Quantum circuits for the Dirac field in 1+1 dimensions

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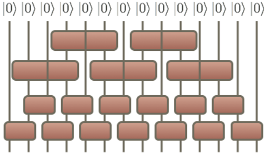
based on arXiv:1905.08821 (with Witteveen, Scholz, Swingle) and
arXiv:1707.06243 (with Haegeman, Swingle, Cotler, Evenbly, Scholz)

Tensor networks

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle$$

Efficient variational classes for many-body quantum states:

matrix product states



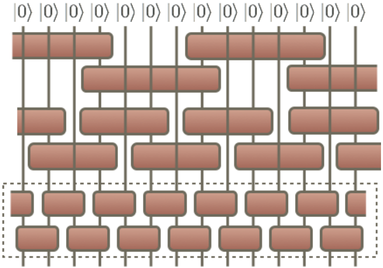
MERA

- can have interpretation as quantum circuit

Useful theoretical formalism:

- geometrize entanglement structure: *generalized area law*
- bulk-boundary dualities: *lift physics to the virtual level*
- quantum phases, topological order, RG, holography, ...

MERA multi-scale entanglement renormalization ansatz (Vidal)



↓ local quantum circuit that prepares state from $|0\rangle^{\otimes N}$

↑ entanglement renormalization

↕ organize q. information by scale

- self-similar layers that are short-depth quantum circuits
- variational class for critical systems in 1D
- interpretation: disentangle & coarse-grain
- network arises from tensor network renormalization:

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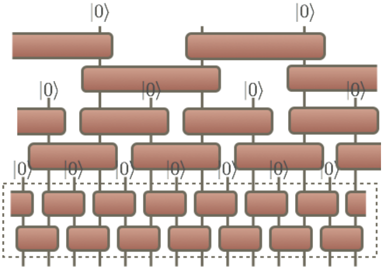

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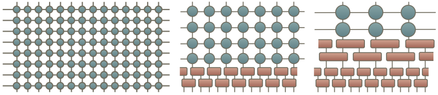

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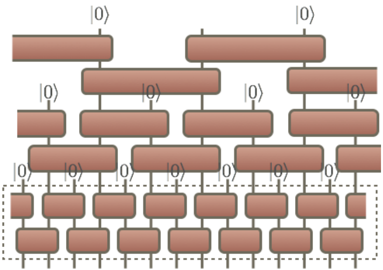

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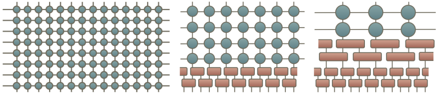
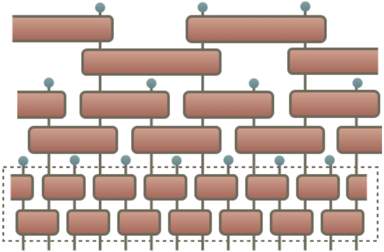


MERA and holography



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- hyperbolic geometry (Swingle)
- starting point for tensor network models of holography (HaPPY; Hayden-. . . -W.)

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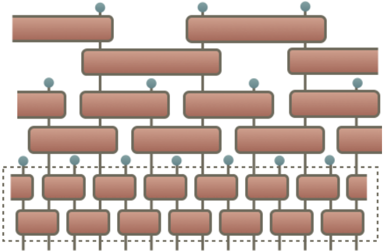

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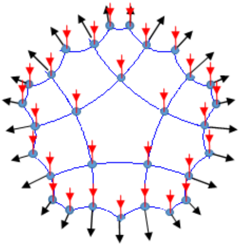

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Tensor networks and quantum field theories

Tensor networks are discrete and finitary representations, while QFTs are infinite and defined in the continuum.

Two successful approaches:

- *continuum* (cMPS, cMERA, ...)
- *lattice* (MPS, PEPS, MERA, ...)

