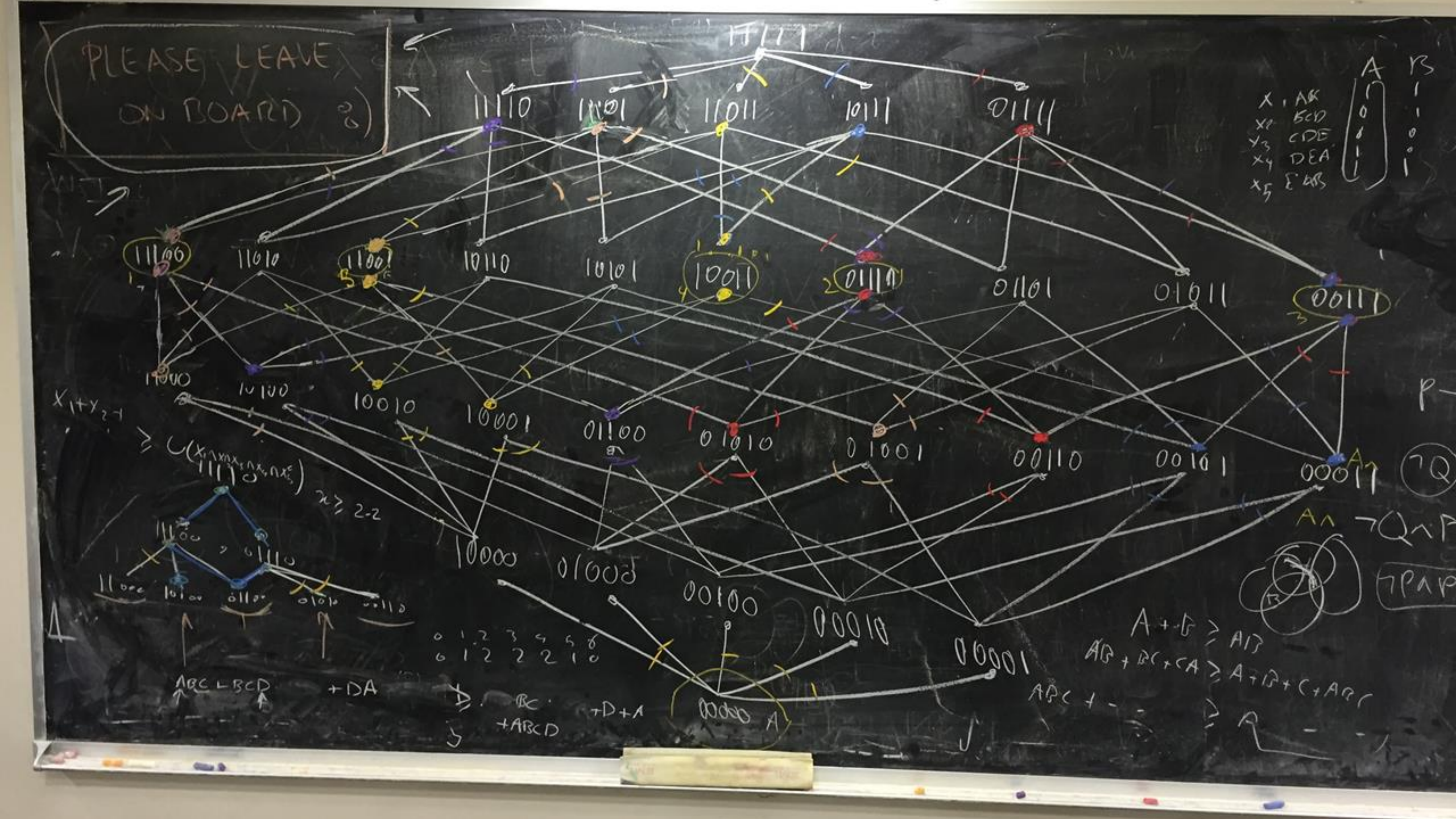
$n = 2k + 1$

Theorem. *We have*

$$\sum_{i=1}^{n} S(A_i | A_{i+1} \dots A_{i+k}) \geqslant S(A_1 \dots A_n)$$

Here, $S(A|B) = S(AB) - S(B)$ is the conditional entropy.

Part of a new, infinite family that generalizes monogamy and SSA.



