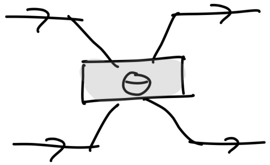

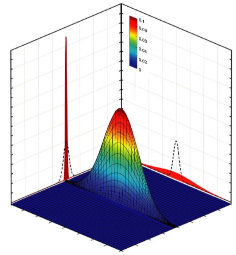

EQuIP TNT

Robust Tests of Gaussianity

Michael Walter

Munich, Oct 2025

Joint works with Filippo Girardi, David Gross, Francesco Mele, Sepehr Nezami, Freek Witteveen, Lennart Bittel, Salvatore Oliviero

RUHR UNIVERSITÄT BOCHUM RUB → LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

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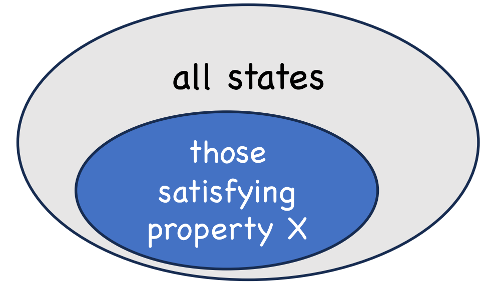
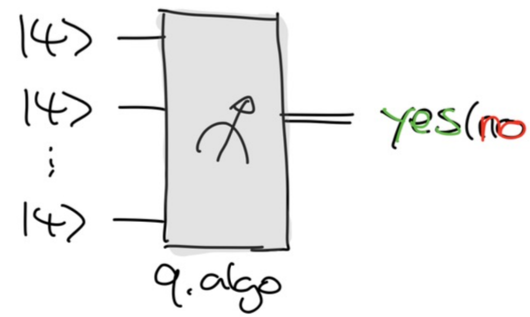
Property testing



For a set X of quantum states, want algo that takes copies of *unknown state* ρ as input and decides between:

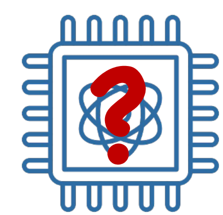
- **Yes**, ρ is in X
- **No**, ρ is ϵ-far from X



Question: Which properties of quantum states can be tested **efficiently**?

small # of samples (copies), simple circuits, ...

Why care? Conceptually interesting, but also tells us which many-body properties that can be *practically* verified...



Many variations, e.g. tolerant testing, streaming model, unitary queries, ...

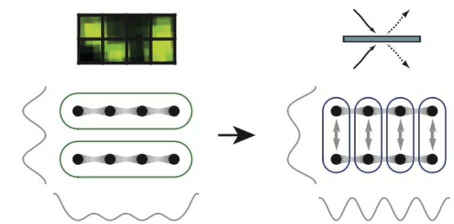
Property testing in the real world

Article | Published: 02 December 2015

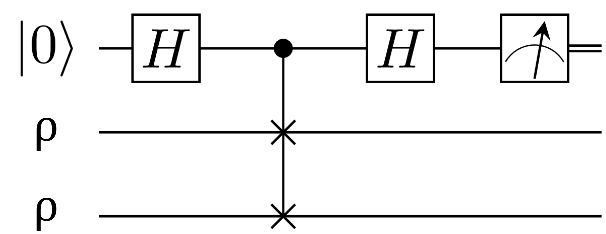
Measuring entanglement entropy in a quantum many-body system

Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli & Markus Greiner

Nature **528**, 77–83 (2015) | Cite this article



Swap Test: uses 2 copies, acceptance probability related to **purity** tr ρ^2



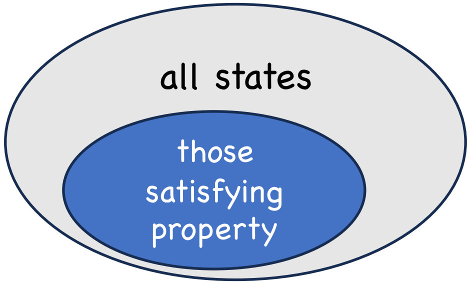
➔ "O(1) copies suffice to test if state is pure (or far from it)"

More precisely: O(1/eps) copies suffice to test if pure or eps-far from pure, with constant probability of error

➔ useful not just in practice, but also in theory!

