Eigenvalue Distributions of Reduced Density Matrices

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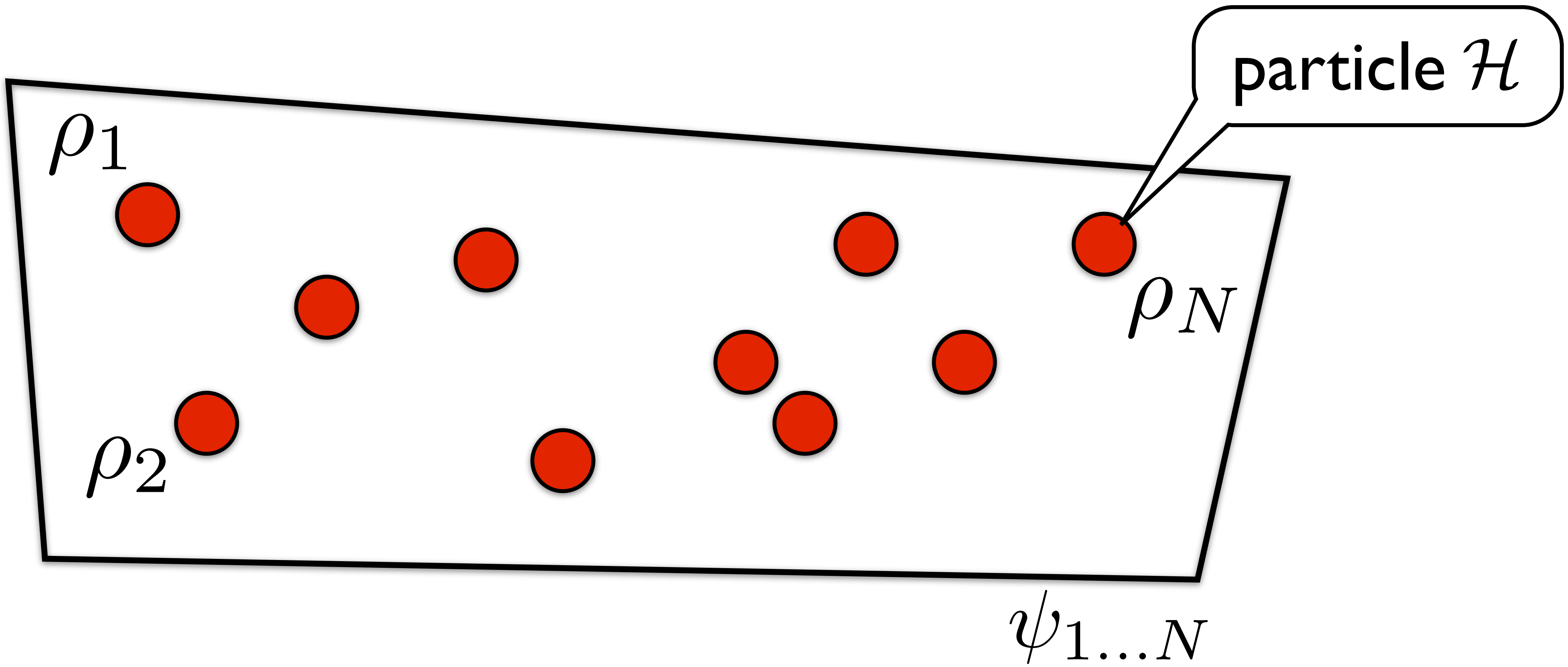
joint work with Matthias Christandl, Brent Doran, Stavros Kousidis

