Computing Multiplicities of Lie Group Representations

Michael Walter

joint work with Matthias Christandl, Brent Doran
(ETH Zürich)

Representation Theory

<u>Representation</u> of a group G: vector space V and group homomorphism $G \to \mathrm{GL}(V)$

$$S_k \curvearrowright \mathbb{C}^k \qquad U(d) \curvearrowright \mathbb{C}^d \qquad G \curvearrowright \mathrm{Sym}^k(V)$$

For "nice" groups: any representation V can be decomposed into a direct sum of <u>irreducible</u> ones:

$$V = \bigoplus_\lambda m_\lambda \cdot V_{G,\lambda}$$

multiplicities

The Branching Problem is in P

$$V_{G,\lambda}\Big|_{H}^{G} = \bigoplus_{\mu} m_{\mu}^{\lambda} \cdot V_{H,\mu}$$

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<u>Main Result:</u> Christandl, Doran, W. (2012)

Poly-time algorithm for any fixed homomorphism $H \to G$ between compact, connected Lie groups.

"matrix groups" like O(d), U(d), Sp(n)

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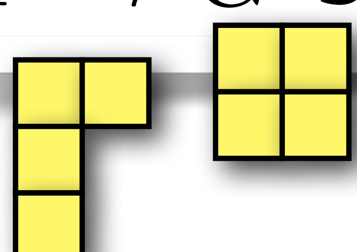

Given highest weights λ, μ encoded as bitstrings, the algorithm computes the multiplicity m_{μ}^{λ} in poly time.

Why interesting?

Long history and important applications in mathematics, quantum physics & information theory...

Kostka numbers
$T(d) \subseteq U(d)$

Littlewood-Richardson coefficients
$U(d) \to U(d) \times U(d)$

Kronecker coefficients
$U(d) \times U(d) \to U(d^2)$

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matrix multiplication, VP vs. VNP, ...

...as well as in algebraic & geometric complexity theory!

Strassen (1983), Mulmuley & Sohoni (2001)

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<u>Poly-time algorithms for fixed d</u>

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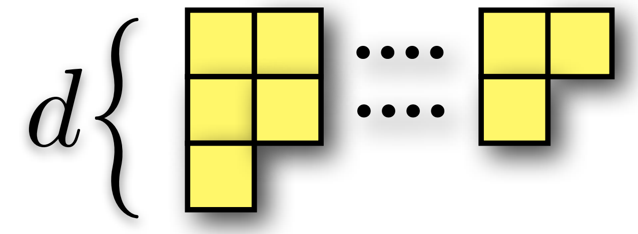

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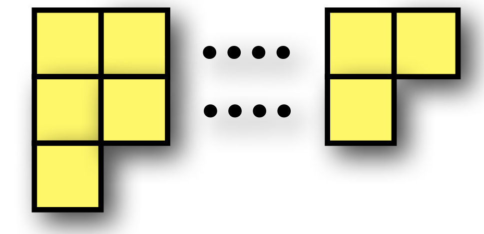
$d\{$ 

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#P-Hardness for variable d

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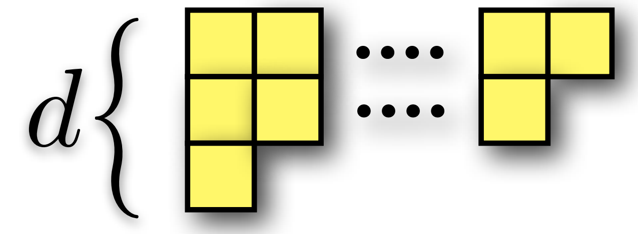

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Conjecture (Mulmuley): Positivity can be decided in poly-time.

Branching Problem for Tori $G = U(1)^r$

all compact, connected Abelian Lie groups

All irreducible representations are one-dimensional and of the form

$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix} \cdot \psi = z_1^{k_1} \cdots z_r^{k_r} \, \psi$$

Labeled by their weight $\omega = (k_1, \ldots, k_r) \in \mathbb{Z}^r$.

Branching Problem for Tori

$$G = U(1)^r$$
$$H = U(1)^s$$

(Thus) any homomorphism $H \to G$ is of the form

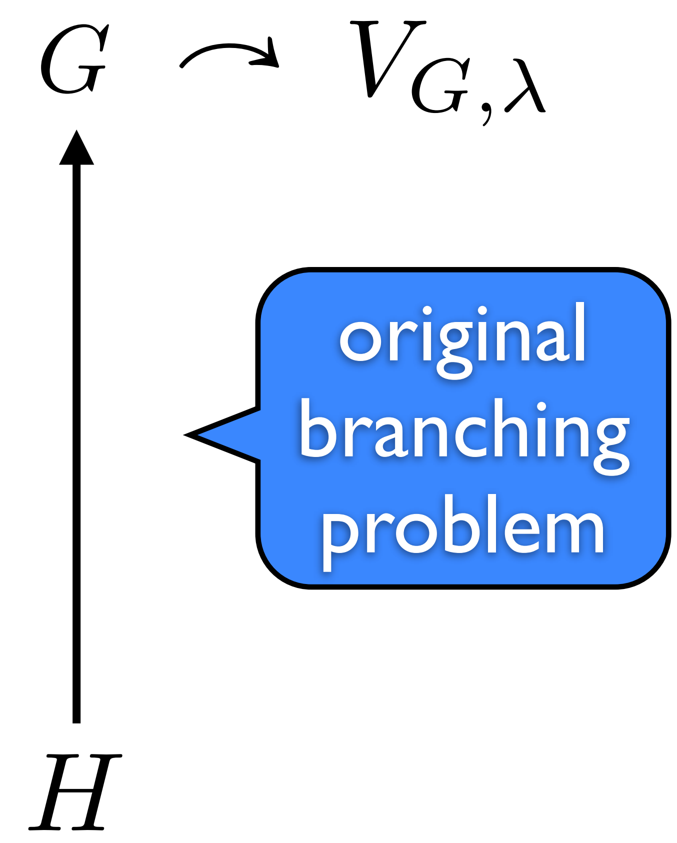
$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_s \end{pmatrix} \to \begin{pmatrix} z_1^{k_{1,1}} \cdots z_s^{k_{s,1}} & & \\ & \ddots & \\ & & z_1^{k_{r,1}} \cdots z_s^{k_{s,r}} \end{pmatrix}$$

for an integer matrix $\Omega = (k_{i,j}) \in \mathbb{Z}^{s \times r}$.

Conclusion: $$V_{G,\omega}\Big|_H^G = V_{H,\Omega\omega}$$

branching problem for tori is trivial

Sketch of Algorithm



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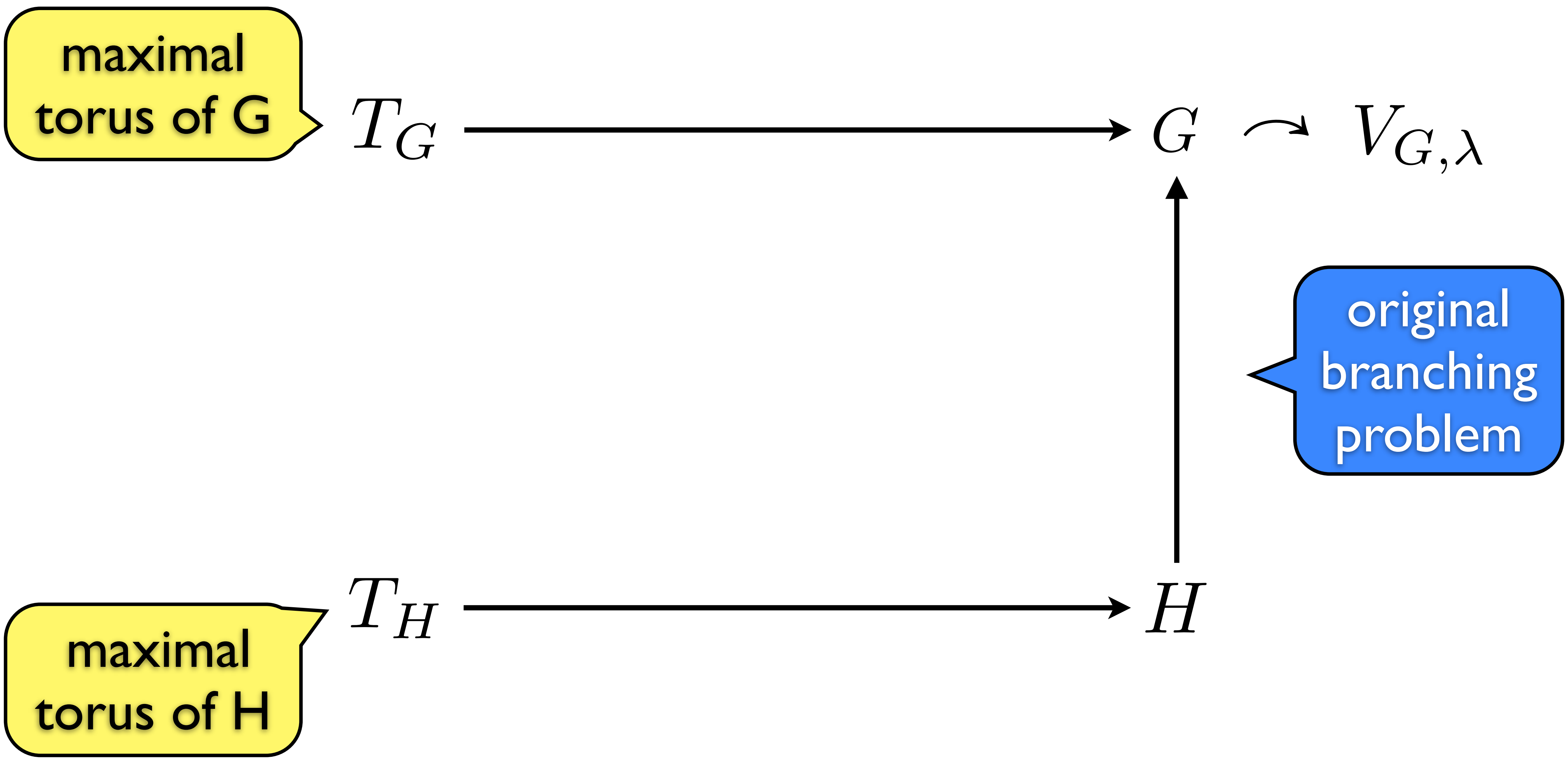

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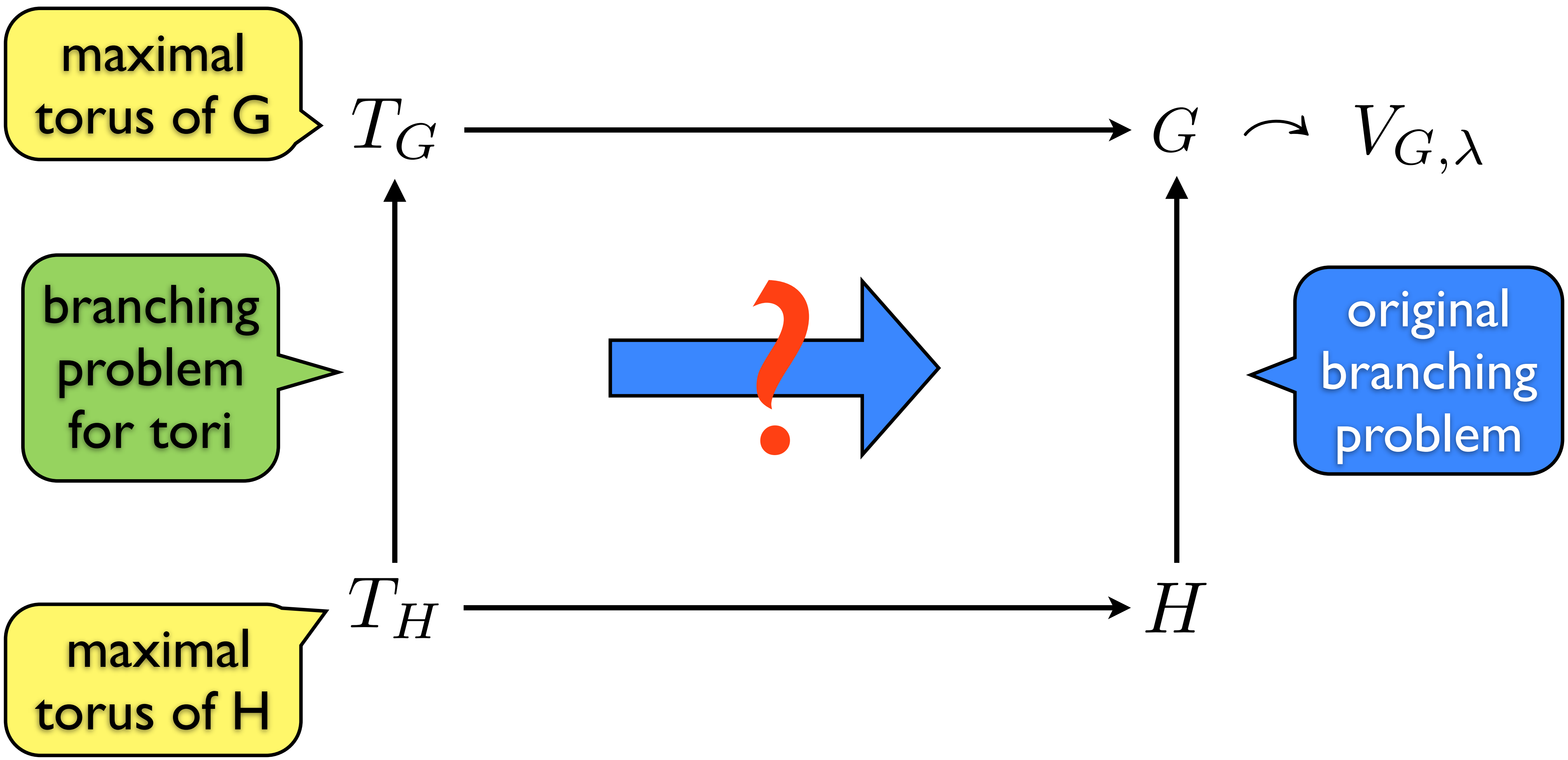