Buhrman et al, Harrow-Montanaro, Watrous, ...

Examples and surprises

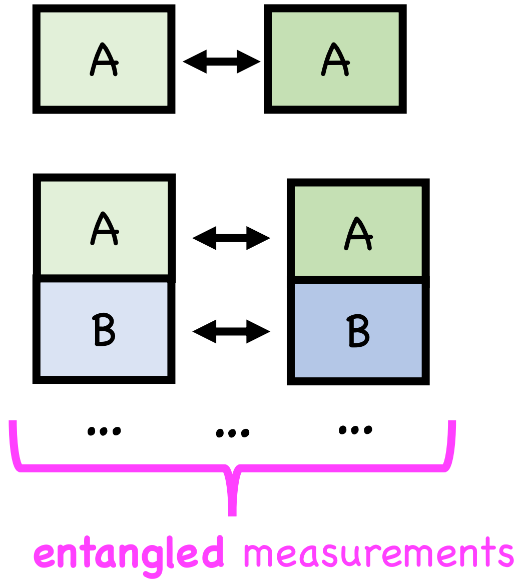


Sometimes few samples suffice, and sometimes not:

Purity: O(1) copies ☺
Buhrman et al

Product: O(1) copies ☺
Brandao-Harrow

Mixedness: $\Theta(2^n)$ copies ☹
Childs et al



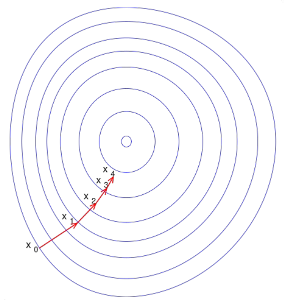
What if we restrict to **single-copy measurements**? In this case there can be an exponential disadvantage! ☹

Qubits vs bosons

Quantum computing is best developed in **finite dimension** $(\mathbb{C}^2)^{\otimes n}$
➔ quantum circuits, universality, complexity theory, ...

For bosonic (a.k.a. continuous-variable) quantum systems
even the right notion of complexity is not so clear (to me). $L^2(\mathbb{R}^n)$

→ talks by Simon, Ulysse, ..., recent work by Robert et al



In contrast, classical researchers routinely design algorithms that work with real numbers – think gradient descent!

Property testing and learning tasks can provide useful proving ground:
sample complexity already interesting, algos often turn out "practical"...

Gaussian states and unitaries

$L^2(R^n)$

$X_1,\ldots,X_n,$
$P_1,\ldots,P_n$

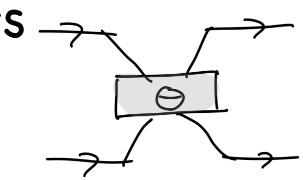
$R_1,\ldots,R_{2n}$

A pure state is Gaussian if given by (complex) multivariate Gaussian wavefunction.

➔ Fully described by 2n-dimensional mean and covariance:

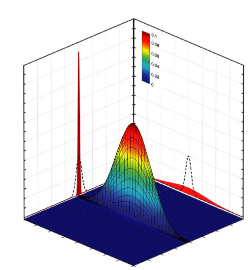
$\mu_j = \mathrm{tr}\ \rho R_j$ $\Sigma_{ij} = \mathrm{tr}\ \rho\{R_j - m_j, R_k - m_k\}$

➔ Generated by Gaussian unitaries a.k.a. linear quantum optics *(beam splitters, squeezing, …):*



$\mu \rightarrow S\mu + d$
$\Sigma \rightarrow S\Sigma S^T$

where S = *symplectic* matrix



➔ Phase

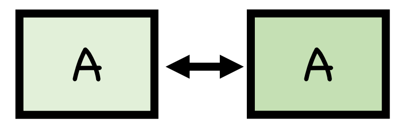
Question: Can we efficiently test if a given bosonic quantum state is Gaussian, or far from it?