PLEASE LEAVE ON BOARD
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10000
01000
00100
00010
00001
00000
A
B
X_1 ABC
X_2 BCD
X_3 CDE
X_4 DEA
X_5 EAB
$X_1 + X_2 - 1$
≥ 2-2
0 1 2 3 4 5 6
0 1 2 2 2 1 0
ABC + BCD
+DA
+ABCD
+D+A
A + B ≥ AB
AB + BC + CA ≥ A + B + C + ABC
ABC
¬Q
A ∧ ¬Q ∧ P
¬P ∧ P

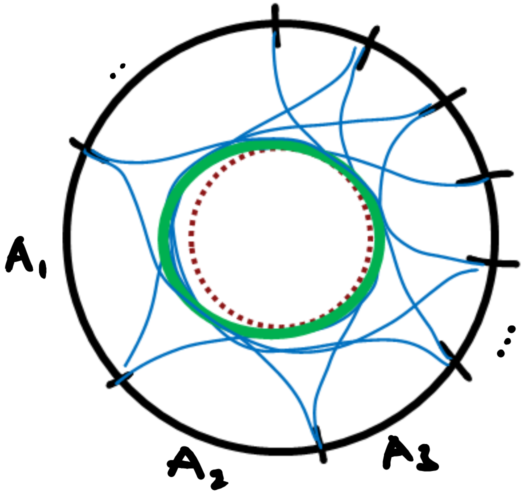
The cyclic inequalities

$$\sum_{i=1}^{n} S(A_i | A_{i+1} \dots A_{i+k}) \geqslant S(A_1 \dots A_n)$$

Interpretation for contiguous boundary regions ($n \to \infty$):

length of enveloping curve of family of geodesics
$\geqslant$ length of horizon

[Balasubramanian et al.]



But our inequality hold for arbitrary boundary regions in arbitrary geometries!

The Holographic Entropy Cone

Entropy Cones

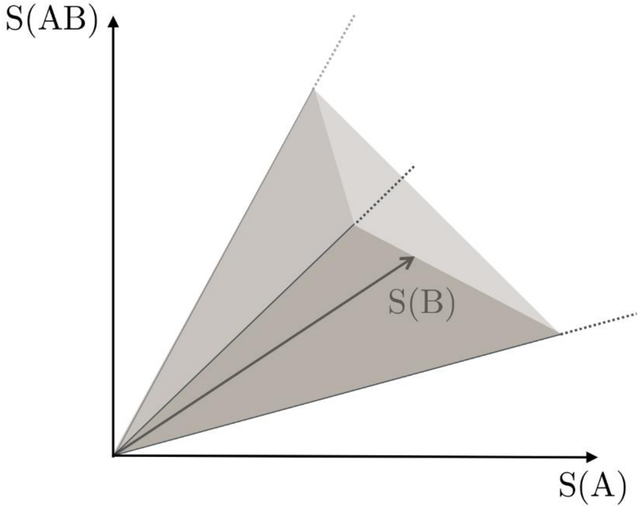
Ryu-Takayanagi entropy formula $S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$

$$\mathcal{C}_n = \{(S(A_1), \ldots, S(A_1A_2), \ldots, S(A_1 \ldots A_n)) \in \mathbb{R}^{2^n-1}\}$$

where we allow for *arbitrary* bulk geometries and boundary regions.

This is a convex cone, the holographic entropy cone.