special thanks to David Gross

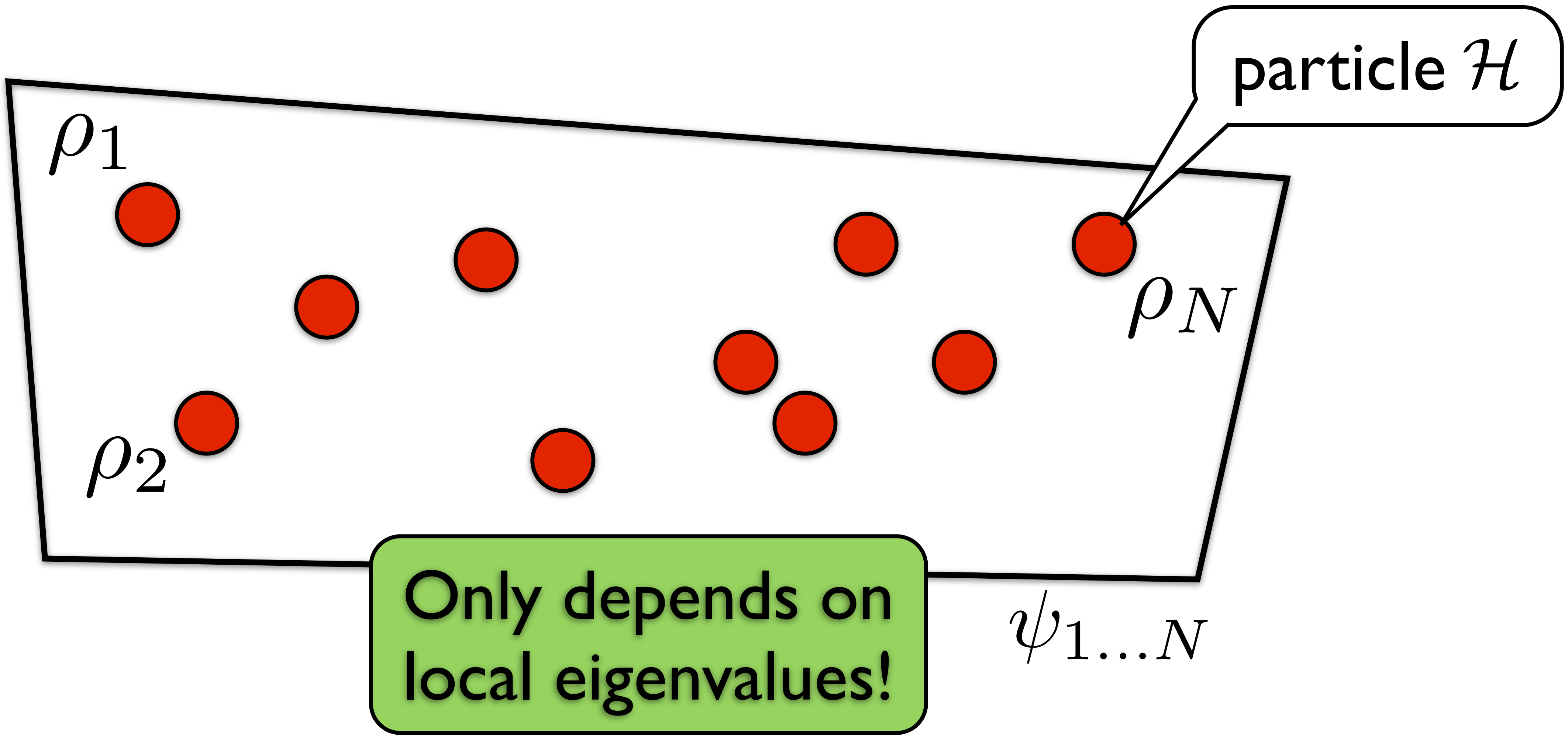
arXiv:1204.0741

One-Body Quantum Marginal Problem



What are the possible tuples of reduced density matrices $\rho_1, \ldots, \rho_N$ of a global pure state $\psi_{1\ldots N}$?