Classically simulable. Very similar to Clifford unitaries & stabilizer states.

Warmup: Testing by symmetry

Recall **purity**:

ρ is pure ⇔ $F\,\rho^{\otimes 2} = \rho^{\otimes 2}$ swap-invariant

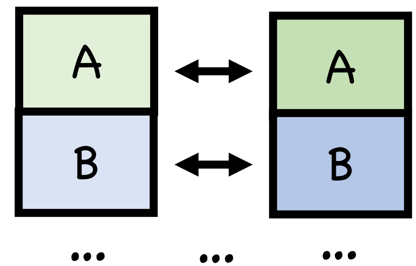


U(d)

S_2

Recall **productness**:

ρ is pure product state ⇔ $F_A\,\rho^{\otimes 2} = F_B\,\rho^{\otimes 2} = \ldots = \rho^{\otimes 2}$ locally swap-invariant



$U(d_A) \times U(d_B) \times$

$S_2 \times S_2 \times$

In both cases:
- states form a single group orbit
- two copies have an enhanced symmetry ➔ natural test
- it is true, but not (fully) obvious that this test is robust

Smaller group ⇔ subset of states ⇔ larger symmetry.

Symmetry of Gaussians

Classical facts: If X is Gaussian with mean μ & covariance Σ...

linear transformations:

$\mu \to L\mu$
$\Sigma \to L\Sigma L^T$

t copies are again Gaussian

$\mu \to \mu \otimes 1_t$
$\Sigma \to \Sigma \otimes I_t$

t copies have **enhanced symmetry**
& this characterizes Gaussians!

permutations in S_t → orthogonal matrices in $O(t)$

stochastic if $\mu \neq 0$

In fact, a **45 degree rotation** is enough (if $\mu=0$). $(X,Y) \to (X+Y,-X+Y)/\sqrt{2}$

Folklore: These are also quantum facts ☺

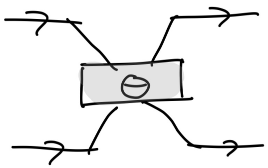
cf. Kac–Bernstein, Diaconis' Gaussian de Finetti, ...

Result: Gaussianity testing by symmetry

Quantum fact: A pure state ρ is Gaussian ⇔ $\rho^{\otimes 2}$ invariant under (stochastic) rotations in $O(t)$

$L^2(\mathbb{R}^n)^{\otimes t}$
$= L^2(\mathbb{R}^{n \times t})$

cf. Springer et al, Wolf et al (Gaussian extremality), König-Smith (entropy power), Leverrier (Gaussian q. de Finetti), Cuesta ("robust" fact), Bu-Li, Hahn-Takagi (test), ..., *hands-on calculation* ☺



We show that this gives rise to an efficient test:

Result: $O(\max(\varepsilon^{-4}, n^8 E^8))$ copies suffice for Gaussianity, via rotation test that uses t=2 (3) copies at time.

E = "energy" per mode

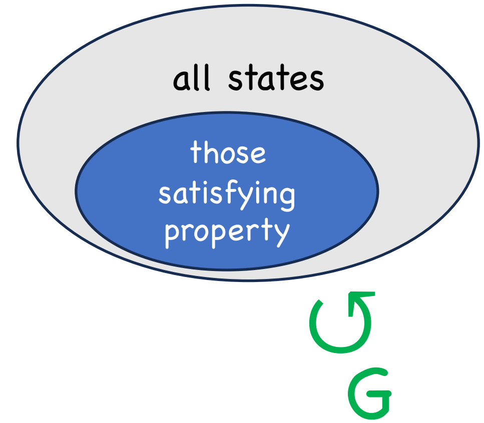
Intuition: G = XP − PX generator of 2d rotations. WLOG **Σ** diagonal. Then:

$$\langle G^2 \rangle = \langle X^2 + P^2 \rangle^2 - 1$$

harmonic oscillator, gapped, ground state Gaussian

We also give a "tolerant" tester with guarantees too ugly to fit the slides.