rescaling
disjoint union



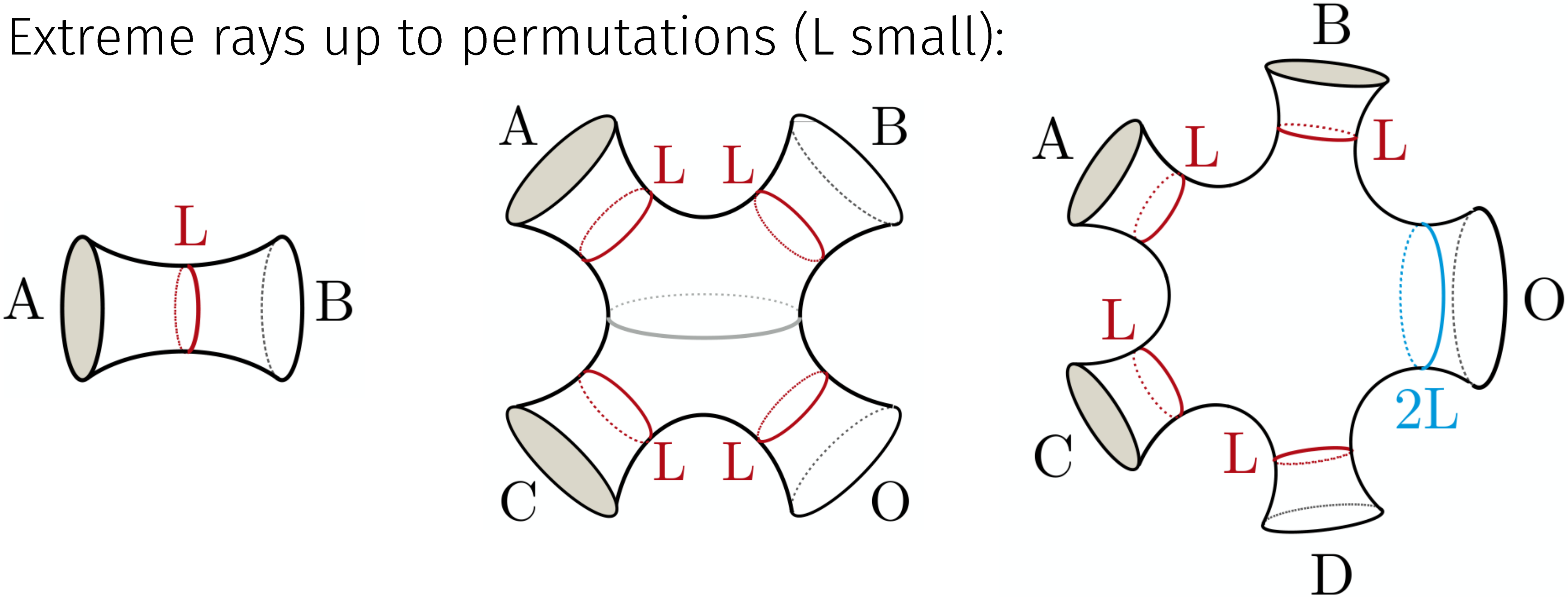
facets: entropy inequalities

extreme rays: most extreme entropy vectors

Few regions

We find that (strong) subadditivity and monogamy are sufficient for n ≤ 4 regions.

Extreme rays up to permutations (L small):

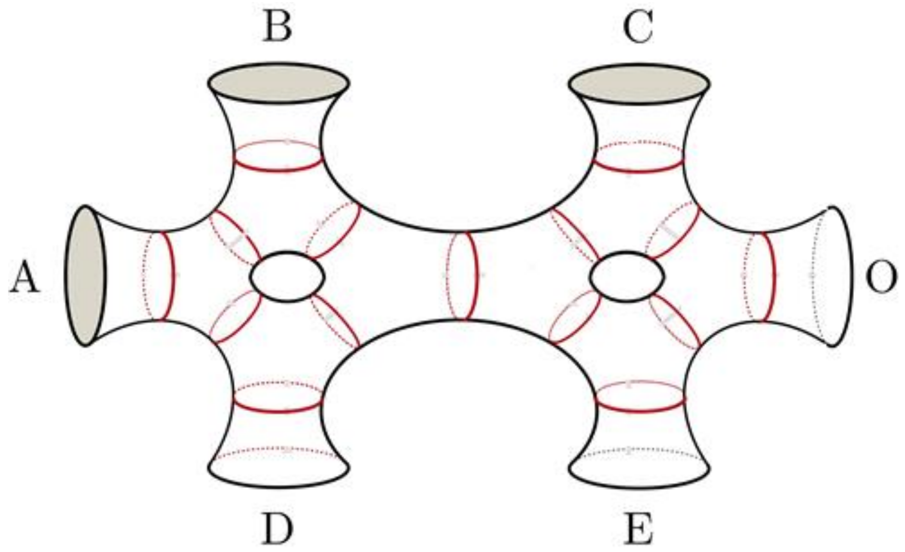


Five and more regions

For $n \geq 5$ regions, we prove several new inequalities (facets) including the family of cyclic inequalities.

New kinds of extreme rays:

- Higher genus
- Interior cycles become relevant



All can be explained by multiboundary wormhole geometries.

Aside: Quantum Error Correction

Tensor network models for AdS/CFT correspondence have been proposed that are realized via stabilizer states.

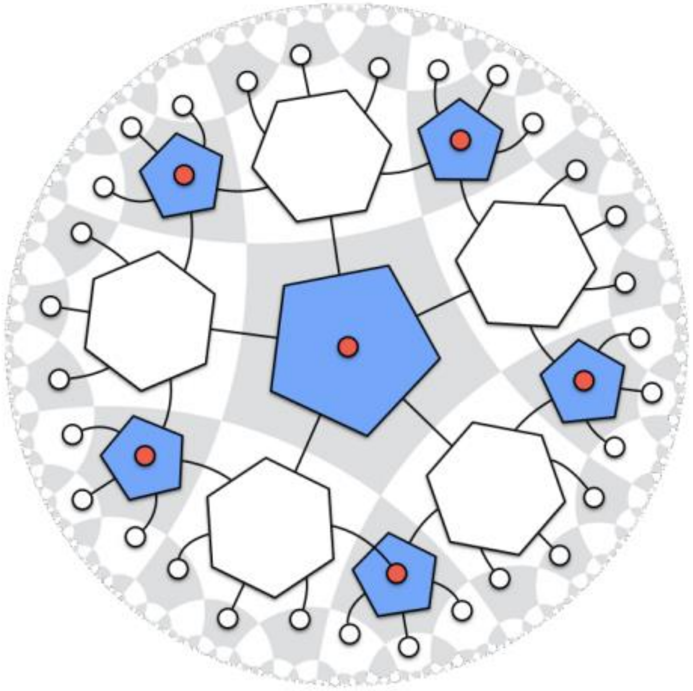


Figure from [Harlow *et al*]