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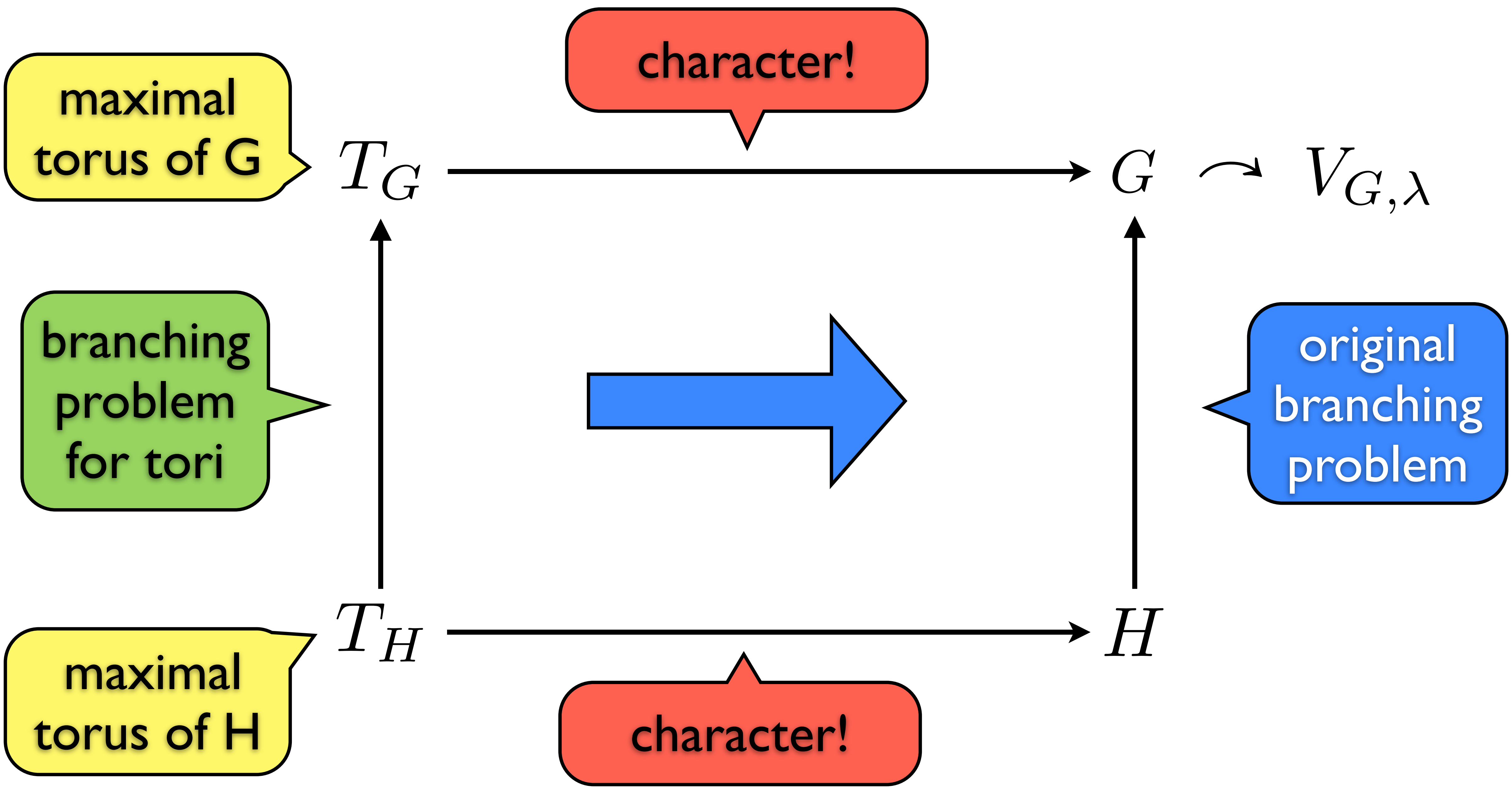

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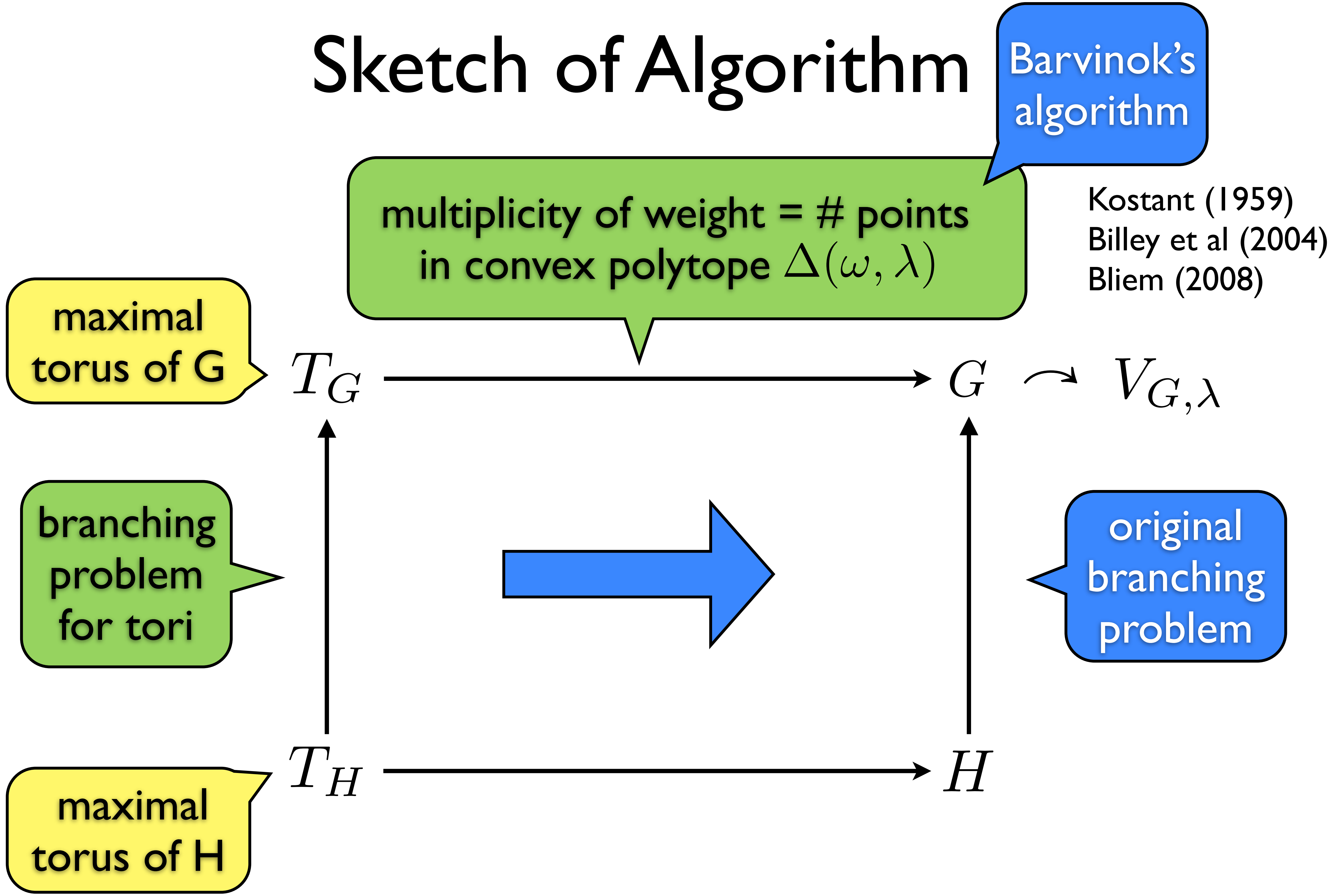

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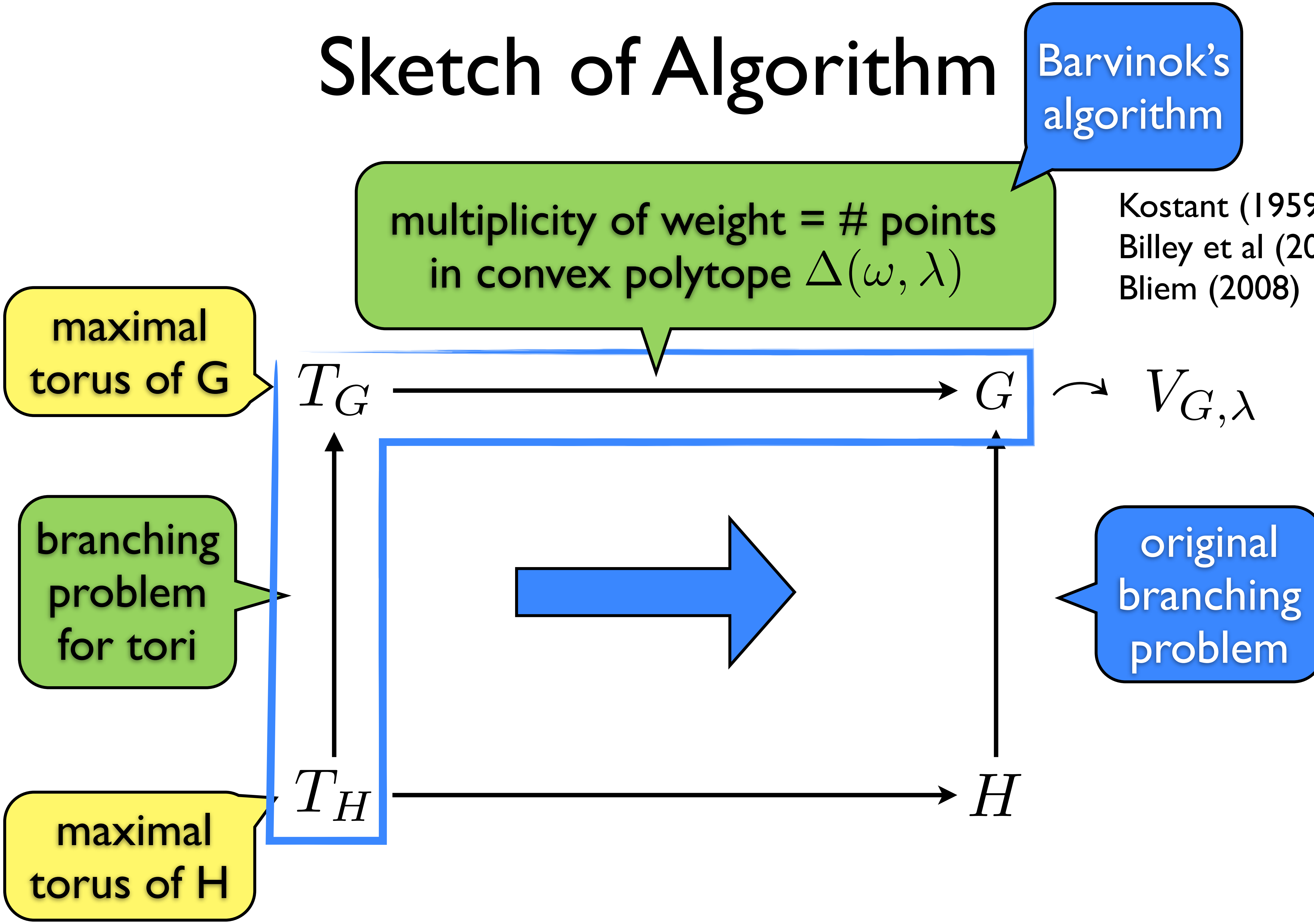

