Quantum circuits for the Dirac field in 1+1 dimensions

Michael Walter

UNIVERSITY OF AMSTERDAM · QuSoft Research Center for Quantum Software · NWO

It from Qubit seminar, Stanford, May 2019

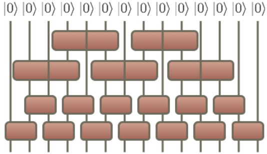
joint work with Freek Witteveen, Volkher Scholz, Brian Swingle (arXiv:1905.08821)

Tensor networks

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle$$

Efficient variational classes for many-body quantum states:

matrix product states



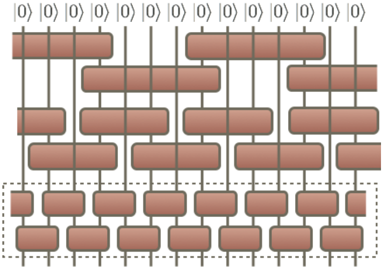
MERA

- can have interpretation as quantum circuit

Useful theoretical formalism:

- geometrize entanglement structure: *generalized area law*
- bulk-boundary dualities: *lift physics to the virtual level*
- quantum phases, topological order, RG, holography, ...

MERA multi-scale entanglement renormalization ansatz (Vidal)



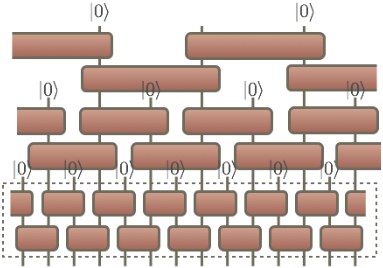
↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

↑ entanglement renormalization

↕ organize q. information by scale

- self-similar layers that are short-depth quantum circuits
- variational class for critical systems in 1D
- interpretation: disentangle & coarse-grain
- network arises from tensor network renormalization:

MERA multi-scale entanglement renormalization ansatz (Vidal)

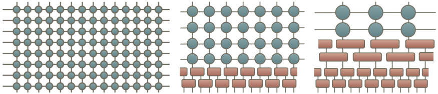


↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

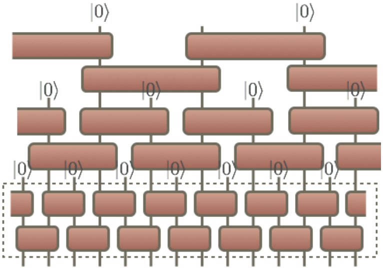
↑ entanglement renormalization

↕ organize q. information by scale

- self-similar layers that are short-depth quantum circuits
- variational class for critical systems in 1D
- interpretation: disentangle & coarse-grain
- network arises from tensor network renormalization:



MERA multi-scale entanglement renormalization ansatz (Vidal)

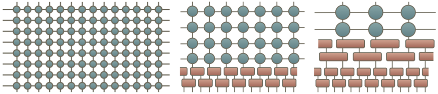


↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

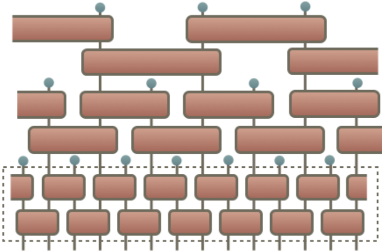
↑ entanglement renormalization

↕ organize q. information by scale

- self-similar layers that are short-depth quantum circuits
- variational class for critical systems in 1D
- interpretation: disentangle & coarse-grain
- network arises from tensor network renormalization:



MERA and holography



$\downarrow$ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

$\uparrow$ entanglement renormalization

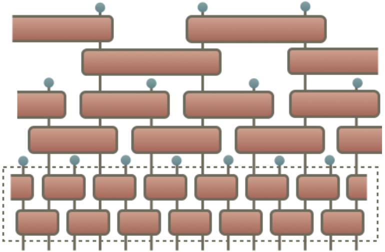
$\updownarrow$ organize q. information by scale

- can always extend to 'holographic' mapping (Qi)

- hyperbolic geometry (Swingle)
- starting point for tensor network toy models of holography (HaPPY; Hayden-...-W.)