Questions:

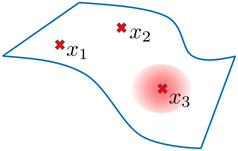
- what do tensor networks capture?
- how to measure goodness of approximation?
- can we give rigorous construction principles?
- why do tensor networks work well?

cf. plethora of results on gapped 1D lattice systems in QIT/cond-mat

Tensor networks for correlation functions

Given many-body system in state ρ and choice of operators $\{O_\alpha\}$, define correlation function:

$$C(\alpha_1, \cdots, \alpha_n) = \mathrm{tr}[\rho O_{\alpha_1} \cdots O_{\alpha_n}]$$



Goal: Design tensor network for correlation functions!

- unified perspective: system can be continuous – discreteness imposed by how we probe it
- tensor network for ρ sufficient (if possible), but likely suboptimal

Examples: Zaletel-Mong (MPS/q. Hall states), König-Scholz (MPS/CFTs), cf. quantum marginal problem

Our results

We construct tensor networks for free fermion theories:

- 1D Dirac fermion in continuum & lattice
- non-relativistic 2D fermions on lattice (Fermi surface)

Key features:

- tensor networks that target correlation functions
- rigorous approximation guarantees
- entanglement renormalization quantum circuits: (branching) MERA
- explicit construction, no variational optimization

We achieve this using tools from signal processing: wavelet theory.

1D Dirac fermion – Lattice result

Fermions hopping on infinite 1D lattice at half filling:

$$H_{1\mathrm{D}} = -\sum_n a_n^\dagger a_{n+1} + h.c.$$

- equivalent to 'staggered' massless Dirac fermions (Kogut-Susskind)
- easily solved using Fourier transform – but *not* using local q. circuit!

We construct MERA networks that target correlation functions:

$$C(\{f_i\}) := \langle a^\dagger(f_1) \cdots a(f_{2N}) \rangle$$

Result (simplified)

$$C_{\mathrm{exact}} \approx C_{\mathrm{MERA}}$$

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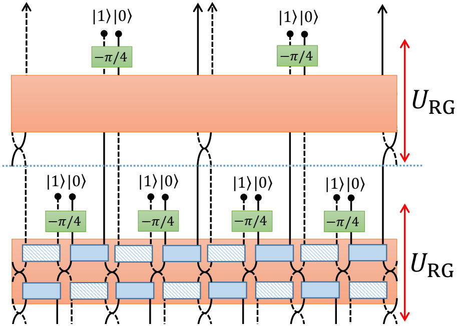

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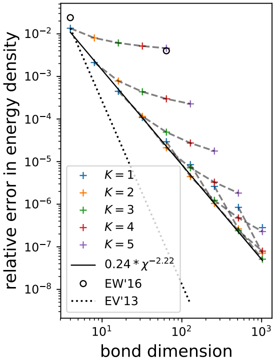
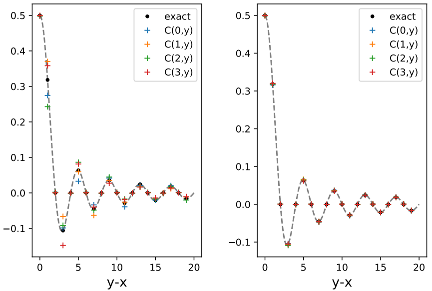
1D Dirac fermion – Numerics

Energy error



bond dimension $= 2^{\text{circuit depth}}$

Green function $C(x, y) = \langle a_x^\dagger a_y \rangle$

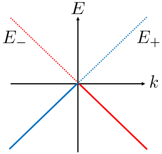


bond dimension 2^2 bond dimension 2^6

1D Dirac fermion – Continuum result

Massless Dirac fermion in 1+1d:

$$i\gamma^\mu \partial_\mu \psi = 0$$



We construct circuits that target vacuum correlation functions in Dirac CFT:

$$C(\{f_i, A_j\}) := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \rangle$$

$\Psi(f_i)$ smeared fields, A_j normal-ordered bilinears (e.g. smeared T, L_n)

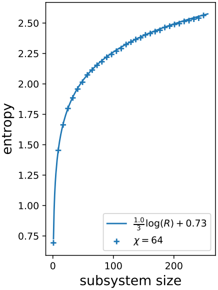
Result (simplified)

$$C_{\text{exact}} \approx C_{\text{MERA}}$$

Rigorous quantum circuit approximation for a QFT!