Kostant (1959)
Billey et al (2004)
Bliem (2008)

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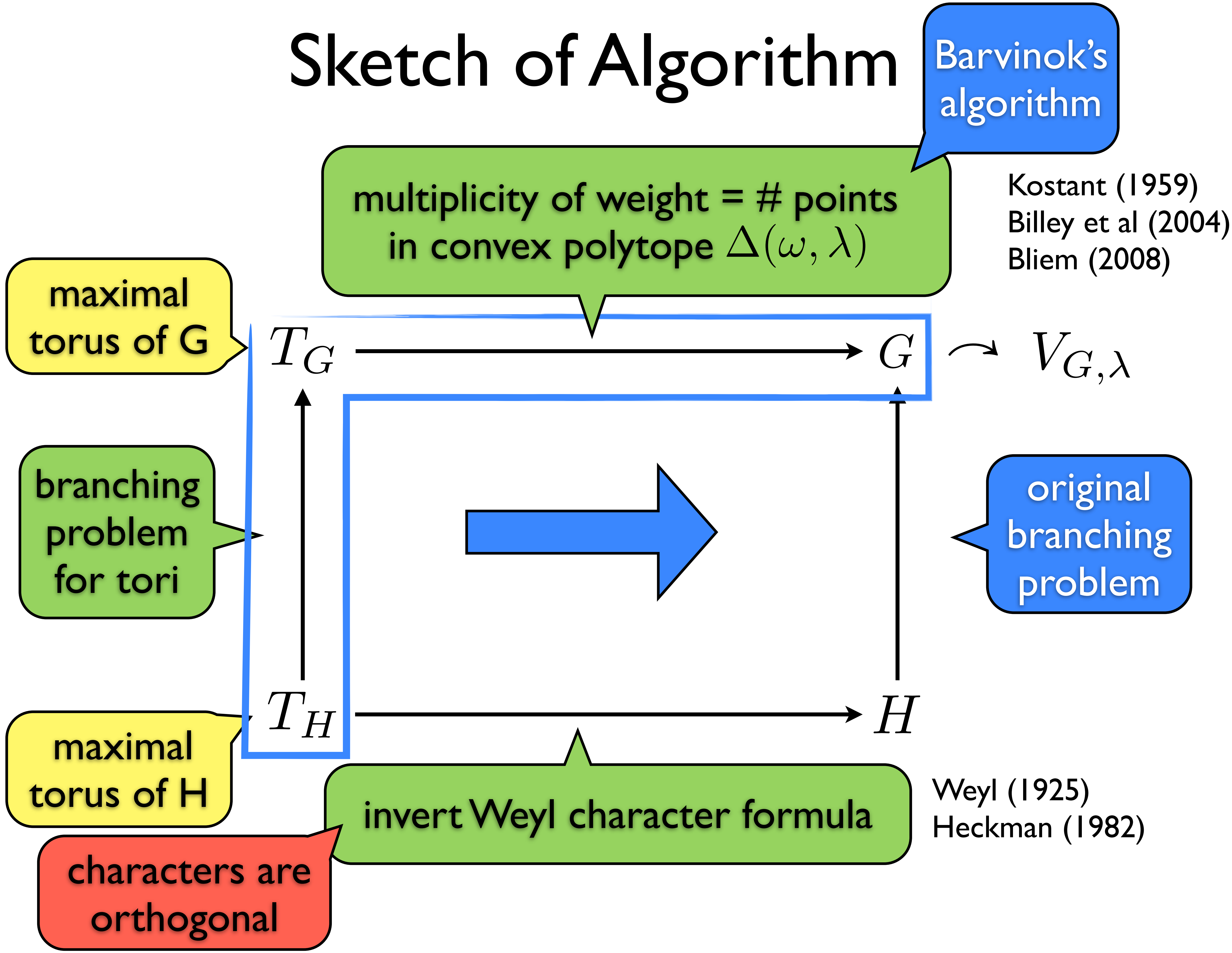

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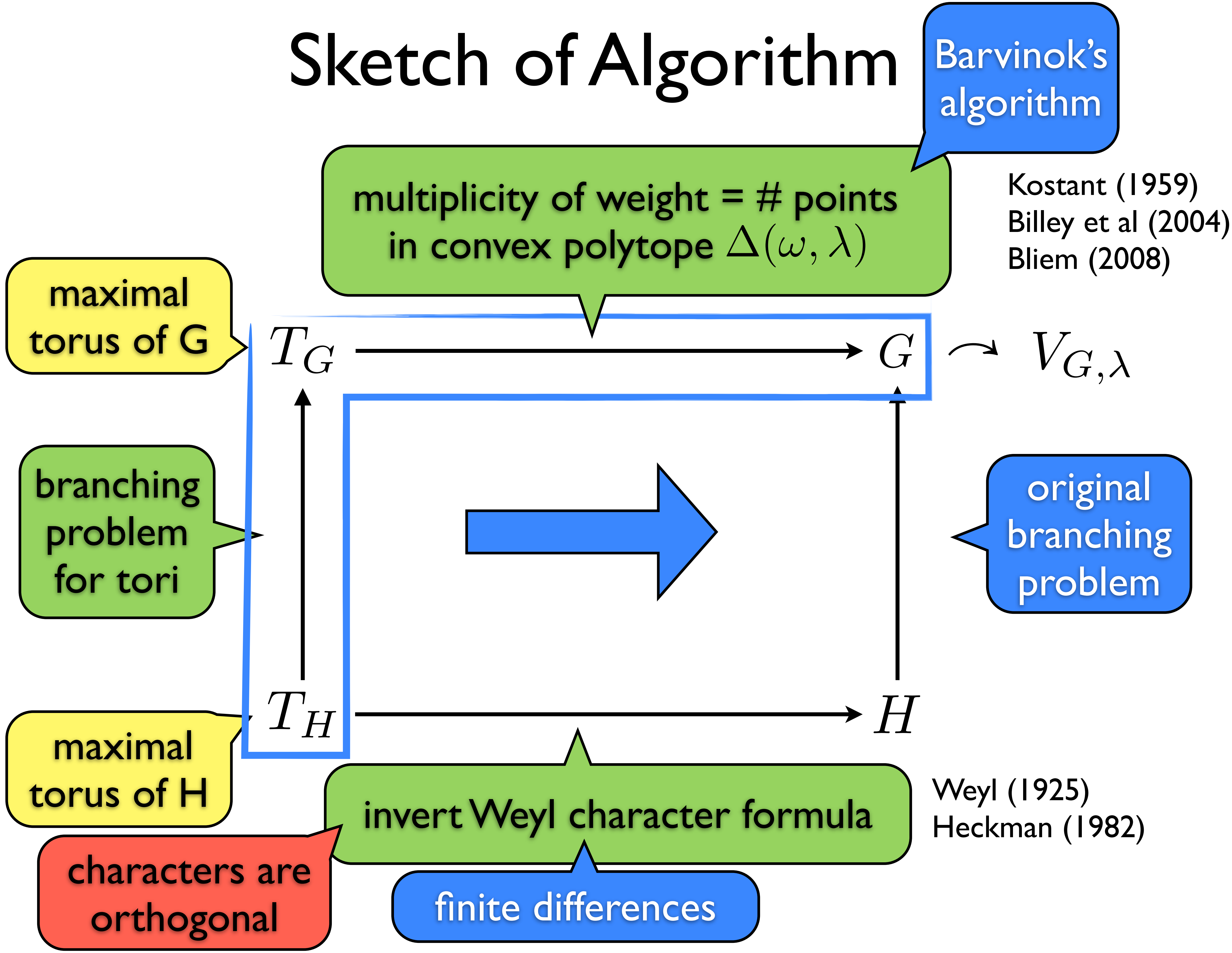
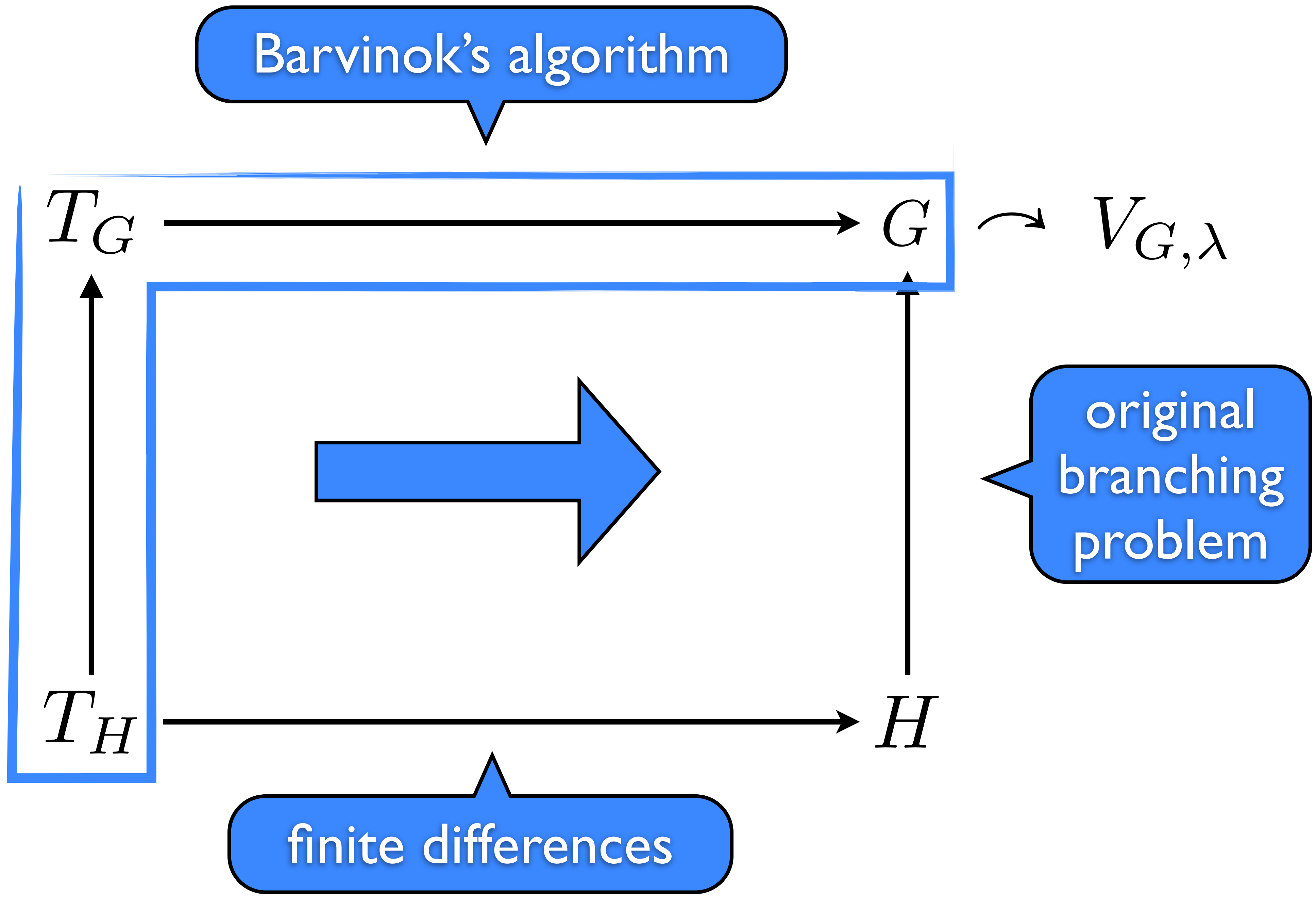


Summary of Algorithm



Variation

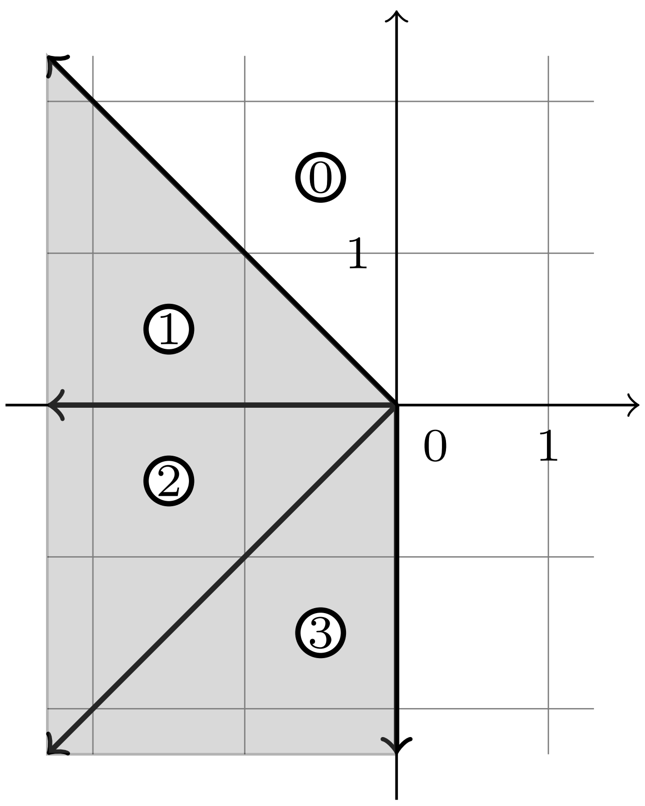
It follows from the proof that m_μ^λ is "piecewise" periodic polynomial function.