- quantum error correction property = noise-resilience on QC (Kim et al)
  $\rightsquigarrow$ important to understand design principles

MERA and holography

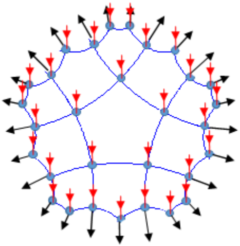


↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

↑ entanglement renormalization

↕ organize q. information by scale

- can always extend to 'holographic' mapping (Qi)



- hyperbolic geometry (Swingle)
- starting point for tensor network toy models of holography (HaPPY; Hayden-...-W.)
- quantum error correction property = noise-resilience on QC (Kim et al)
  $\rightsquigarrow$ important to understand design principles

Tensor networks and quantum field theory

Tensor networks are discrete and finitary representations, while QFTs have infinite # of degrees of freedom and are defined in the continuum.

Two successful approaches:

- *modify ansatz* $\rightsquigarrow$ *continuum tensor networks* (cMPS, cMERA, ...)
- *connect discrete ansatz to continuum theory* (MPS, PEPS, MERA, ...)

Questions:

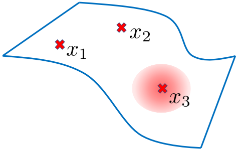
- what do tensor networks capture?
- how to measure goodness of approximation?
- can we give rigorous construction principles?
- why do tensor networks work well?

cf. plethora of results on gapped 1D lattice systems in QIT/cond-mat

Tensor networks for correlation functions

Given many-body system in state ρ and choice of operators $\{O_\alpha\}$, consider correlation functions:

$$C(\alpha_1, \cdots, \alpha_n) = \mathrm{tr}[\rho O_{\alpha_1} \cdots O_{\alpha_n}]$$



Goal: Design tensor network for correlation functions!

- unified perspective: system can be continuous – discreteness imposed by how we probe it
- tensor network for ρ sufficient (if possible), but likely suboptimal

Examples: Zaletel-Mong (MPS/q. Hall states), König-Scholz (MPS/CFTs), cf. *quantum marginal problem*

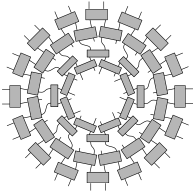
Our results

Result (informal)

We construct tensor networks for the Dirac CFT in 1+1 dimensions.

Key features:

- tensor networks target correlation functions
- rigorous approximation guarantees
- entanglement renormalization quantum circuits
- explicit construction, no variational optimization



We achieve this using tools from signal processing: multiresolution analysis and discrete/continuum duality in wavelet theory.

Majorana and Ising CFT from sub-circuits. In prior work, we constructed (branching) MERA for free fermions on 1d and 2d square lattice (Fermi surface!).

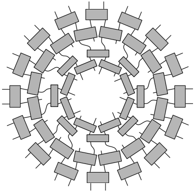
Our results

Result (informal)

We construct tensor networks for the Dirac CFT in 1+1 dimensions.

Key features:

- tensor networks target correlation functions
- rigorous approximation guarantees
- entanglement renormalization quantum circuits
- explicit construction, no variational optimization



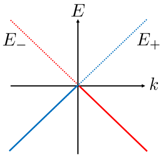
We achieve this using tools from signal processing: multiresolution analysis and discrete/continuum duality in wavelet theory.

Majorana and Ising CFT from sub-circuits. In prior work, we constructed (branching) MERA for free fermions on 1d and 2d square lattice (Fermi surface!).

In more detail…

Massless Dirac fermion in 1+1d: $i\gamma^\mu \partial_\mu \psi = 0$

Easily solved using Fourier transform. But *not* using geometrically local quantum circuit/tensor network…



We construct networks that target vacuum correlation functions:

$$C(\{O_i\}) := \langle O_1 \cdots O_n \rangle$$

of smeared fields or normal-ordered bilinears (e.g., T, L_n)

Result (simplified)

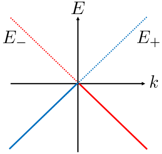
$$C_{\text{exact}} \approx C_{\text{MERA}}$$

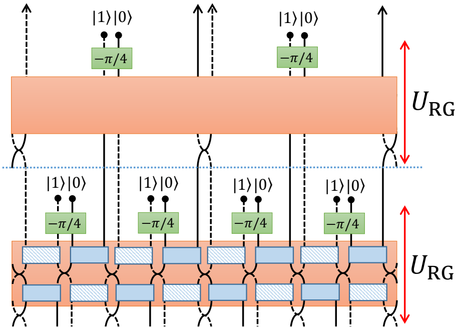
Goodness depends on quality parameter and # layers. Rigorous approximation!

In more detail...