1D Dirac fermion – Verifying conformal data

- central charge: $S(R) = \frac{c}{3} \log R + c'$



- usual procedure: identify fields by searching for operators that coarse-grain to themselves

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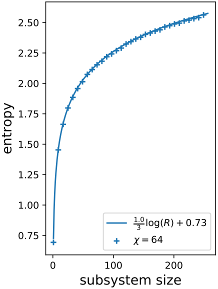

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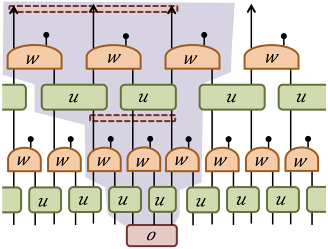

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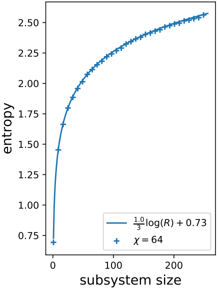

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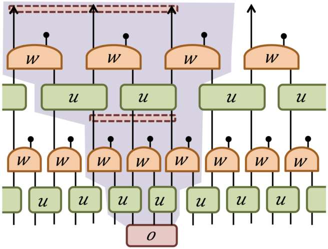

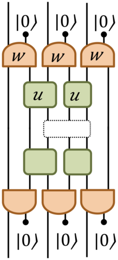

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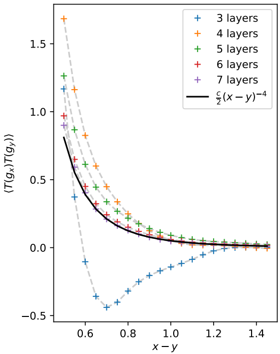
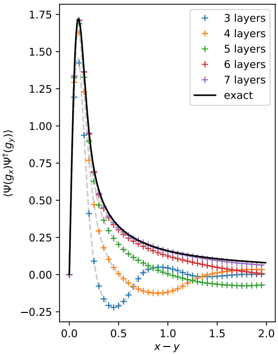
1D Dirac fermion – Numerics

Two-point functions:

$\langle T(x)T(y)\rangle$



$\Im\langle\Psi(x)\Psi(y)^\dagger\rangle$

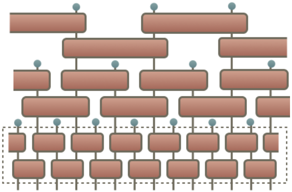


Similarly: OPE coefficients.

How to construct a free-fermion (= Gaussian) MERA?

Free-fermion ground states are Fermi seas filled with negative energy modes of single-particle Hamiltonian. This begs the question:

How to perform entanglement renormalization on the single-particle level? Is there a single-particle variant of MERA?



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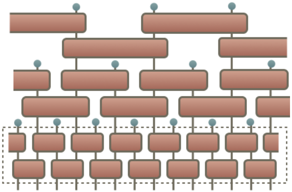

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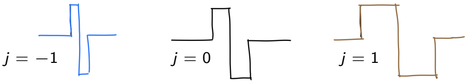
Wavelets and renormalization

Fourier basis resolves signal into scales, but is completely nonlocal.
In contrast, can also generate basis by scalings and translates of single localized wave packet – a wavelet:



Then we can recursively resolve signal into different scales:

where

$$W_j = \text{span of wavelets at scale } j$$
$$V_j = \text{signals at scale up to } j = W_j \oplus W_{j+1} \oplus \dots$$

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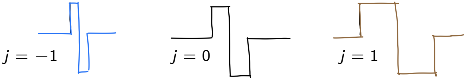