Meinrenken & Sjamaar (1999)

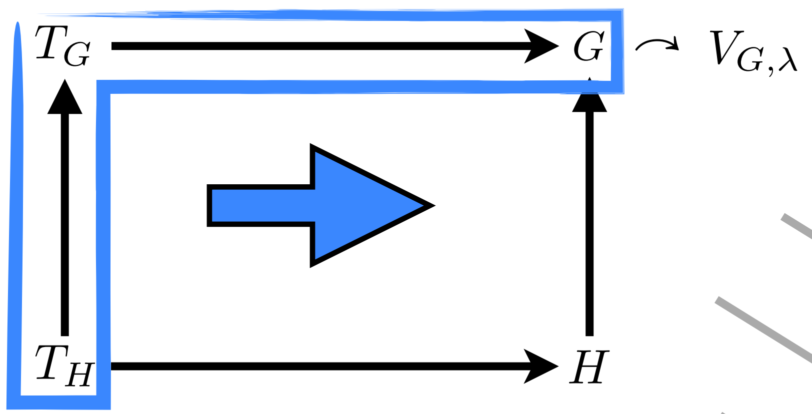
polynomials on sublattices

polyhedral domains

Parametric Algorithm: Can in fact precompute these polynomials *once and for all* (e.g., using parametric version of Barvinok).



Summary (of main part)



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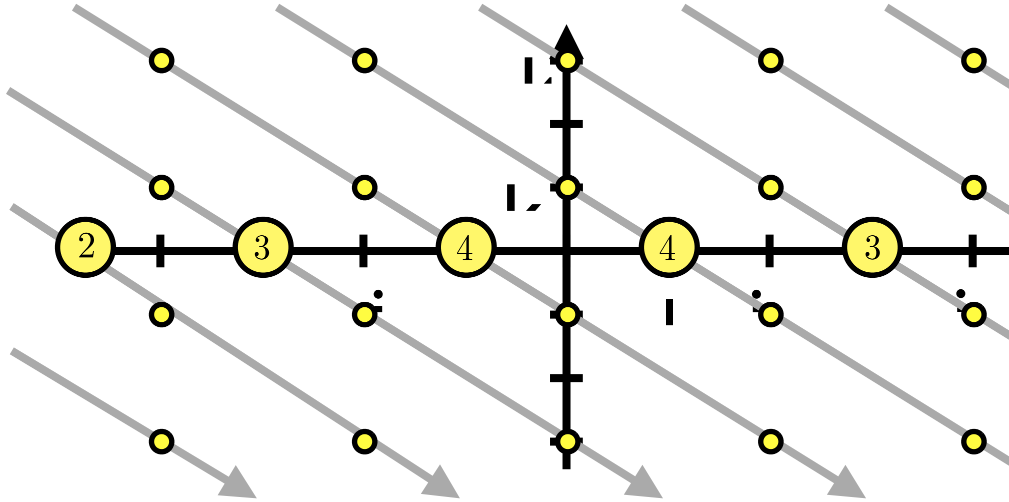

Poly-time algorithm for any branching problem of compact connected Lie groups.

Aside on Asymptotics

<u>Geometric Complexity Theory</u>: Lower bounds from $X \not\subseteq Y$ for certain varieties X, Y (e.g., orbit closure of perm/det of certain size).

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Idea: Study coordinate rings, find obstruction

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multiplicity of V_λ in degree k of coordinate ring of X

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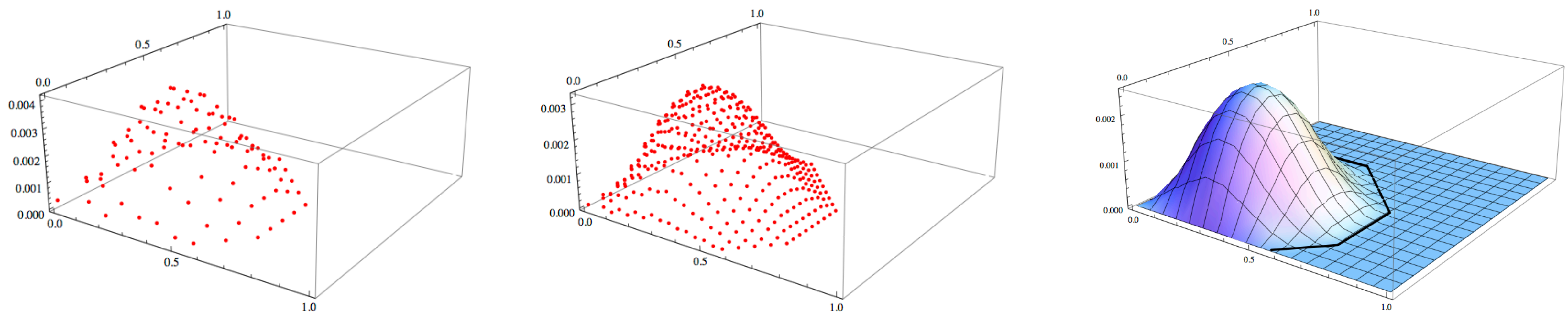
Difficult! Study asymptotics? E.g. asymptotic support (moment polytope).

Bürgisser & Ikenmeyer (2011)

Aside on Asymptotics



Asymptotic growth rate (<u>Duistermaat-Heckman measure):</u>

$$\mathrm{DH}_X = \lim_{k \to \infty} \frac{1}{k^{d_X}} \sum_{\lambda} m_{G,X,k}(\lambda)\, \delta_{\lambda/k}$$

<u>Our observation:</u> No crit. for obstructions, since $d_X \neq d_Y$.

Example: SU(2)

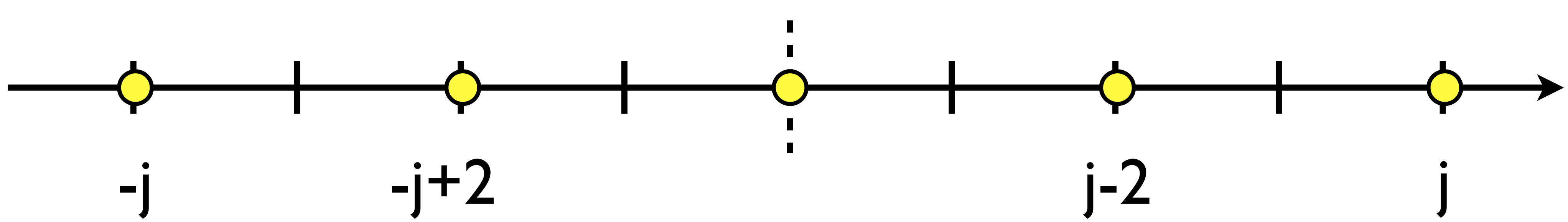
The irreducible representations of SU(2) are

$$V_j = \mathrm{Sym}^j(\mathbb{C}^2)$$

labeled by their <u>spin</u> $j = 0, 1, \ldots$

Maximal torus $\{\begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}\}$ is isomorphic to U(1), and its irred. representations labelled by <u>weight</u> $k \in \mathbb{Z}$.

Weights of V_j (all multiplicities are one):



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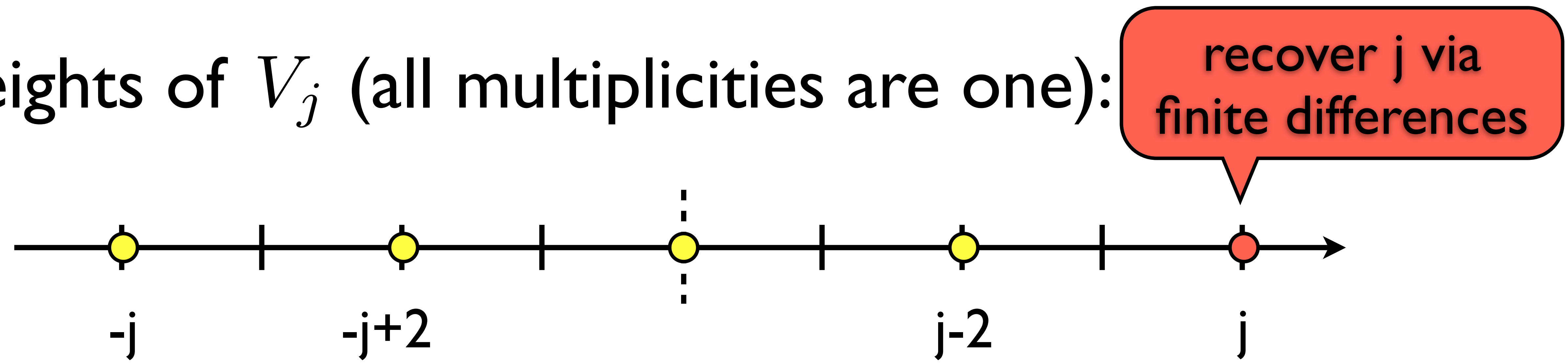

Example: Littlewood-Richardson for SU(2)

Goal: decompose a tensor product of SU(2)-irreps:

$$\begin{array}{ccc} U(1) \times U(1) & \longrightarrow & SU(2) \times SU(2) \curvearrowright V_j \otimes V_k \\ \uparrow & & \uparrow \\ U(1) & \longrightarrow & SU(2) \end{array}$$

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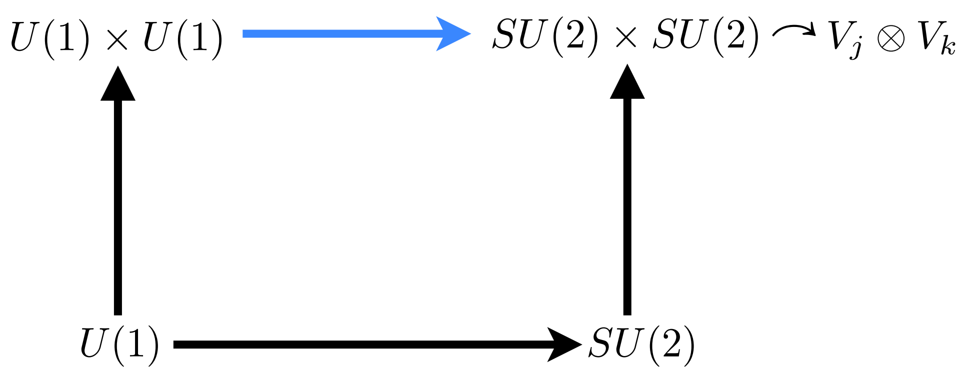
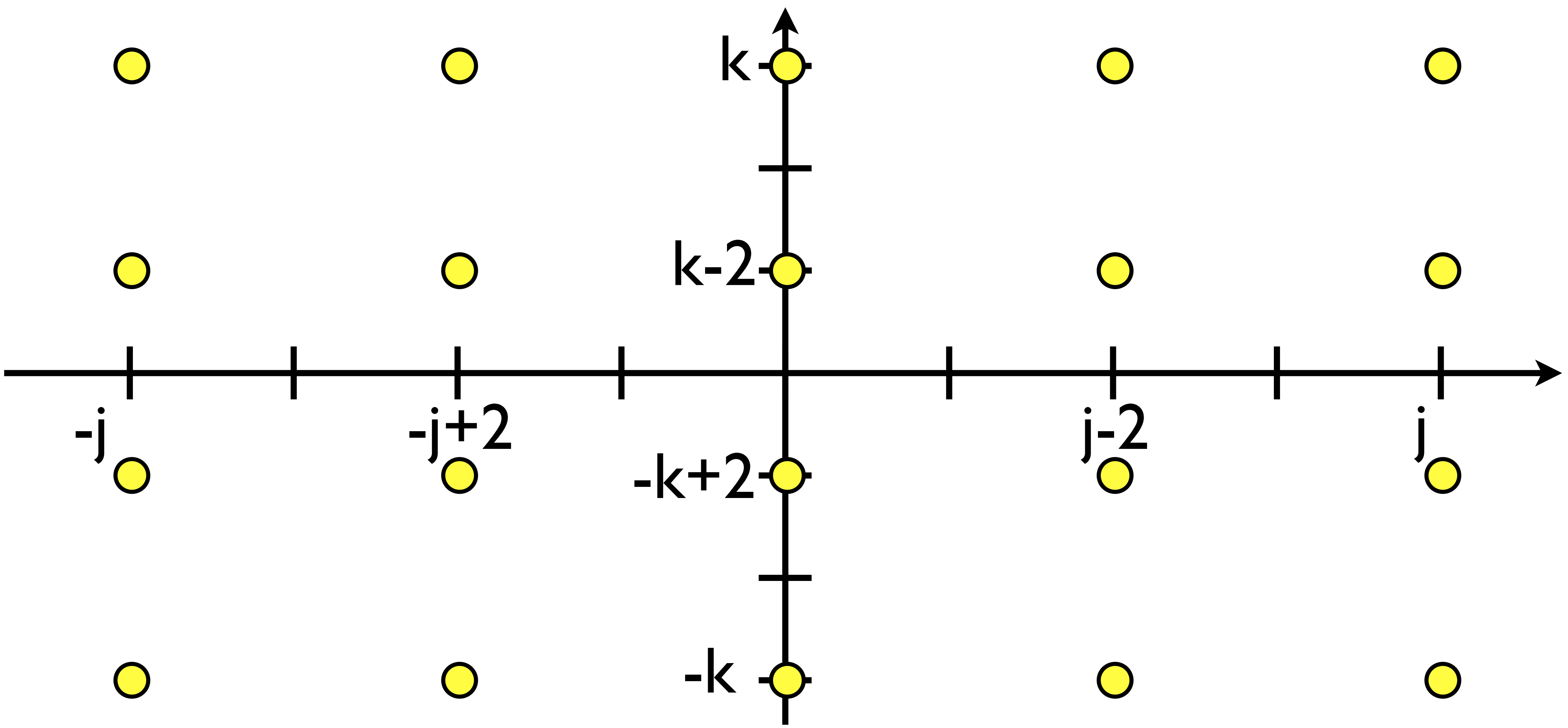