Massless Dirac fermion in 1+1d: $i\gamma^\mu \partial_\mu \psi = 0$

Easily solved using Fourier transform. But *not* using geometrically local quantum circuit/tensor network...





We construct networks that target vacuum correlation functions:

$$C(\{O_i\}) := \langle O_1 \cdots O_n \rangle$$

of smeared fields or normal-ordered bilinears (e.g., T, L_n)

Result (simplified)

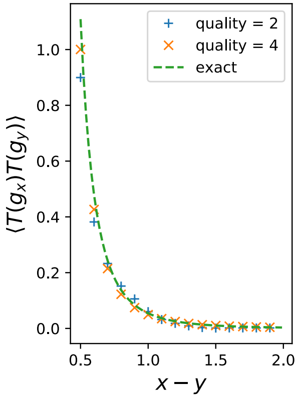
$$C_{\text{exact}} \approx C_{\text{MERA}}$$

Goodness depends on quality parameter and # layers. Rigorous approximation!

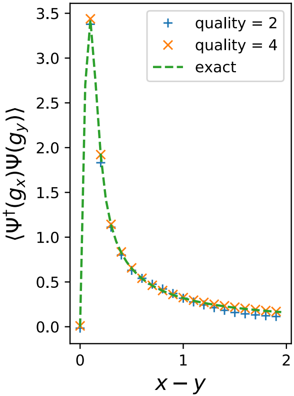
1D Dirac fermion – Numerics for two-point functions

For different values of quality parameter and large number of layers:

$\langle T(x)T(y)\rangle$



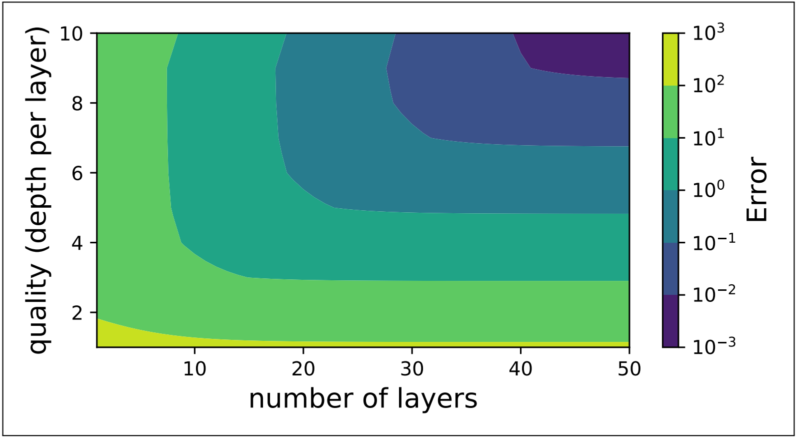
$\langle \Psi(x)^\dagger \Psi(y)\rangle$



Similarly for higher-point functions.

1D Dirac fermion – Numerics for two-point functions

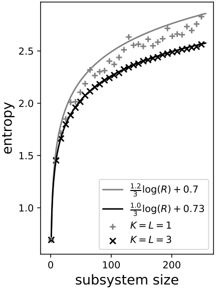
For different values of quality parameter and number of layers:



Similarly for higher-point functions.

1D Dirac fermion – Verifying conformal data

- central charge: $S(R) = \frac{c}{3} \log R + c'$



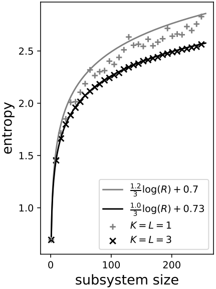
- usual procedure: identify fields by searching for operators that coarse-grain to themselves

$\rightsquigarrow$ diagonalize 'scaling superoperator' (Evenbly-Vidal)

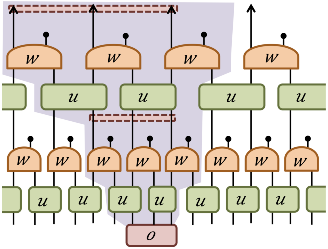
- in our case, no need to diagonalize – theorem contains 'dictionary'
- some scaling dimensions exact (fermion fields)

1D Dirac fermion – Verifying conformal data

- central charge: $S(R) = \frac{c}{3} \log R + c'$



- usual procedure: identify fields by searching for operators that coarse-grain to themselves