This would imply that holographic entropies are stabilizer entropies.

$$\mathcal{C}_n \subseteq \mathcal{C}_n^{\text{stabilizer}} \subseteq \mathcal{C}_n^{\text{quantum}}$$

We find that all known stabilizer entropy inequalities are implied by holographic ones (n ≤ 5). ✓

Interestingly, stabilizer states have an effective classical description, too (discrete phase space).

Graph Models & Lorentzian Wormholes

Graph Models

$$S(A) = \frac{1}{4G_N} \min_{A' \sim A} |A'|$$

Goal: Combinatorial description of holographic entropies.

vertices → bulk pieces

edges → boundary pieces

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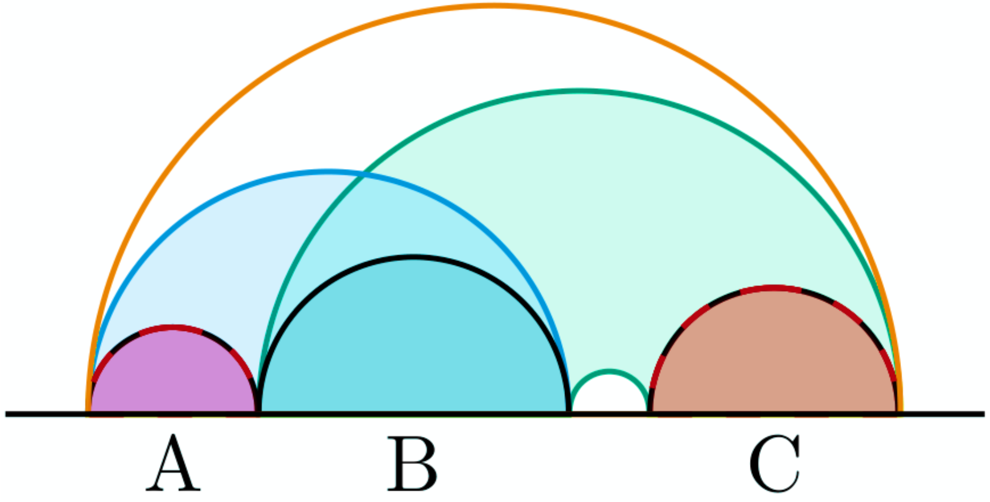

vertices → bulk pieces

edges → boundary pieces

Consider bulk pieces cut out by *all* minimal surfaces.

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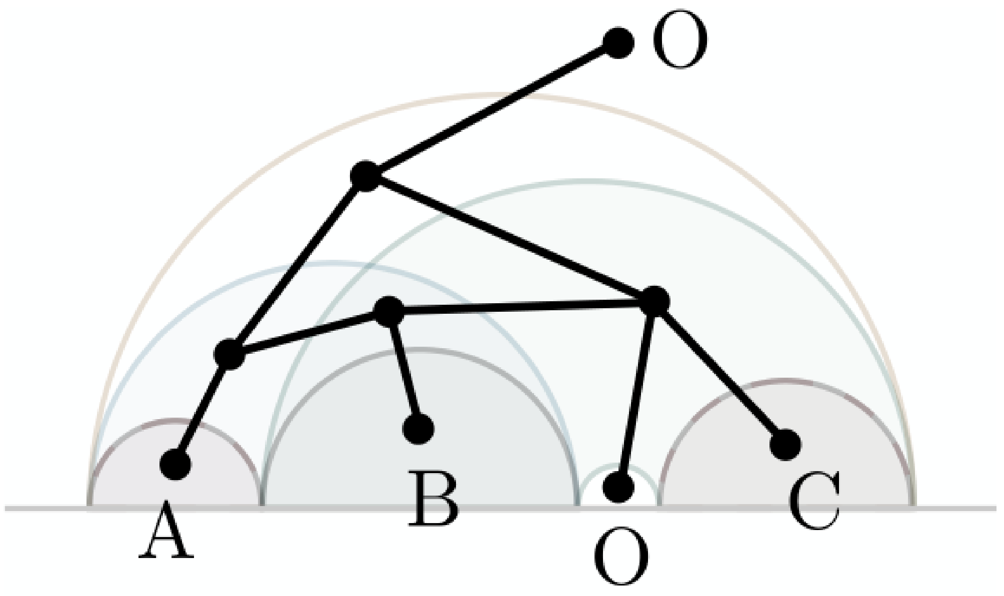