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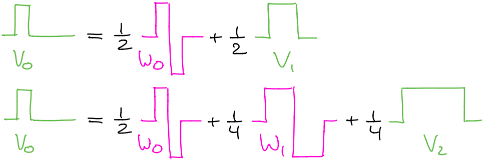

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The basis transformation

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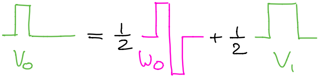
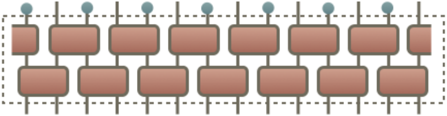


is implemented by a *classical circuit* acting on *single-particle* Hilbert space:



Second quantization yields layer of a Gaussian MERA!

- in fact, obtain 'holographic' mapping (Qi)
- depth of classical circuit = depth of quantum circuit (Evenbly-White)

Upshot: To construct free-fermion ground state, design wavelet transform that targets positive/negative energy modes.

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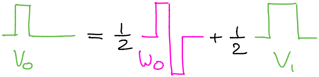

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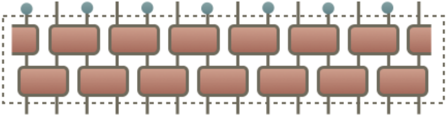

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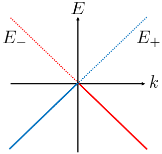
Massless Dirac equation in 1+1d:

$$i\gamma^{\mu}\partial_{\mu}\psi = 0$$



Negative energy modes:

- $\chi_{\pm}$ supported on $k < 0$ / $k > 0$
- $\psi_{1,2}$ related by $-\mathrm{i}\,\mathrm{sign}(k)$ at $t = 0$ (*Hilbert transform*)
- can choose *any* basis of Fermi sea...

Goal: Design pair of wavelets related by Hilbert transform!

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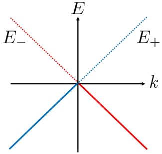

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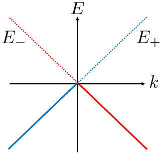

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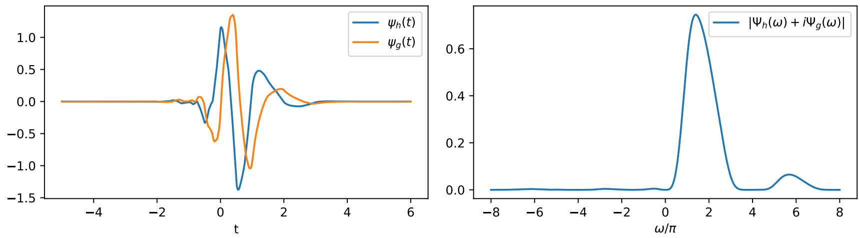
1D Dirac fermion – Hilbert wavelet pairs

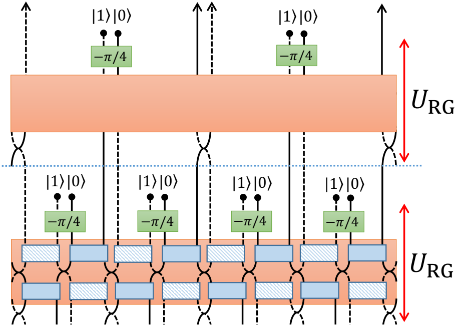
Such wavelet pairs have been studied in the signal processing community:

- motivated by *directionality* and *shift-invariance* (!)
- impossible exactly with local circuit, but possible to arbitrary accuracy (Selesnick)



After second quantizing and careful analysis, obtain tensor network with rigorous approximation guarantee. . .