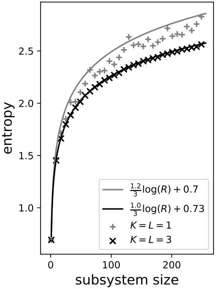


$\rightsquigarrow$ diagonalize 'scaling superoperator' (Evenbly-Vidal)

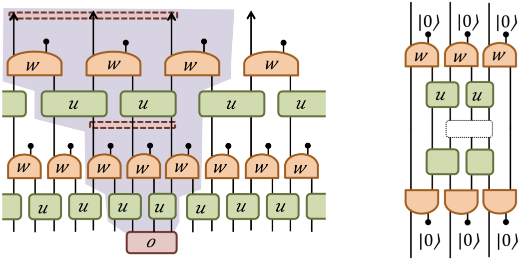
- in our case, no need to diagonalize – theorem contains 'dictionary'
- some scaling dimensions exact (fermion fields)

1D Dirac fermion – Verifying conformal data

- central charge: $S(R) = \frac{c}{3} \log R + c'$



- usual procedure: identify fields by searching for operators that coarse-grain to themselves



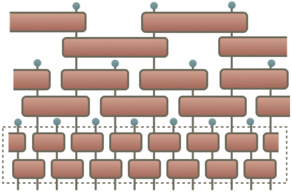
$\rightsquigarrow$ diagonalize 'scaling superoperator' (Evenbly-Vidal)

- in our case, no need to diagonalize – theorem contains 'dictionary'
- some scaling dimensions exact (fermion fields)

How to construct a free-fermion (= Gaussian) MERA?

Need to construct Fermi sea of negative single-particle energy modes.

How to perform entanglement renormalization on the single-particle level?
Is there a single-particle variant of MERA?

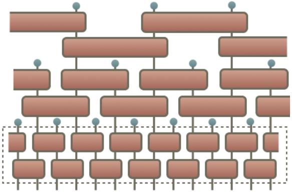


Yes – wavelet transforms!

How to construct a free-fermion (= Gaussian) MERA?

Need to construct Fermi sea of negative single-particle energy modes.

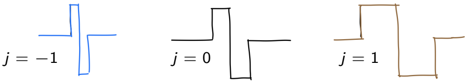
How to perform entanglement renormalization on the single-particle level?
Is there a single-particle variant of MERA?



Yes – wavelet transforms!

Wavelets and renormalization

Fourier basis resolves signal into scales, but is completely nonlocal.
In contrast, can also generate bases by scalings and translates of single localized wave packet – a wavelet:



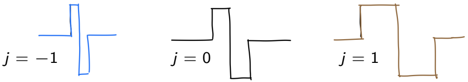
Then we can recursively resolve signal into different scales:

where

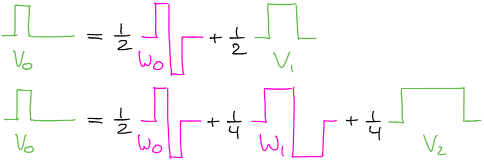
$$W_j = \text{span of wavelets at scale } j$$
$$V_j = \text{signals at scale up to } j = W_j \oplus W_{j+1} \oplus \dots$$

Wavelets and renormalization

Fourier basis resolves signal into scales, but is completely nonlocal.
In contrast, can also generate bases by scalings and translates of single localized wave packet – a wavelet:



Then we can recursively resolve signal into different scales:



where

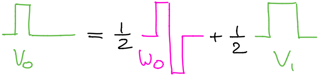
$$W_j = \text{span of wavelets at scale } j$$

$$V_j = \text{signals at scale up to } j = W_j \oplus W_{j+1} \oplus \ldots$$

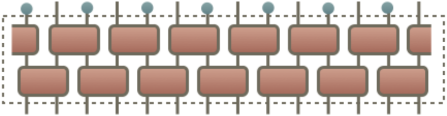
Wavelets and MERA

The basis transformation

$$V_j \rightarrow W_j \oplus W_{j+1}$$



is implemented by a *classical circuit* acting on *single-particle* Hilbert space:



Discrete circuit resolves continuous signal by scale!

Second quantization yields layer of a Gaussian MERA!

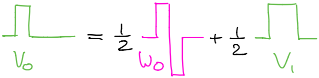
- in fact, obtain 'holographic' mapping (Qi)
- depth of classical circuit = depth of quantum circuit (Evenbly-White)

Still need to design wavelet transform that targets negative energy modes.