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Define corresponding dual graph. Then:

$S(A) =$ weight of minimal cut ← discrete entropy

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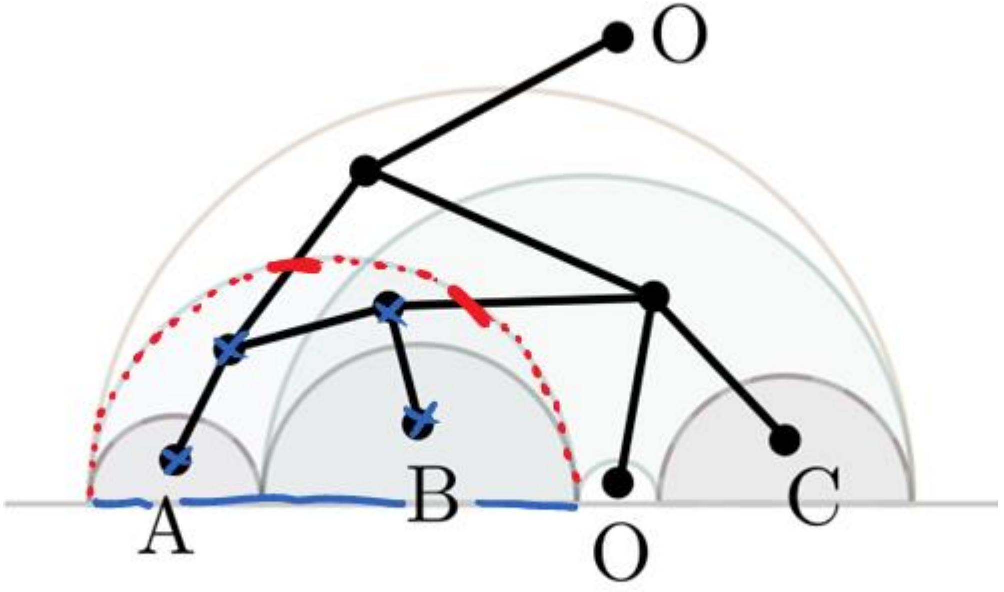

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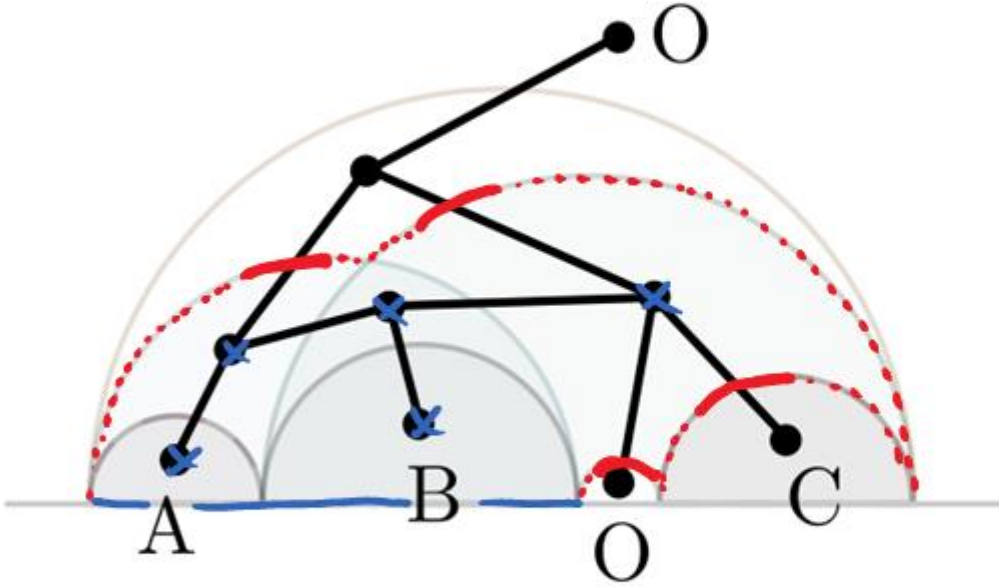