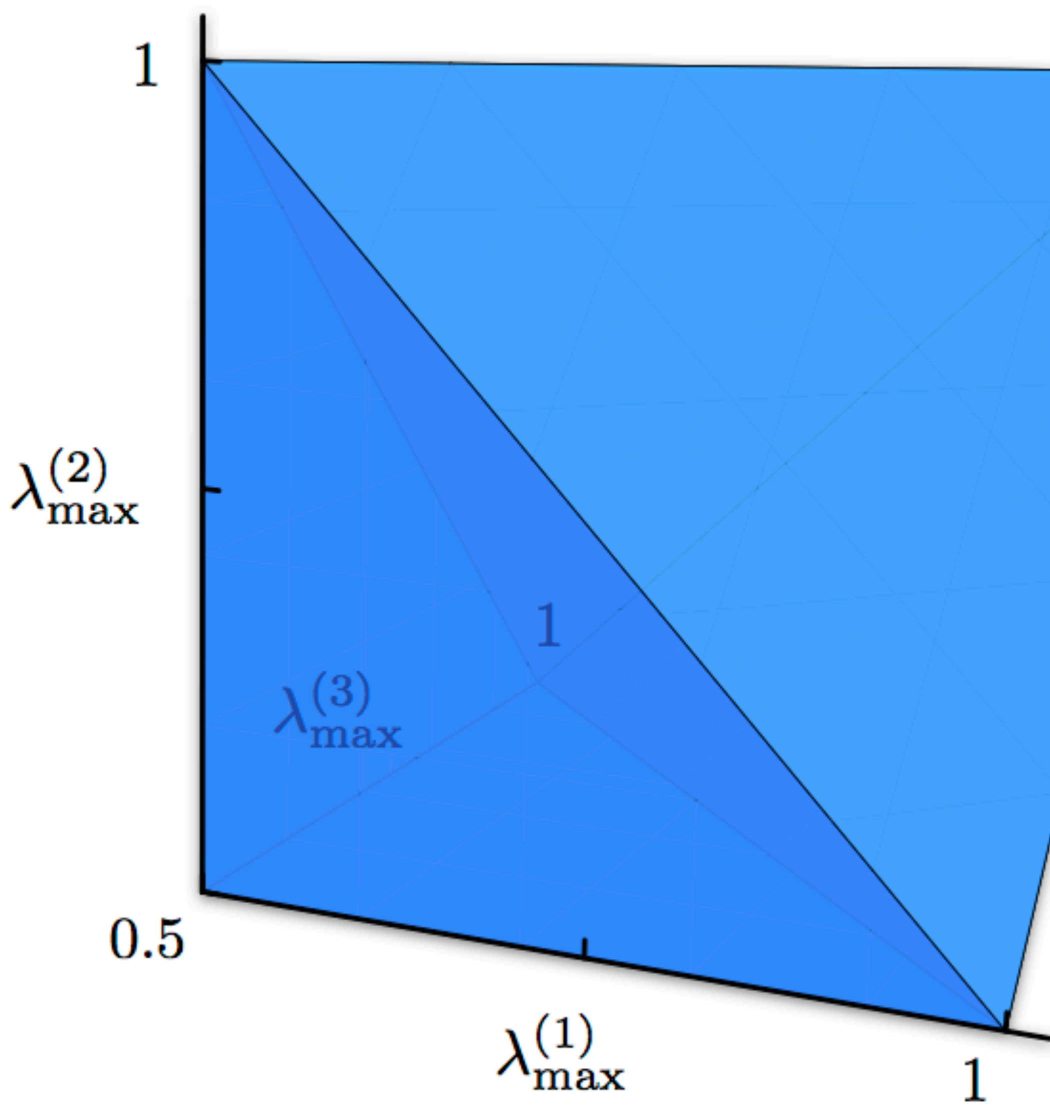
One-Body Quantum Marginal Problem



What are the possible tuples of reduced density matrices $\rho_1, \ldots, \rho_N$ of a global pure state $\psi_{1\ldots N}$?