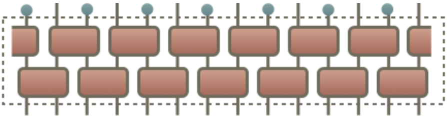
Wavelets and MERA

The basis transformation

$$V_j \to W_j \oplus W_{j+1}$$



is implemented by a *classical circuit* acting on *single-particle* Hilbert space:



Discrete circuit resolves continuous signal by scale!

Second quantization yields layer of a Gaussian MERA!

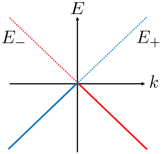
- in fact, obtain 'holographic' mapping (Qi)
- depth of classical circuit = depth of quantum circuit (Evenbly-White)

Still need to design wavelet transform that targets negative energy modes.

1D Dirac fermion – Vacuum state

Massless Dirac equation in 1+1d:

$$i\gamma^\mu \partial_\mu \psi = 0$$



Negative energy modes:

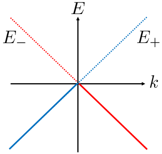
- $\chi_\pm$ supported on $k < 0$ / $k > 0$
- $\psi_{1,2}$ related by $-\mathrm{i}\,\mathrm{sign}(k)$ at $t = 0$ (*Hilbert transform*)

Can choose *any* basis of Fermi sea. So let's design pair of wavelets related by Hilbert transform!

1D Dirac fermion – Vacuum state

Massless Dirac equation in 1+1d:

$$\begin{bmatrix}\psi_1(x,t)\\ \psi_2(x,t)\end{bmatrix} = \begin{bmatrix}\chi_+(x-t)+\chi_-(x+t)\\ \mathrm{i}\chi_+(x-t)-\mathrm{i}\chi_-(x+t)\end{bmatrix}$$



Negative energy modes:

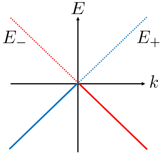
- $\chi_\pm$ supported on $k<0$ / $k>0$
- $\psi_{1,2}$ related by $-\mathrm{i}\,\mathrm{sign}(k)$ at $t=0$ (*Hilbert transform*)

Can choose *any* basis of Fermi sea. So let's design pair of wavelets related by Hilbert transform!

1D Dirac fermion – Vacuum state

Massless Dirac equation in 1+1d:

$$\begin{bmatrix}\psi_1(x,t)\\ \psi_2(x,t)\end{bmatrix} = \begin{bmatrix}\chi_+(x-t)+\chi_-(x+t)\\ \mathrm{i}\chi_+(x-t)-\mathrm{i}\chi_-(x+t)\end{bmatrix}$$



Negative energy modes:

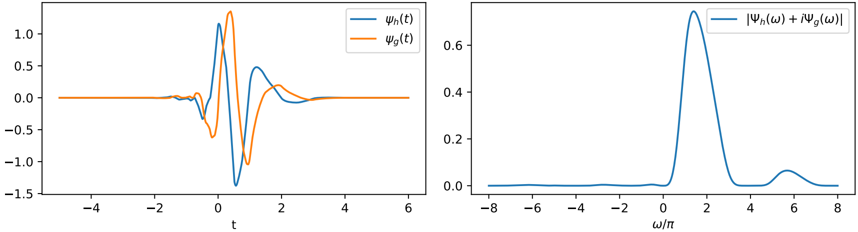
- $\chi_\pm$ supported on $k<0$ / $k>0$
- $\psi_{1,2}$ related by $-\mathrm{i}\,\mathrm{sign}(k)$ at $t=0$ (*Hilbert transform*)

Can choose *any* basis of Fermi sea. So let's design pair of wavelets related by Hilbert transform!

1D Dirac fermion – Hilbert wavelet pairs

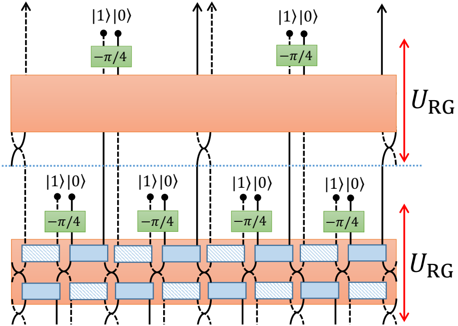
Such wavelet pairs have been studied in the signal processing community:

- motivated by *directionality* and *shift-invariance* (Selesnick)
- impossible exactly with local circuit, but possible to arbitrary accuracy



After second quantizing and careful analysis, obtain tensor network with rigorous approximation guarantee. . .