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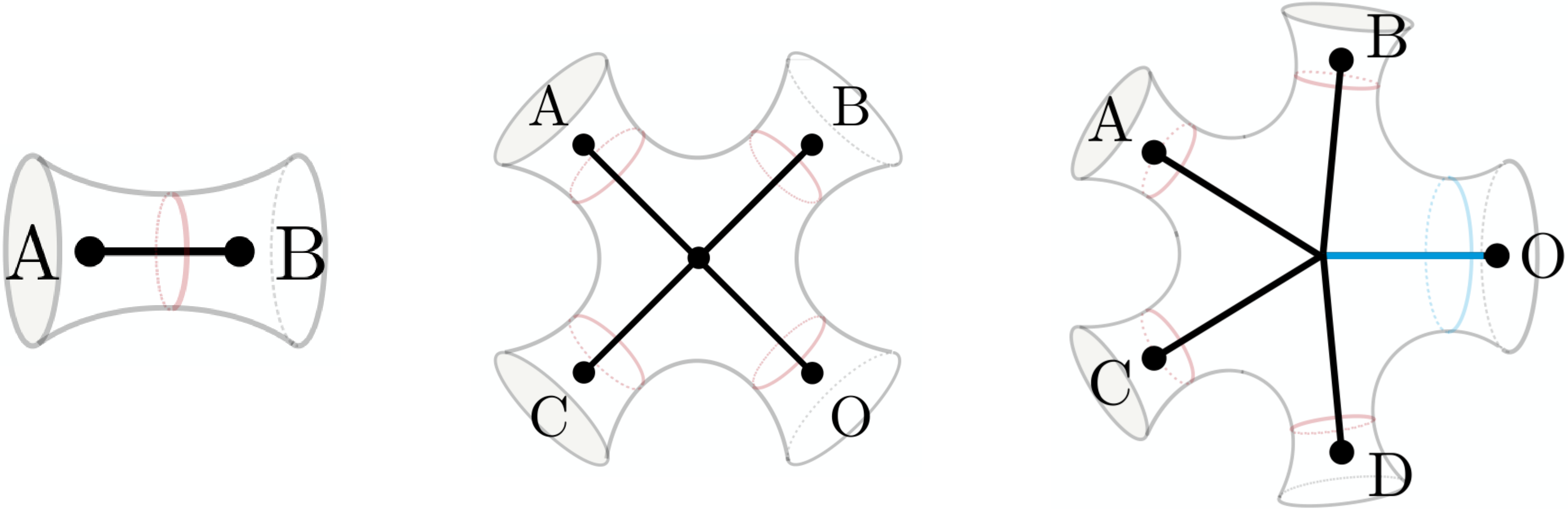
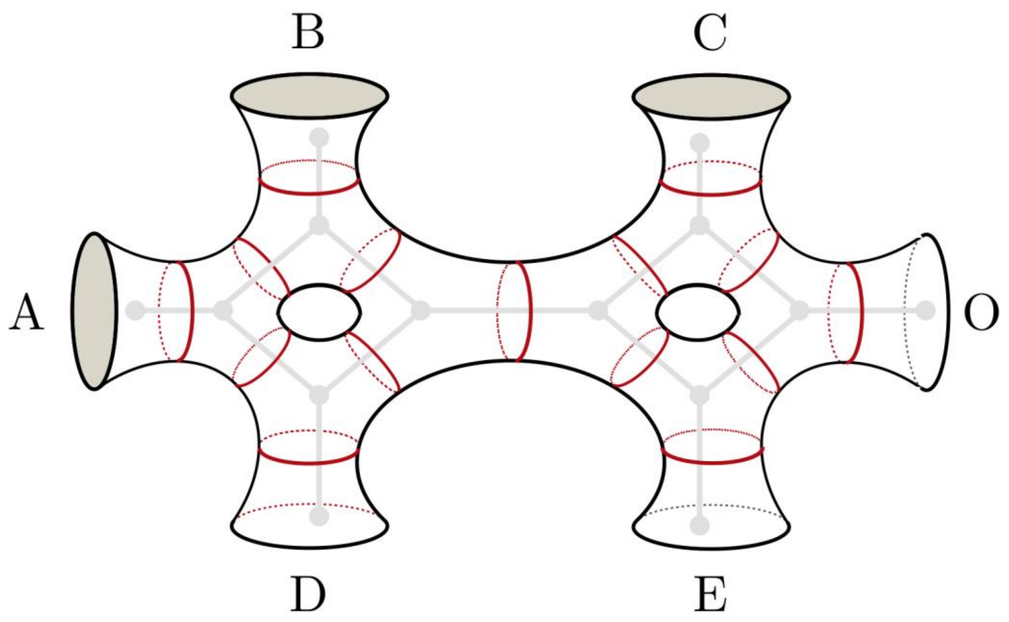
Examples of Graph Models



Discrete Entropy = Ryu-Takayanagi Entropy

Trivalent graphs determine “pair of pants” decomposition of a hyperbolic surface:



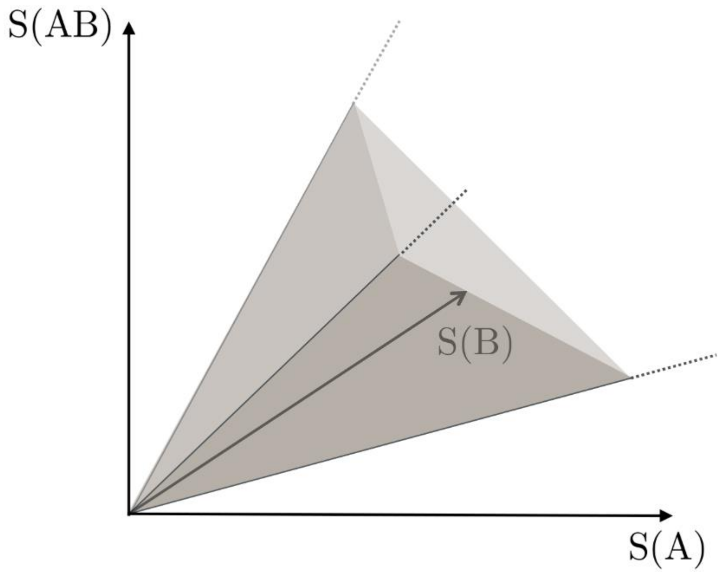
K ≪ 0: thin collars
→ S(A) = S*(A)

Graph models provide a completely equivalent, combinatorial description of holographic entropy.

Structural Insights

Holographic entropy cones are polyhedral:

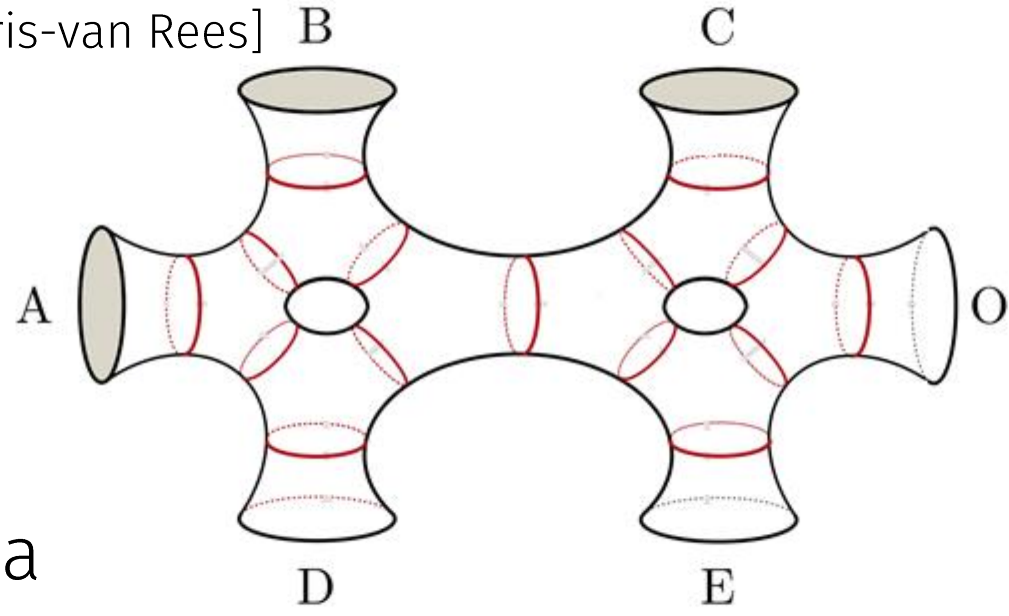
- finitely many entropy inequalities
- finitely many extreme rays



Holographic Entropy Cone & CFT

The geometries constructed from graph models can be thought of time slices of Lorentzian wormholes: [Skenderis-van Rees]

- Each asymptotic region looks like BTZ black hole
- Minimal surfaces can probe deep into the bulk
- No divergences in Ryu-Takayanagi formula



Any holographic entropy vector can be explained by multiboundary wormhole geometry.

Fine print: dual CFT state ≠ Hartle-Hawking state; spacetimes not static, but RT can be argued to hold.

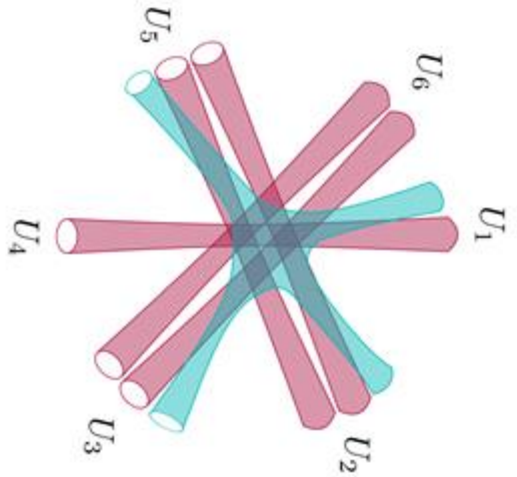
ER ≥ EPR?