Yoga of the commutant



General setup:

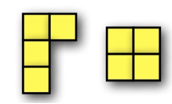
- A group G acts on the single-copy Hilbert space H
- Property is G-invariant (e.g., a single orbit)

Principle: Optimal t-copy test can always be taken in commutant of t^{th} tensor-power action.

$g^{\otimes t} \circlearrowleft H^{\otimes t}$

$[???, g^{\otimes t}] = 0$

Moreover, "generic" operator is natural candidate for test!



➔ purity and product testing: $U^{\otimes t}$ *vs* S_t — Schur-Weyl

➔ Gaussianity testing: ${U_{Gaussian}}^{\otimes t}$ *vs* stochastic $O(t)$ — Kashiwara-Vergne-Howe

also for fermions

In fact, same strategy applied to **stabilizer testing** motivated this work in the first place.

Gross-Nezami-**W**, Nebe-Scheeren, Bittel et al

➔ many applications in quantum TCS, many-body physics, …

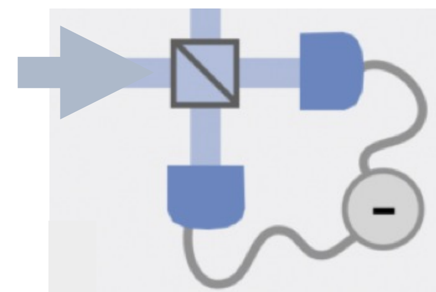
Result: Unentangled measurements

Recall that Gaussian states are described by covariance **Σ** (and mean).

Minimal uncertainty principle: For Gaussian states, the *symplectic* eigenvalues of **Σ** are = 1, and otherwise > 1.

$\Sigma \geq i\,\Omega$

Idea: Tomograph **Σ** using "homodyne" measurements, and test if symplectic eigenvalues ≈ 1.



Mele et al

Result: ε^{-8} poly(E,n) copies suffice to test Gaussianity *using single-copy measurements.*

Similarly, O(n) copies suffice for single-copy stabilizer testing. Hinsche-Helsen

Result: Lower bounds

We saw: Gaussianity can be tested efficiently, using poly(n,E) copies.

Question: Is Gaussianity testing even possible with # of copies that is *independent* of # of modes and energy?

Partial answer: Yes, if $\varepsilon \leq \varepsilon_0$ using the 45-degree rotation test.

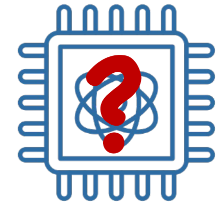
There are also Gaussian mixed states. Can those be tested efficiently?

"No go" result: Even restricted to bounded energy states, exp(n) copies are required to test if a mixed state is Gaussian or 1/poly(n)-far from it.

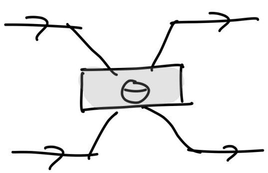
can this be constant?

Rough idea: Valiant-Valiant construct hard-to-distinguish classical distributions, from "any" starting distribution.
➔ Apply to squared amplitudes of thermal state
➔ Good *quantum* Gaussianity test would imply *classical* contradiction.

Summary



Property testing asks which many-body properties can be practically verified, and which *cannot*.



Here we focused on Gaussian states, which are of conceptual interest very widely used.

We found new mathematical tools and quantum protocols to robustly verify Gaussianity, and a "no go" for mixed states.

Symmetry and learning theory techniques that could be of independent interest. Many interesting open problems...

Thank you for your attention!