1. Weight multiplicities for $V_j \otimes V_k$:



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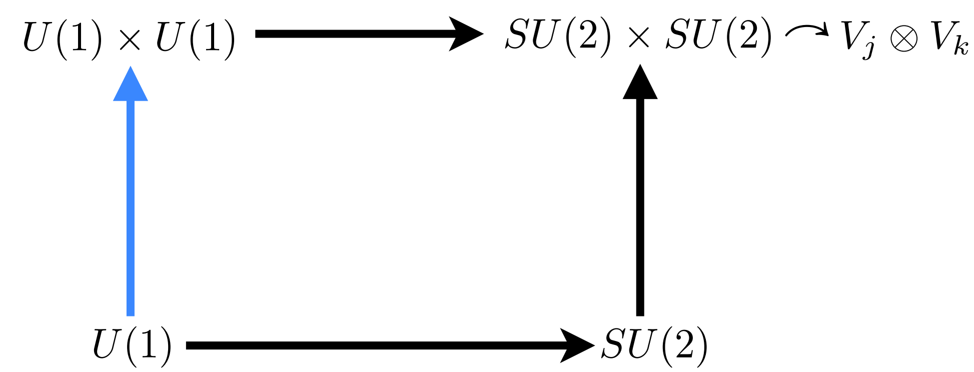

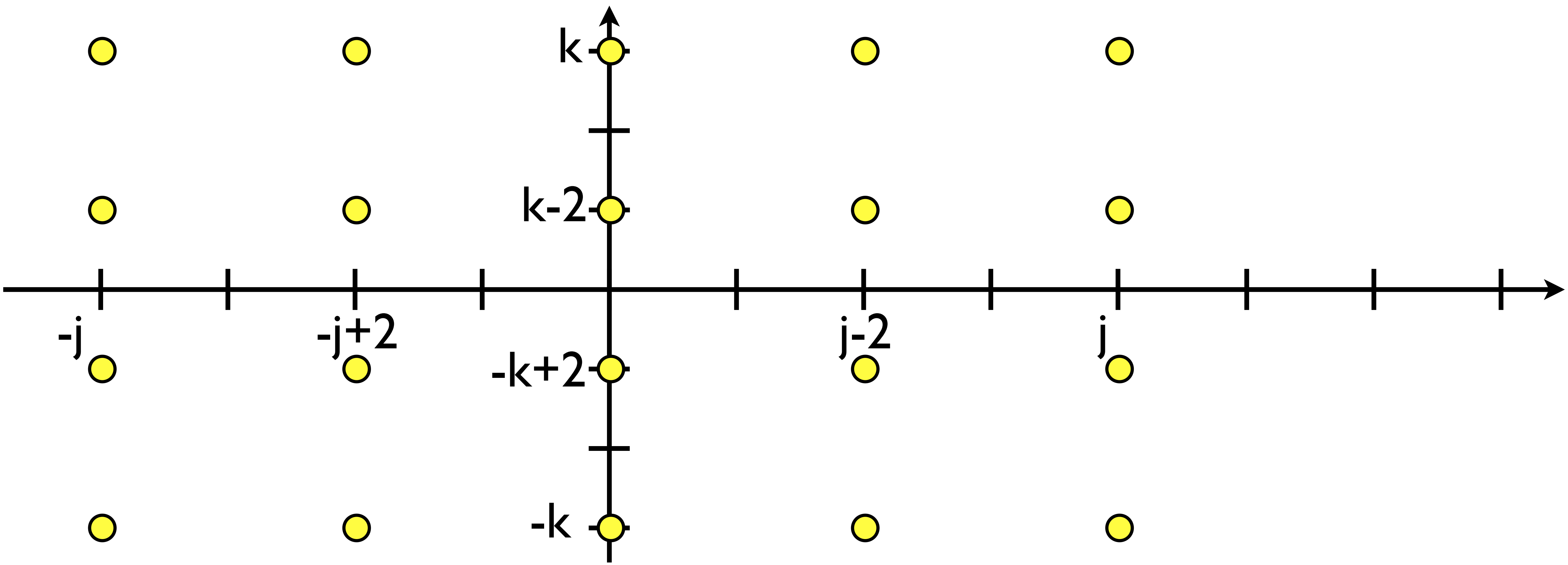

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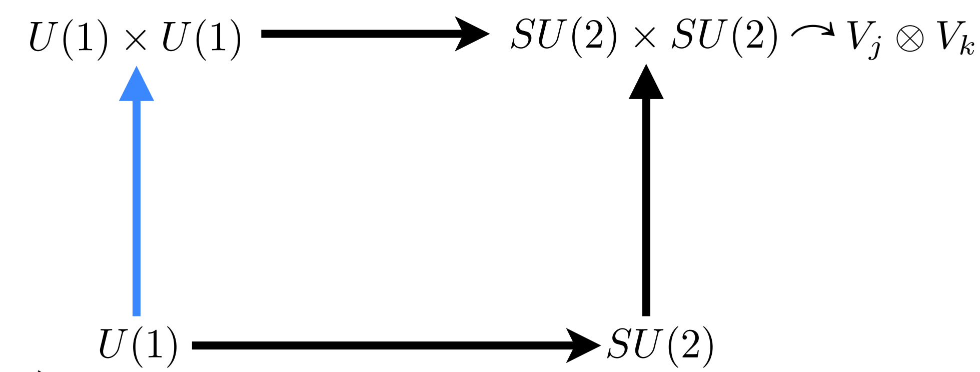

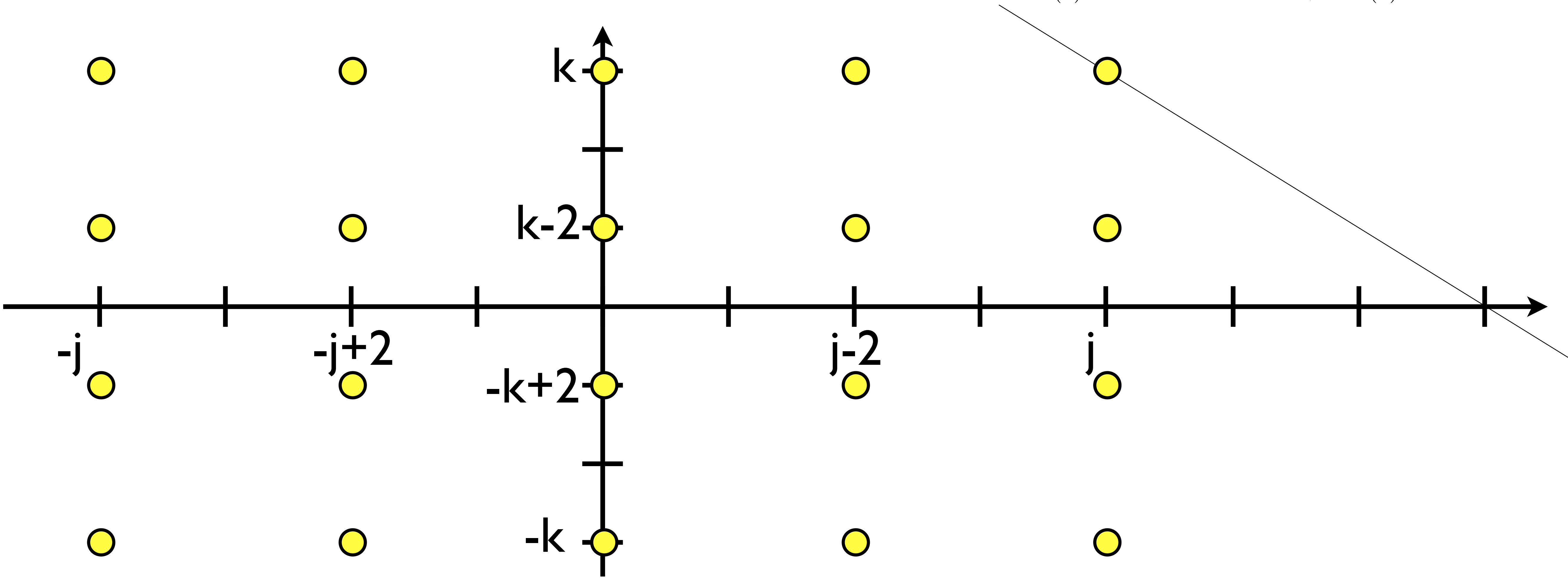

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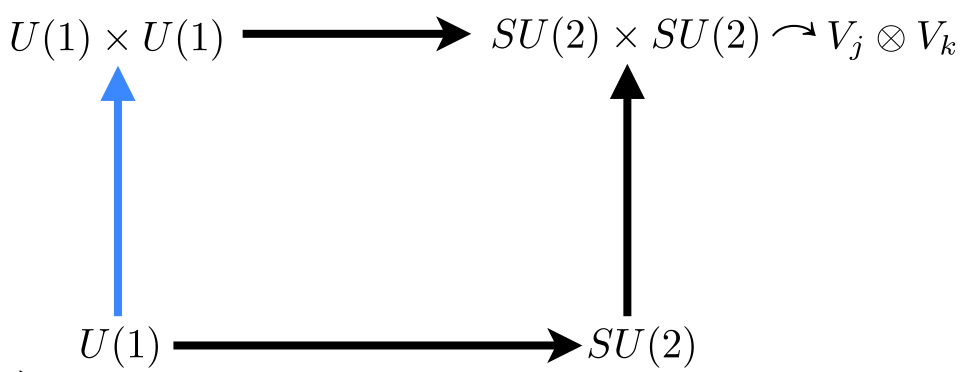