1D Dirac fermion – Result



Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of Hilbert pair
- Γ – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

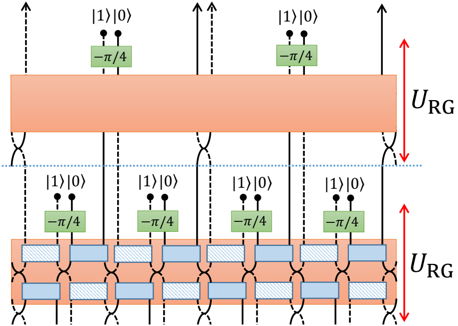
$$C(\{f_i, A_j\}) := \left\langle \Psi^\dagger(f_1)\cdots\Psi(f_{2N})A_1\cdots A_M \right\rangle$$

Result (simplified)

$$\left|C_{\text{exact}} - C_{\text{MERA}}\right| \leq \Gamma \max\{2^{-\mathcal{L}/3}, \varepsilon \log \frac{1}{\varepsilon}\}$$

In particular, all conformal symmetries approximately inherited.

1D Dirac fermion – Result



Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of Hilbert pair
- Γ – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

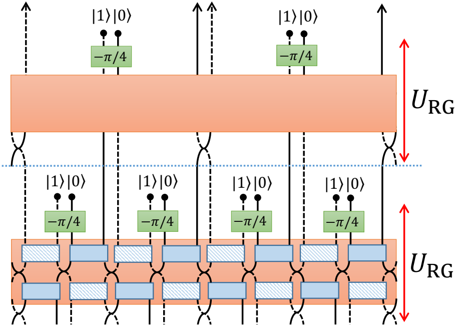
$$C(\{f_i, A_j\}) := \left\langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \right\rangle$$

Result (simplified)

$$\left| C_{\text{exact}} - C_{\text{MERA}} \right| \leq \Gamma \max\{2^{-\mathcal{L}/3}, \varepsilon \log \frac{1}{\varepsilon}\}$$

In particular, all conformal symmetries approximately inherited.

1D Dirac fermion – Result



Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of Hilbert pair
- Γ – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

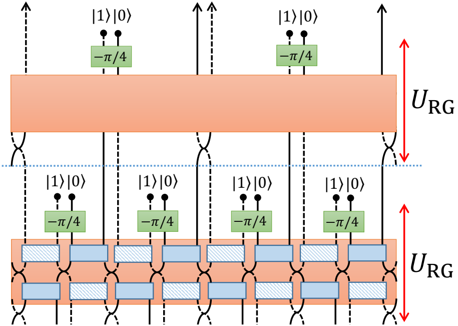
$$C(\{f_i, A_j\}) := \left\langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \right\rangle$$

Result (simplified)

$$\left| C_{\text{exact}} - C_{\text{MERA}} \right| \leq \Gamma \max\{2^{-\mathcal{L}/3}, \varepsilon \log \frac{1}{\varepsilon}\}$$

In particular, all conformal symmetries approximately inherited.

1D Dirac fermion – Result



Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of Hilbert pair
- Γ – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

$$C(\{f_i, A_j\}) := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \rangle$$

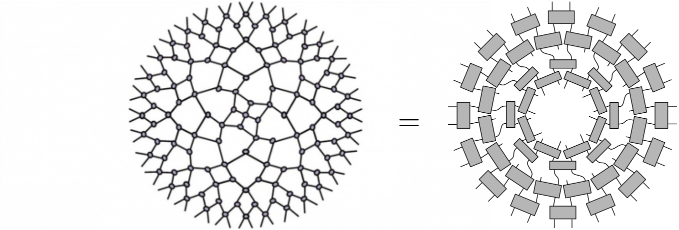
Result (simplified)

$$\left| C_{\mathrm{exact}} - C_{\mathrm{MERA}} \right| \leq \Gamma \max\{2^{-\mathcal{L}/3}, \varepsilon \log \frac{1}{\varepsilon}\}$$

In particular, all conformal symmetries approximately inherited.