Example: Three Qubits

$$N = 3$$
$$\mathcal{H} = \mathbb{C}^2$$



$$\begin{cases} \lambda^{(1)}_{\max} + \lambda^{(2)}_{\max} \le 1 + \lambda^{(3)}_{\max} \\ \lambda^{(1)}_{\max} + \lambda^{(3)}_{\max} \le 1 + \lambda^{(2)}_{\max} \\ \lambda^{(2)}_{\max} + \lambda^{(3)}_{\max} \le 1 + \lambda^{(1)}_{\max} \end{cases}$$

Higuchi, Sudbery & Szulc (2003)

One-Body Quantum Marginal Problem

Mathematical structure: Symplectic Geometry

$$\psi_{1\ldots N} \mapsto (\text{eig } \rho_1, \ldots, \text{eig } \rho_N)$$

moment map

- Result is always a convex polytope (Kirwan)
- Explicit linear inequalities — Klyachko (2004), Daftour & Hayden (2004), Berenstein & Sjamaar (2000), Ressayre (2007)
- Representation theory — Christandl & Mitchison (2004), Klyachko (2004), Christandl, Harrow & Mitchison (2006)

Eigenvalue Distributions

Given a random pure state $\psi_{1\ldots N}$, what is the joint distribution of its local eigenvalues $\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)}$?

Main Result: Christandl, Doran, Kousidis, W. (2012)

Algorithm to compute exact distribution for any N and $\mathcal{H}$, and arbitrary statistics.

Hayden, Leung, Shor & Winter (2004 & 2006)

Motivation: typical entanglement & entropies, statistical physics, black hole information

Lloyd & Pagels (1988), Popescu, Short & Winter (2006)

Eigenvalue Distributions

Given a <u>random</u> pure state $\psi_{1\ldots N}$, what is the joint distribution of its local eigenvalues $\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)}$?

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Duistermaat-Heckman measure

Task: Compute pushforward of Liouville measure along moment map

Motivatio… …ies,
…formation

Lloyd & Pagels (1988), Popescu, Short & Winter (2006)

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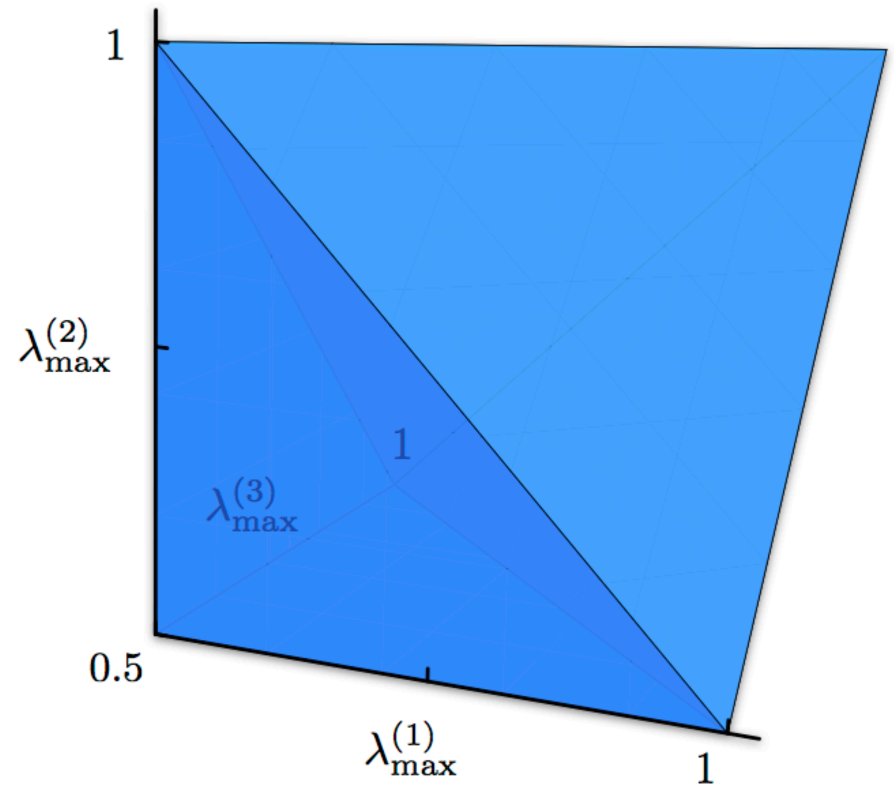