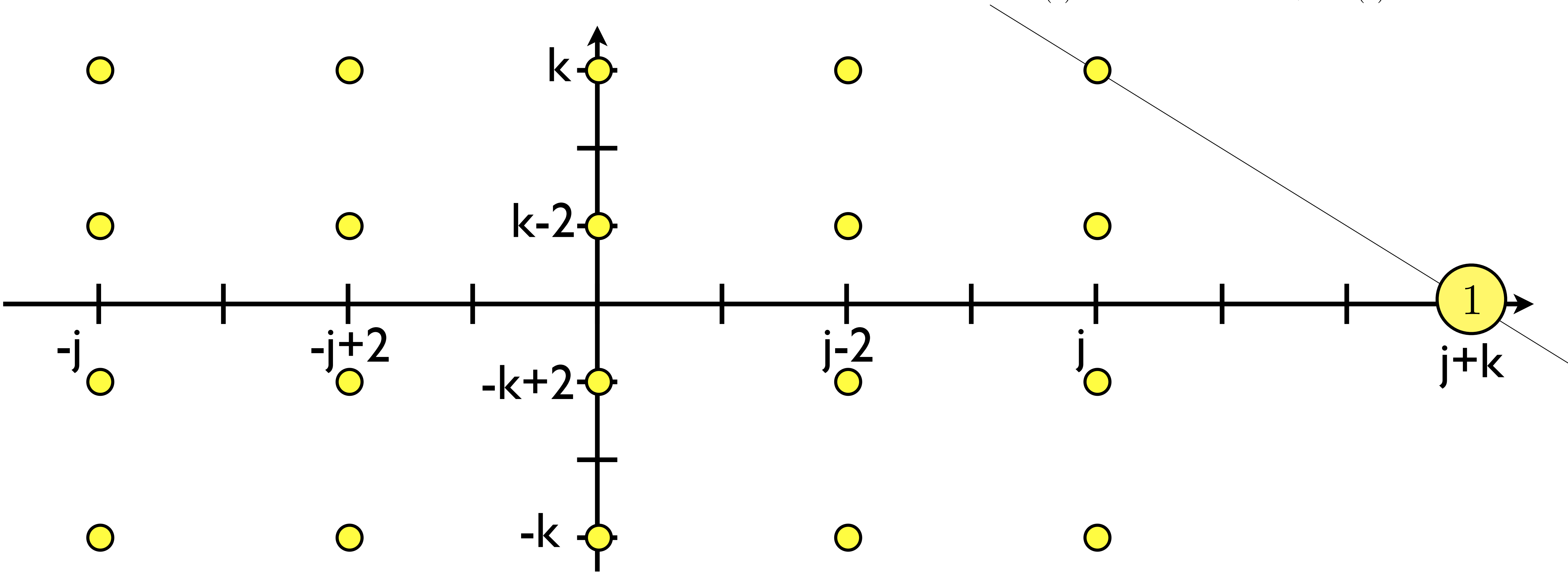

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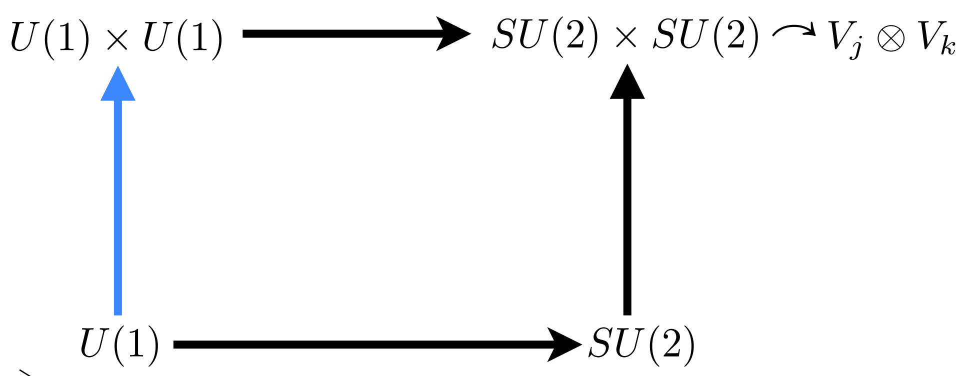

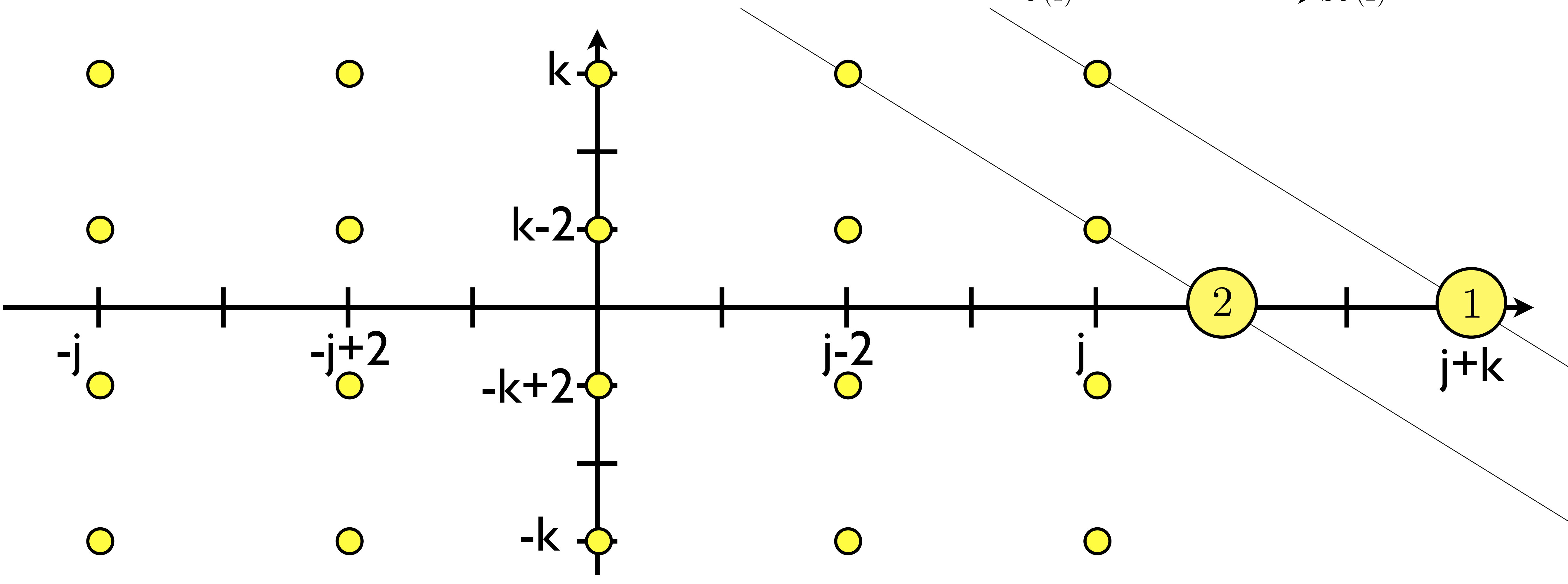

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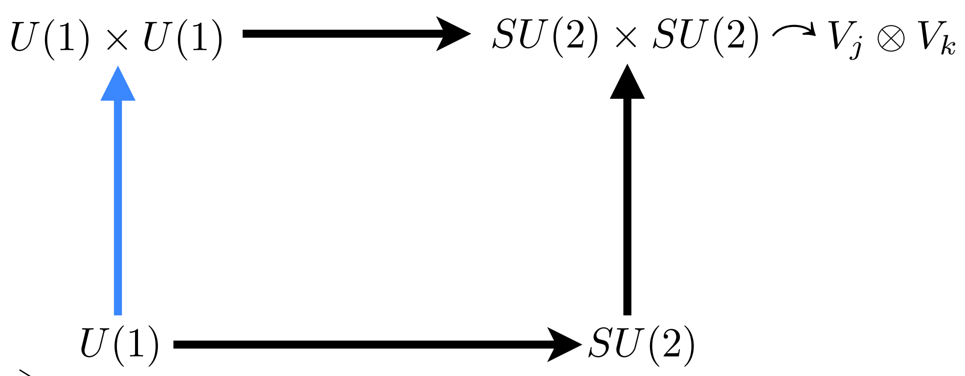

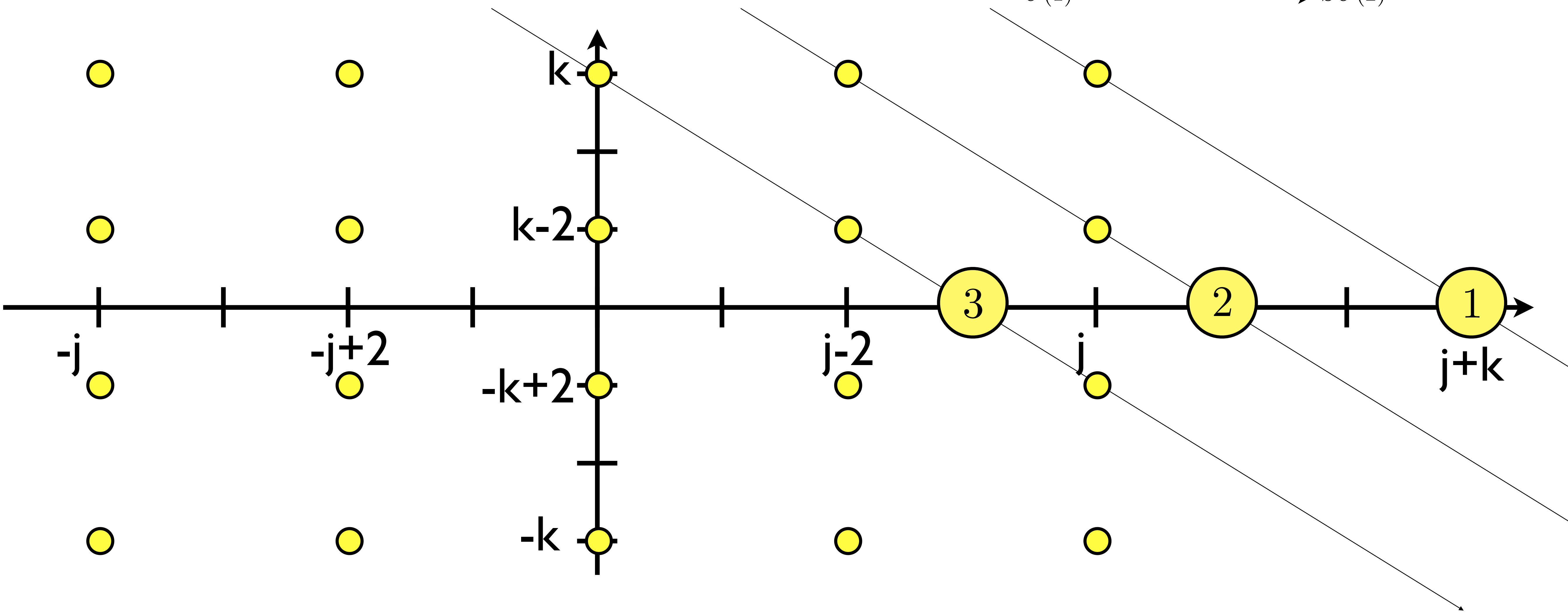

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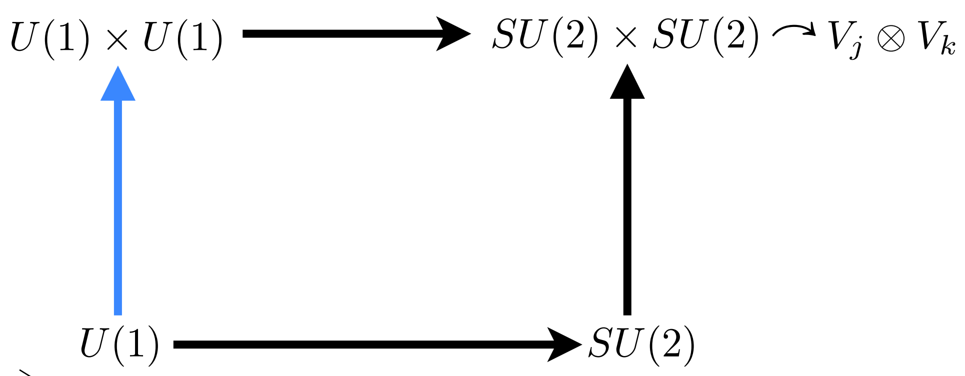