1D Dirac fermion – Circle

Construction also works for Dirac fermion on circle:



- finite number of layers once UV cut-off fixed
- systematic construction by (anti)periodizing wavelets

Non-relativistic 2D fermions – Lattice model

When put on lattice, massless Dirac fermion becomes: (Kogut-Susskind)

$$H_{1\mathrm{D}} \cong -\sum_n a_n^\dagger a_{n+1} + h.c.$$

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{2\mathrm{D}} = -\sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c.$$

Fermi surface:

- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
- Green's function factorizes w.r.t. rotated axes

Non-relativistic 2D fermions – Lattice model

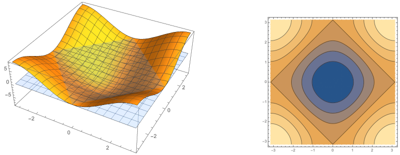
When put on lattice, massless Dirac fermion becomes: (Kogut-Susskind)

$$H_{1\mathrm{D}} \cong -\sum_n a_n^\dagger a_{n+1} + h.c.$$

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{2\mathrm{D}} = -\sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c.$$

Fermi surface:



- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
- Green's function factorizes w.r.t. rotated axes

Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

$$W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \quad \rightsquigarrow \quad (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh}$$

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):

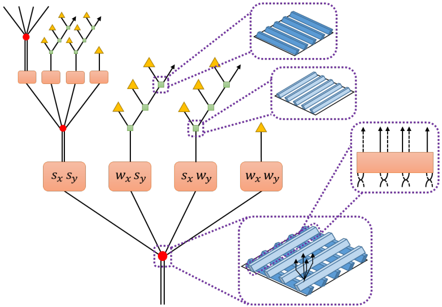
$\rightsquigarrow$ approximation theorem (with Haegeman, Swingle, Cotler, Evenbly, Scholz).

Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

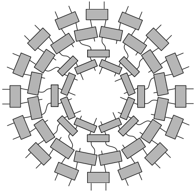
$$W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \quad \rightsquigarrow \quad (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh}$$

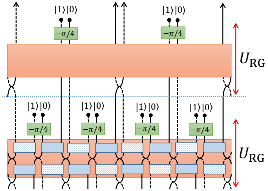
After second quantization, obtain variant of branching MERA (Evenbly-Vidal):



$\rightsquigarrow$ approximation theorem (with Haegeman, Swingle, Cotler, Evenbly, Scholz).

Summary and outlook





- entanglement renormalization quantum circuits for 1D Dirac CFT
- explicit construction with rigorous guarantees

Outlook:

- thermofield double, massive theories, Dirac cones, ...
- building block for more interesting CFTs? starting point for perturbation theory or variational optimization?
- lift wavelet theory to quantum circuit level!

Thank you for your attention!

How to build an approximate Hilbert pair [Selesnick]

Wavelets are built from filters $g[n]$ that relate functions at different scale:

$$\phi_{j-1}(x) = \sum_{n \in \mathbb{Z}} g[n]\, \phi_j(x - 2^{-j} n)$$

Necessary and sufficient to obtain orthonormal basis (roughly speaking):

$$|G(\theta)|^2 + |G(\theta + \pi)|^2 = 2, \quad G(0) = \sqrt{2}$$

Wavelets are related by Hilbert transform iff filters related by half-shift:

$$G(\theta) = H(\theta) e^{-i\theta/2}$$

To achieve this, find explicit approximation

$$e^{-i\theta/2} \approx e^{-iL\theta} \frac{D(-\theta)}{D(\theta)}.$$

Then, $H(\theta) = Q(\theta) D(\theta)$ and $G(\theta) = Q(\theta) e^{-iL\theta} D(-\theta)$ are approximately related by half-shift for any choice of $Q(\theta)$.

How to build an approximate Hilbert pair [Selesnick]

Wavelets are built from filters $g[n]$ that relate functions at different scale:

$$\phi_{j-1}(x) = \sum_{n\in\mathbb{Z}} g[n]\,\phi_j(x - 2^{-j}n)$$

Necessary and sufficient to obtain orthonormal basis (roughly speaking):

$$|G(\theta)|^2 + |G(\theta+\pi)|^2 = 2, \quad G(0) = \sqrt{2}$$

Wavelets are related by Hilbert transform iff filters related by half-shift:

$$G(\theta) = H(\theta)e^{-i\theta/2}$$

To achieve this, find explicit approximation

$$e^{-i\theta/2} \approx e^{-iL\theta}\frac{D(-\theta)}{D(\theta)}.$$

Then, $H(\theta) = Q(\theta)D(\theta)$ and $G(\theta) = Q(\theta)e^{-iL\theta}D(-\theta)$ are approximately related by half-shift for any choice of $Q(\theta)$.