Eigenvalue distribution:

upper pyramid

$$\left(\lambda^{(1)}_{\max} - \tfrac{1}{2}\right)\left(\lambda^{(2)}_{\max} - \tfrac{1}{2}\right)\left(\lambda^{(3)}_{\max} - \tfrac{1}{2}\right)\begin{cases} 2\min \lambda^{(k)}_{\max} + 1 - \lambda^{(1)}_{\max} - \lambda^{(2)}_{\max} - \lambda^{(3)}_{\max} \\ 2\min \lambda^{(k)}_{\max} - 1 \end{cases}$$

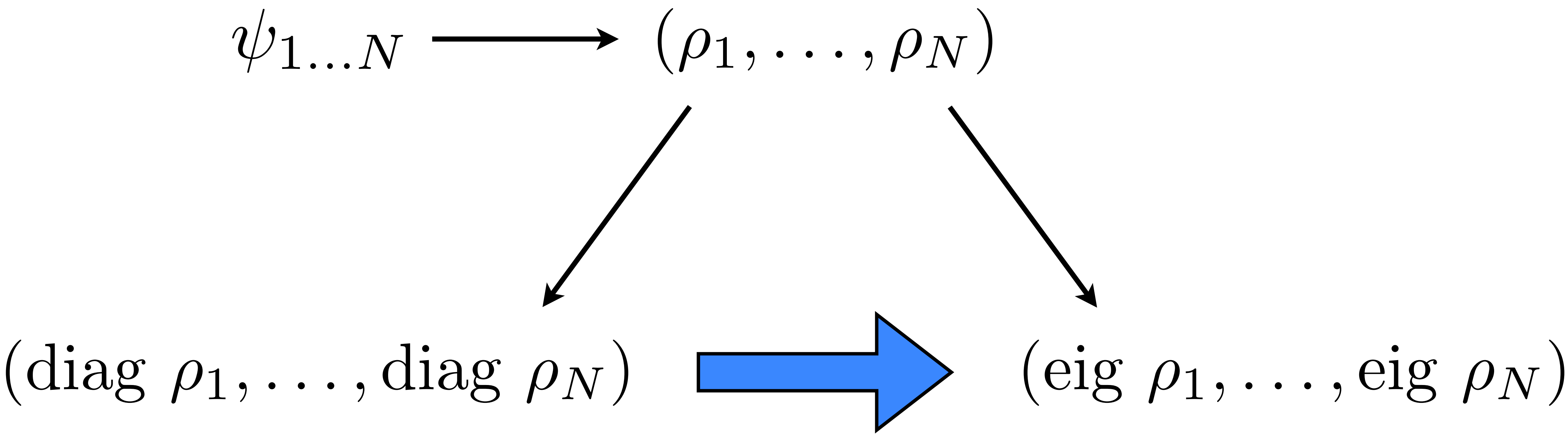
lower pyramid

Sketch of Technique

$$\psi_{1\ldots N} \longrightarrow (\rho_1, \ldots, \rho_N)$$

$$(\text{diag } \rho_1, \ldots, \text{diag } \rho_N) \Rightarrow (\text{eig } \rho_1, \ldots, \text{eig } \rho_N)$$