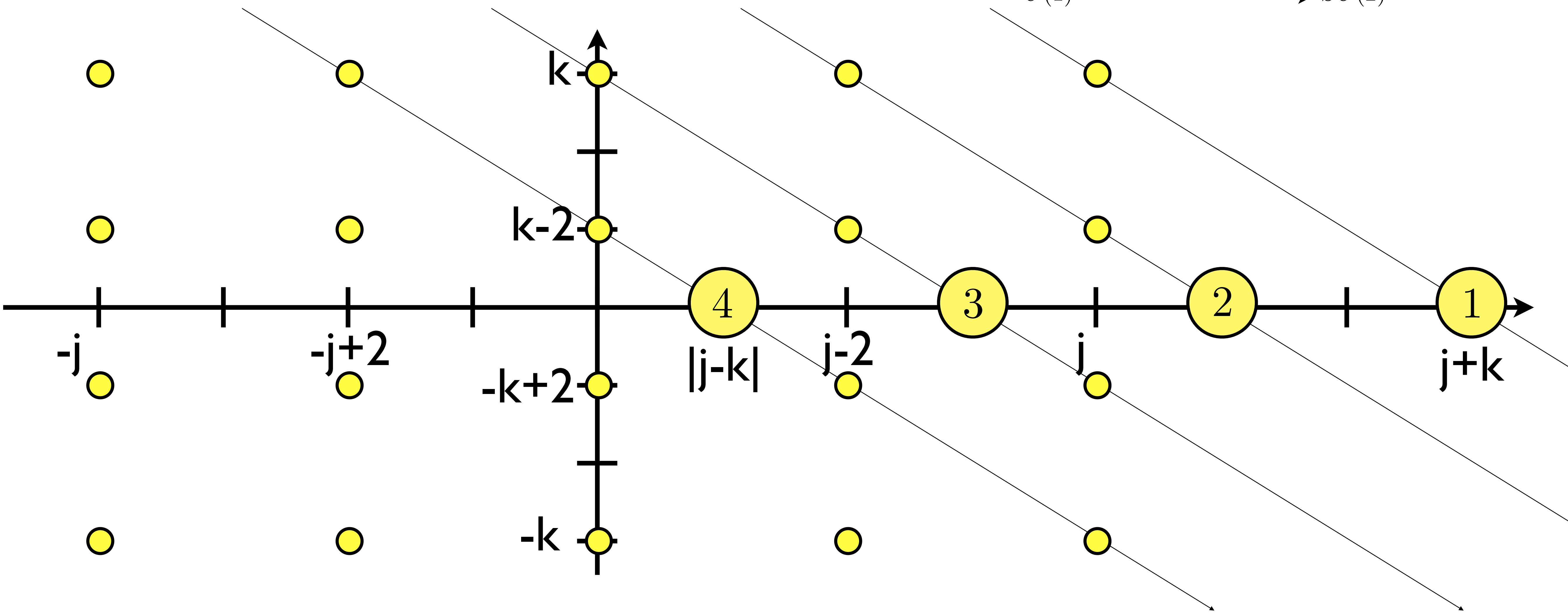

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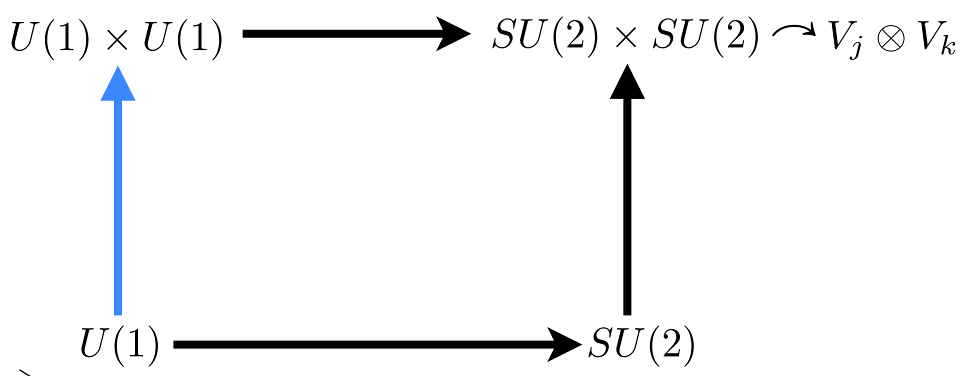

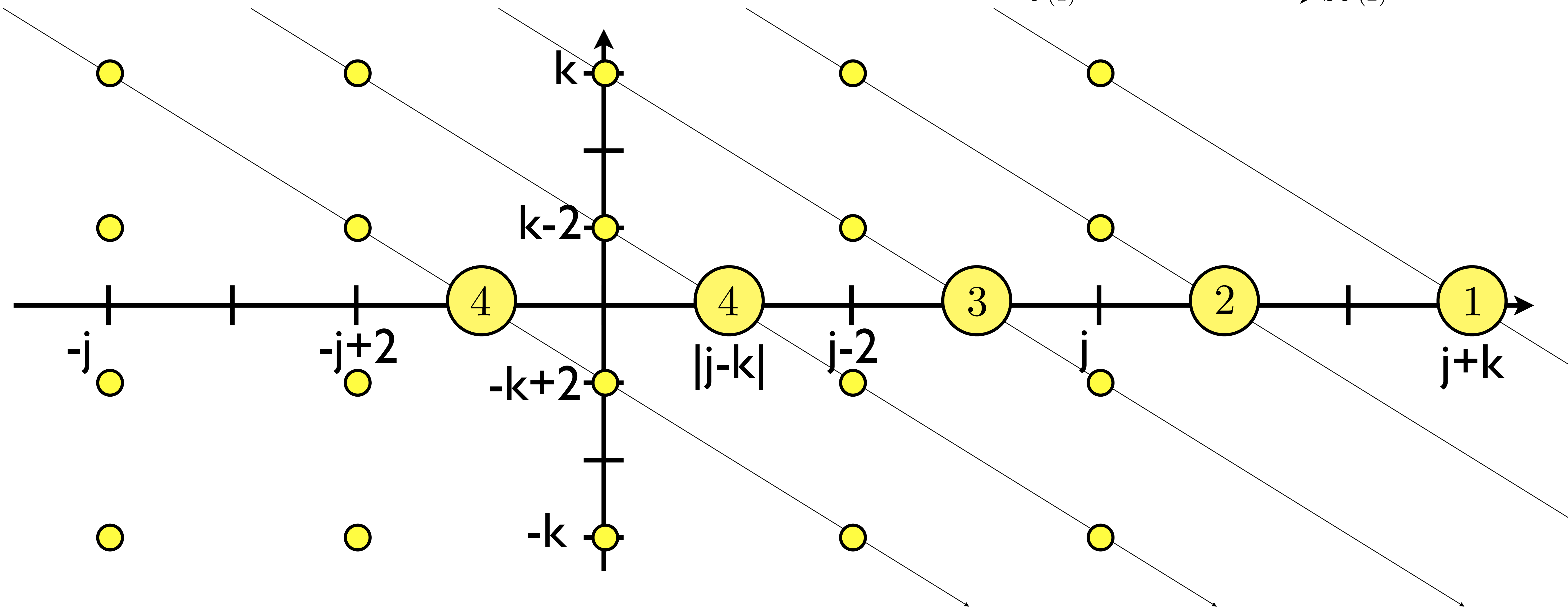

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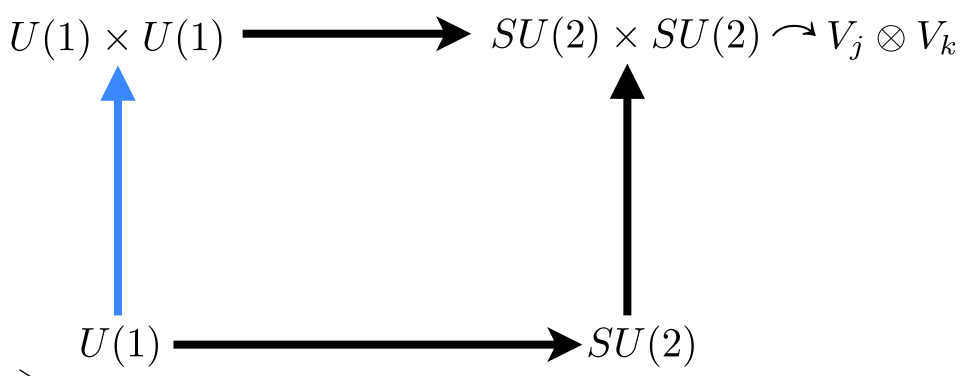

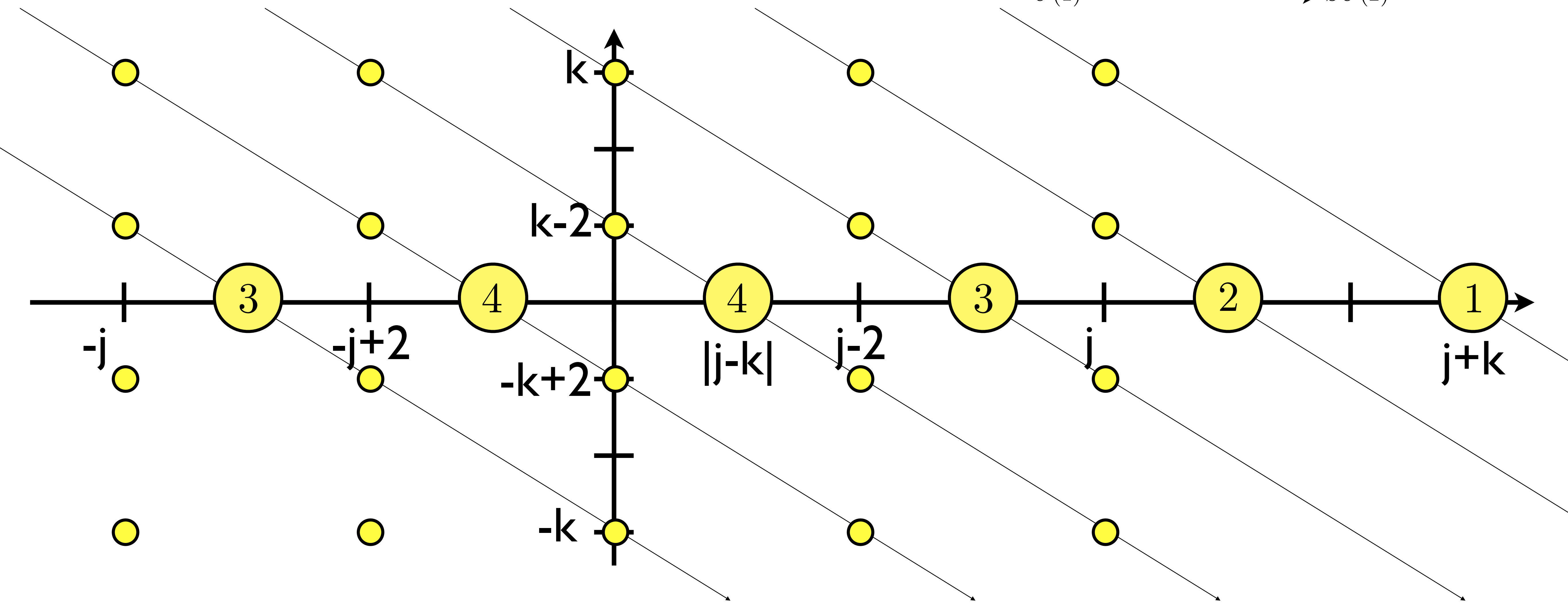

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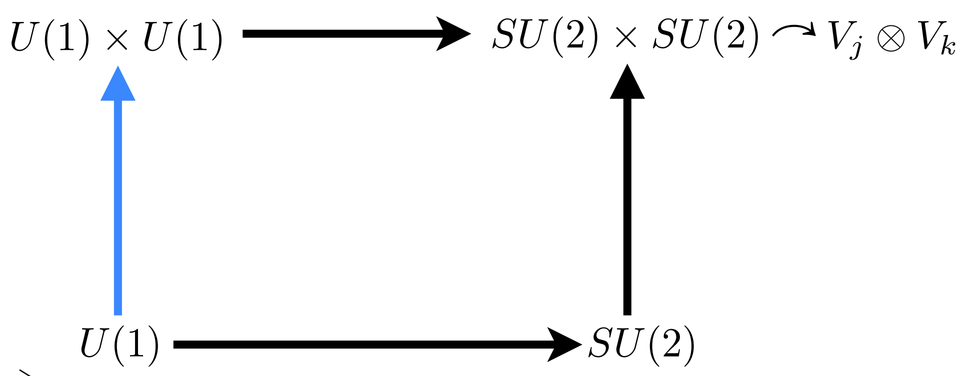