1D Dirac fermion – Result



Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of Hilbert pair
- Γ – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

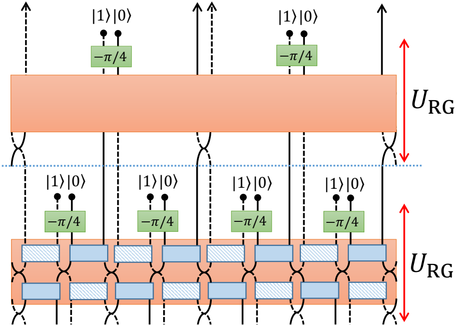
$$C(\{f_i, A_j\}) := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \rangle$$

Result (simplified)

$$|C_{\mathrm{exact}} - C_{\mathrm{MERA}}| \leq \Gamma \max\{2^{-\mathcal{L}/4}, \varepsilon \log \frac{1}{\varepsilon}\}$$

In particular, all conformal symmetries approximately inherited.

1D Dirac fermion – Result



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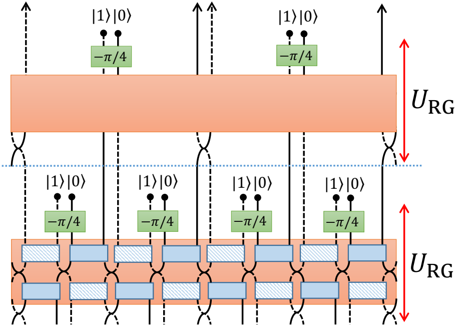

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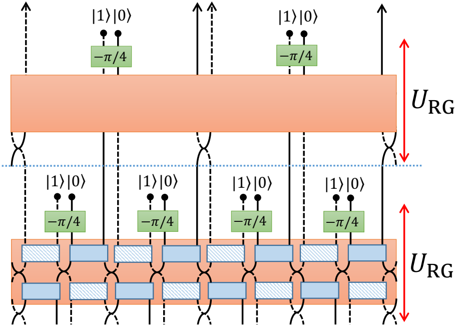

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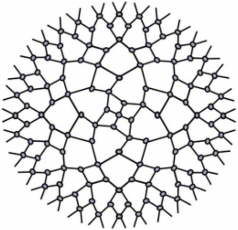
1D Dirac fermion – Circle

Construction also works for Dirac fermion on circle:



- finite number of layers once UV cut-off fixed
- systematic construction by (anti)periodizing wavelets
- only top layers change – wavelet modes start 'wrapping around'

Non-relativistic 2D fermions – Lattice model

$$H_{\mathrm{1D}} \cong -\sum_n a_n^\dagger a_{n+1} + h.c.$$

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{\mathrm{2D}} = -\sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c.$$

Fermi surface:

- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
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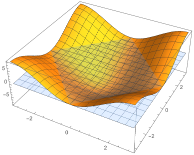

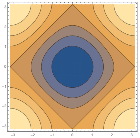

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Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

$$W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \quad \rightsquigarrow \quad (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh}$$

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):

Similar approximation theorem holds.

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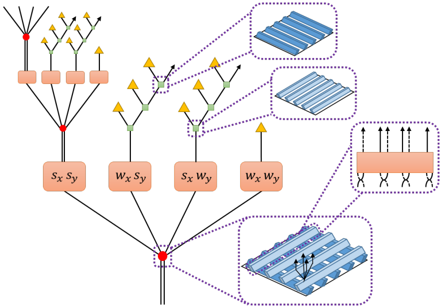
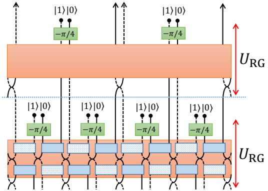
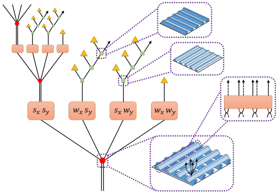


Similar approximation theorem holds.

Summary and outlook





- entanglement renormalization quantum circuits for free fermions
- explicit construction with rigorous guarantees (lattice + continuum)

Outlook:

- thermofield double, massive theories, Dirac cones, ...
- building block for more interesting CFTs? starting point for perturbation theory or variational optimization?

Thank you for your attention!