Sketch of Technique



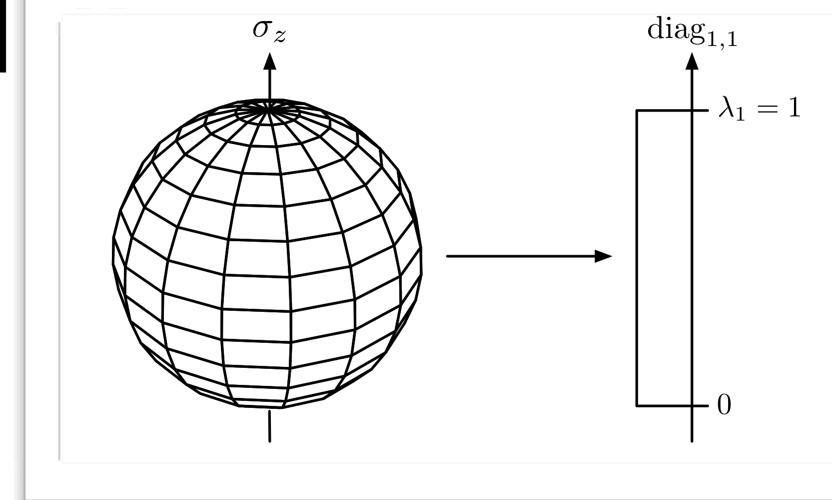
<u>Derivative Principle</u>

$$P_{\text{eig}} = v(\lambda) \left(\prod_{\alpha>0} -\partial_\alpha \right) P_{\text{diag}}$$

well-known polynomial

differences of local eigenvalues

Sketch of Technique



Archimedes (230 BC),
Harish-Chandra (1957)

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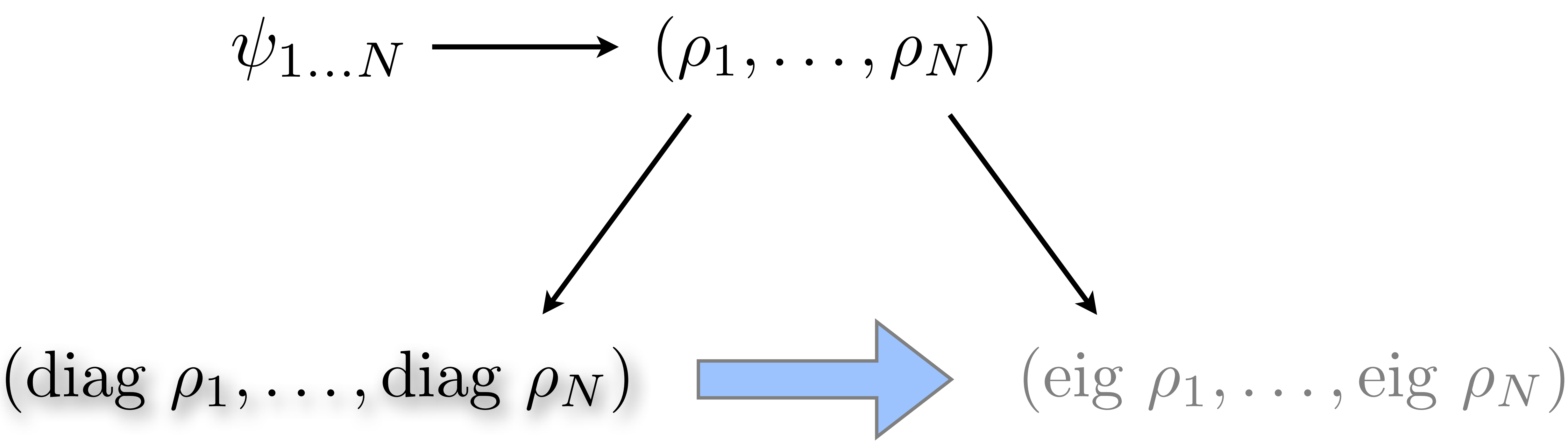