Extreme rays = entropic building blocks (for fixed n)

- some require multipartite entanglement

Convex combination = disjoint union

- explains entropies, but not very physical



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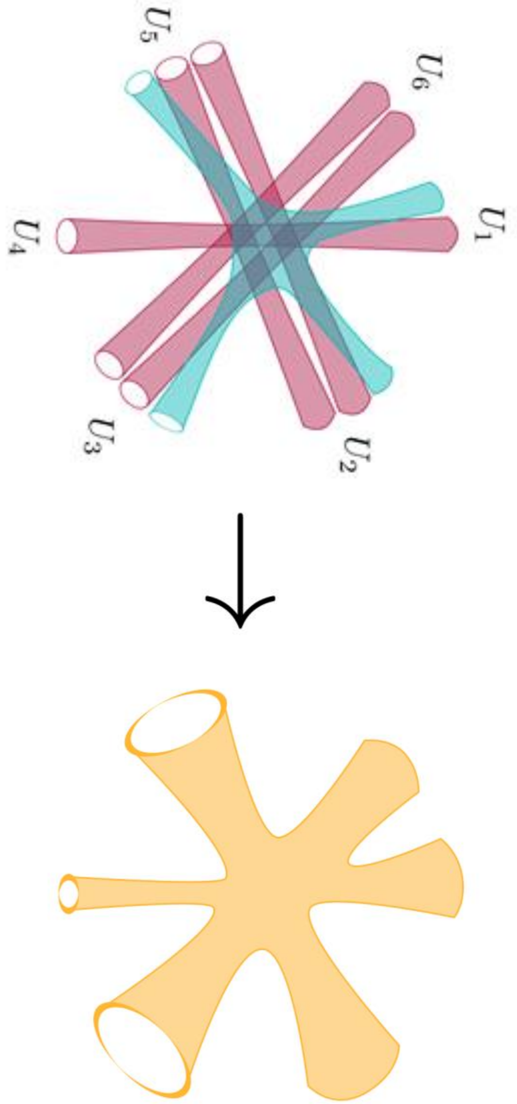


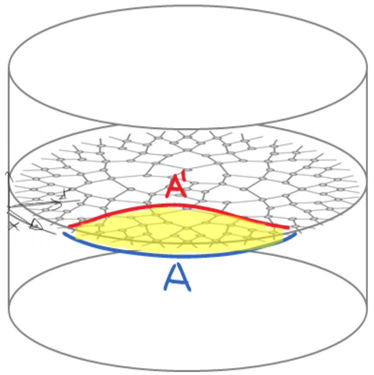
Can we use local unitaries on the CFT state to “stitch together” the geometry?
Can all smooth geometries be obtained in this way?
Can we identify building blocks in a stronger sense?

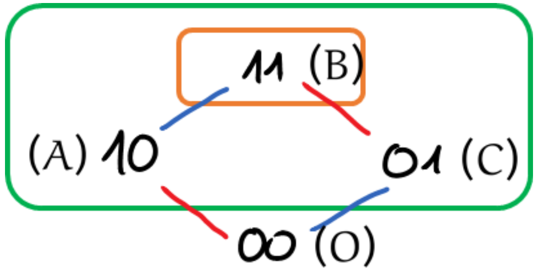
Summary

arXiv:1505.xxxx

Holographic entropy inequalities:

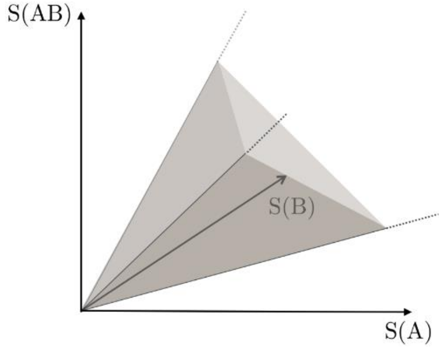
- hypercube proofs
- found several new inequalities

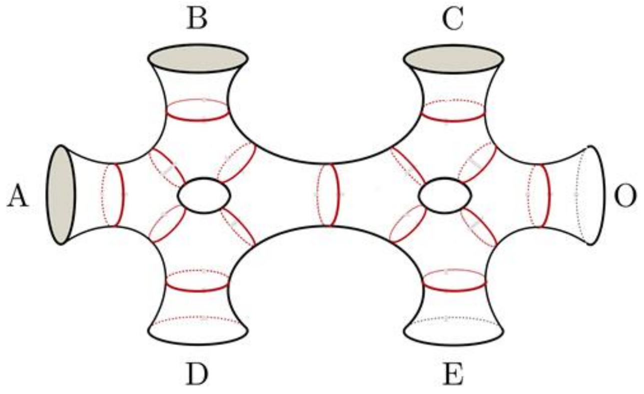




Holographic entropy cone

- Surprising new features for n≥5
- Graph models &
  Lorentzian wormholes





Many open questions: HRT, multipartite entanglement, cond-mat, ...

Thank you for your attention