- Piecewise polynomial density
- <u>Wall-crossing formula:</u>

<u>Classical</u>
Marginal Problem

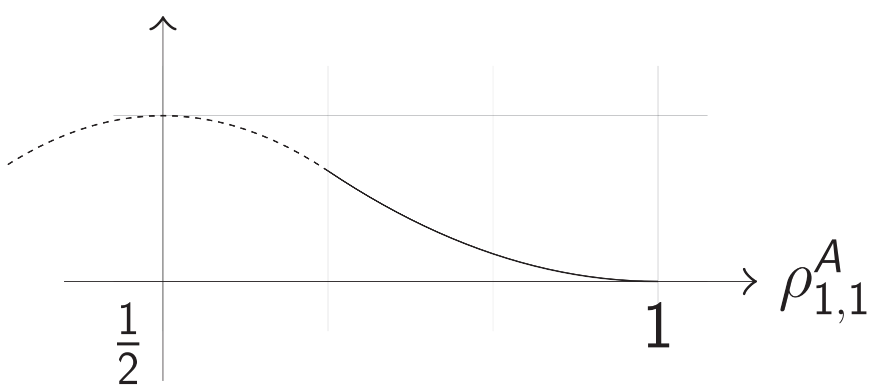
$$f_+(\delta) - f_-(\delta) = \mathrm{Res}\big|_{z=0} \left(\hat{f}_{\hat{W}}(\partial_x, \partial_y) \frac{e^{z\langle\delta-\omega_0,\xi\rangle+\langle\delta,x\rangle+y}}{\prod_{k=m}^{n} z\langle\omega_k - \omega_0, \xi\rangle + \langle\omega_k, x\rangle + y} \right)_{x=0,y=0}$$

based on Boysal & Vergne (2009)

Example: Three Bosonic Qubits

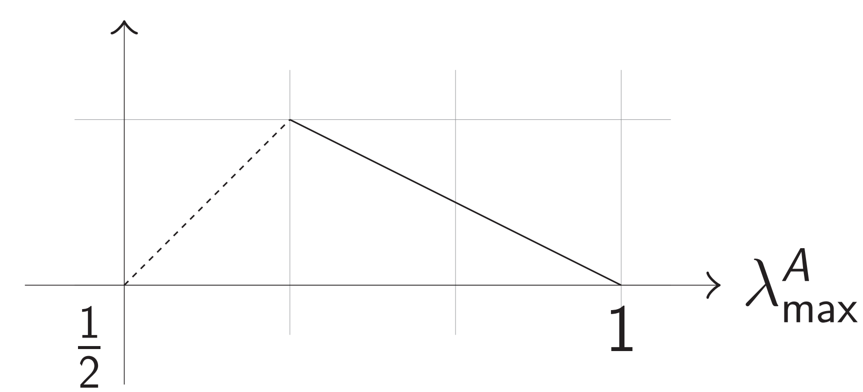
$\psi_{123} \in \mathrm{Sym}^3(\mathbb{C}^2)$

1. Distribution of diagonal entries:

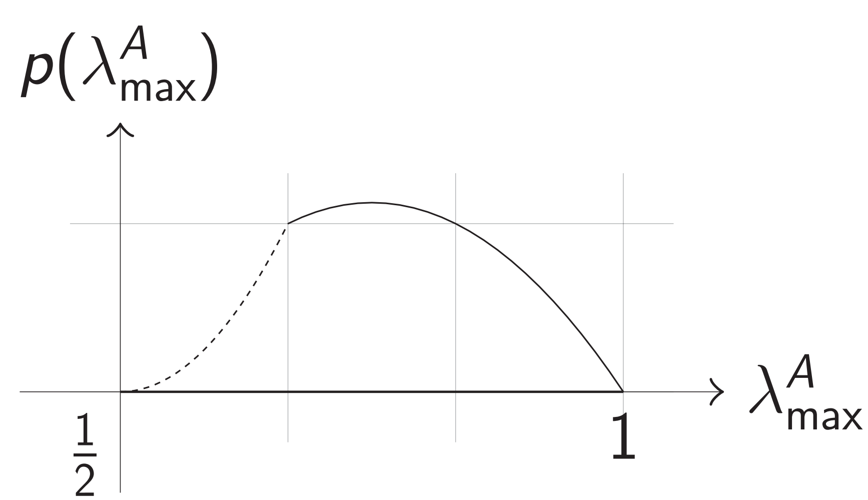


piecewise polynomials

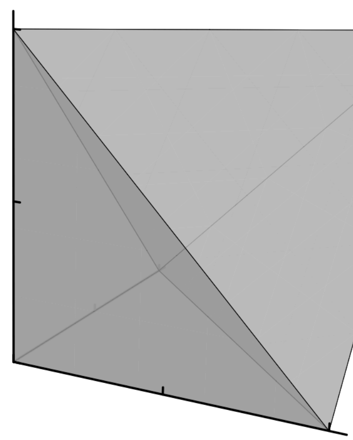
2. Take derivative:

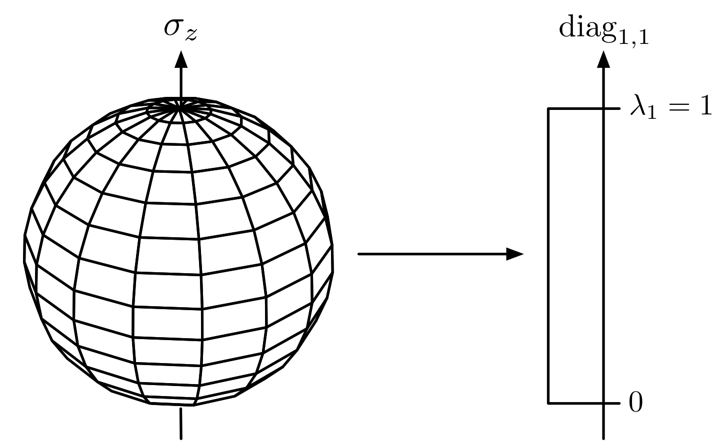


3. Multiply by $v(\lambda_{\max}) = \lambda_{\max} - \frac{1}{2}$:



Conclusion





Quantum Marginal Problem ⟷ Classical Marginal Problem

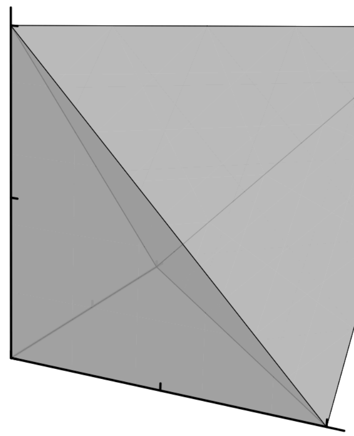
Derivative Principle

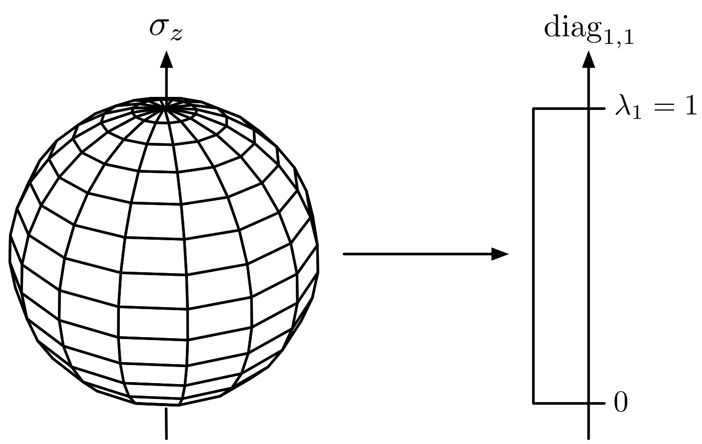
Algorithm for computing eigenvalue distributions of reduced density matrices

arXiv:1204.0741

Conclusion

QMP & Entanglement:
talk by David Gross





Quantum Marginal Problem ⟷ Classical Marginal Problem

Derivative Principle

Algorithm for computing eigenvalue distributions of reduced density matrices

Quantized Algorithm:
arXiv:1204.4379, FOCS 2012

arXiv:1204.0741