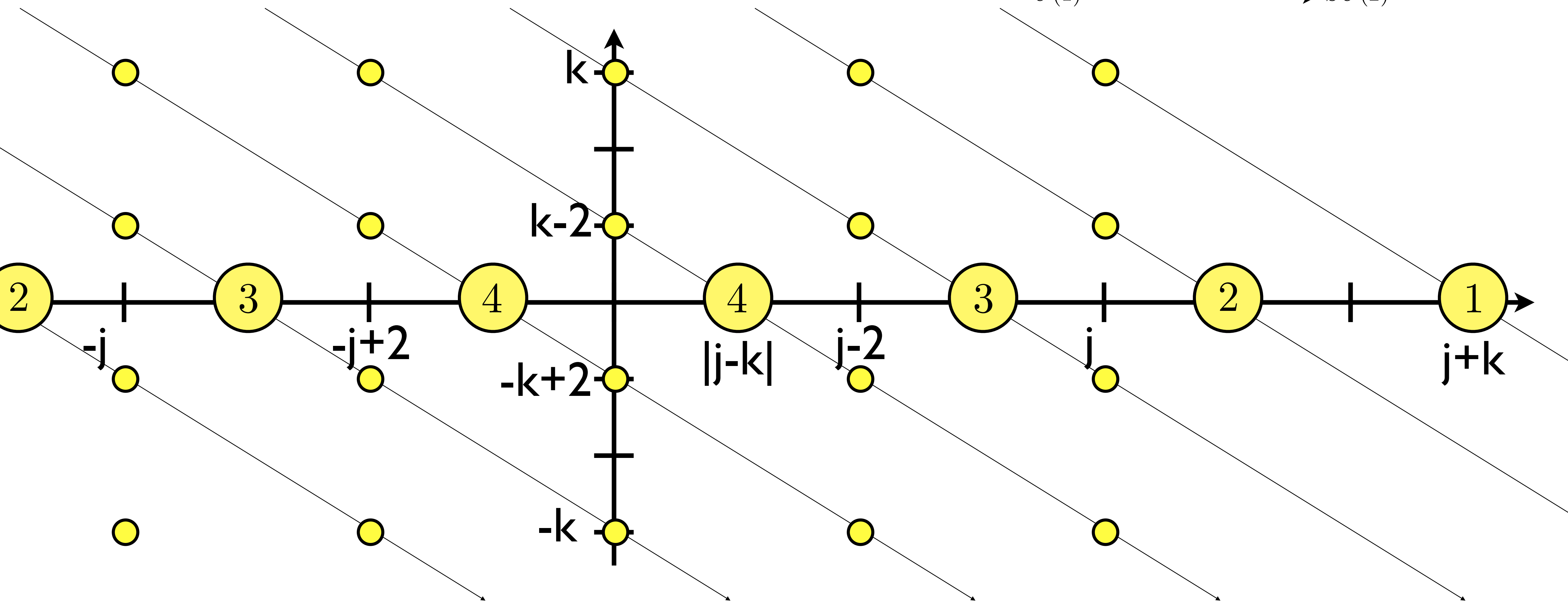

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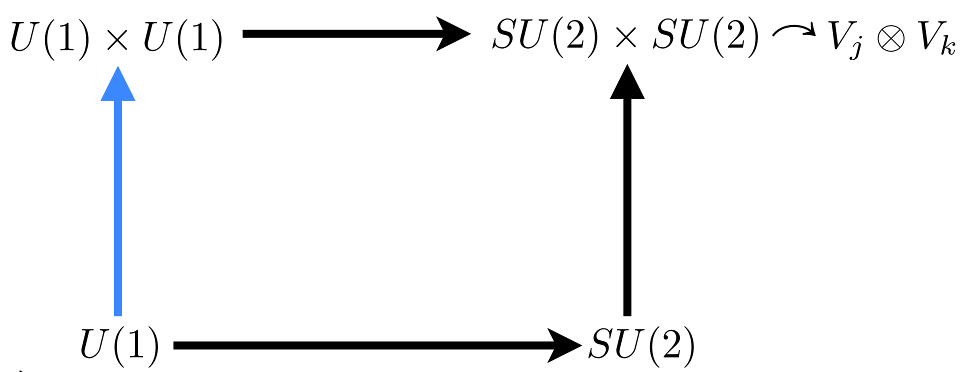

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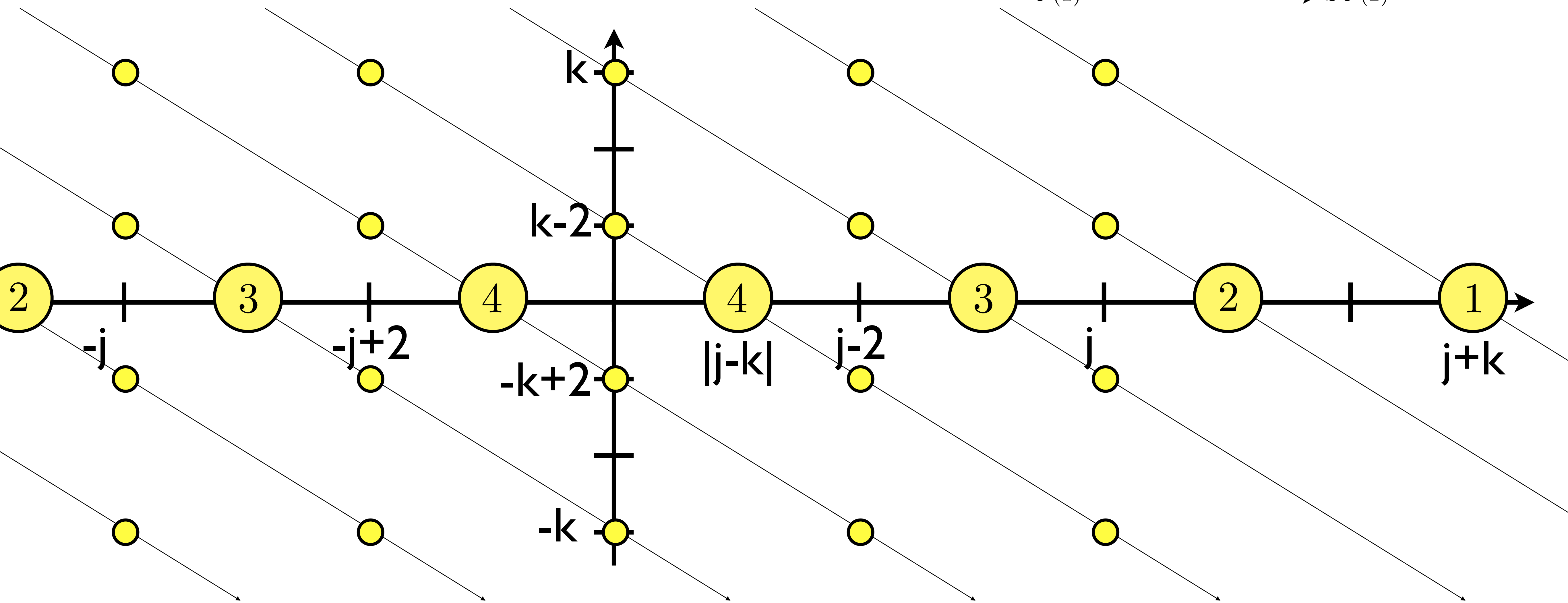

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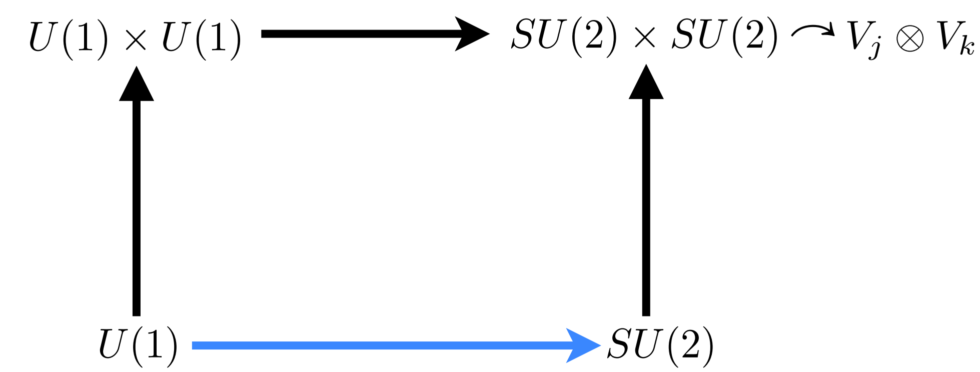
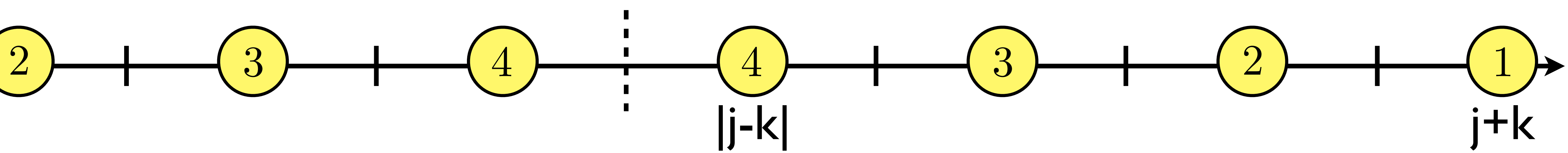


3. Reconstruct decomposition from weight multiplicities:



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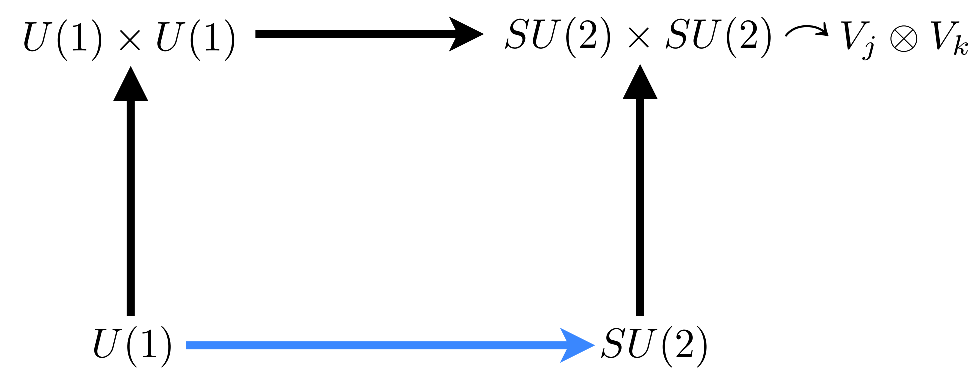

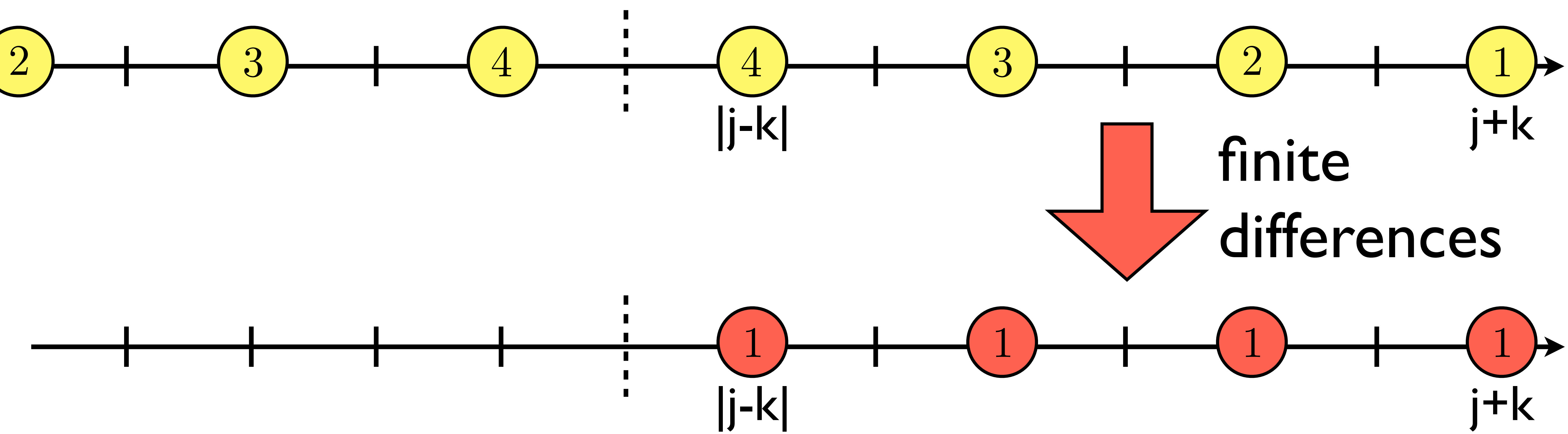

Example: Littlewood-Richardson for U(2)

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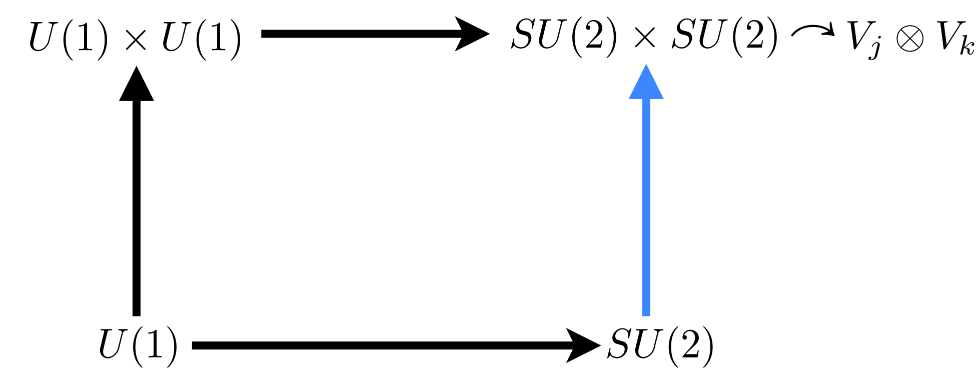
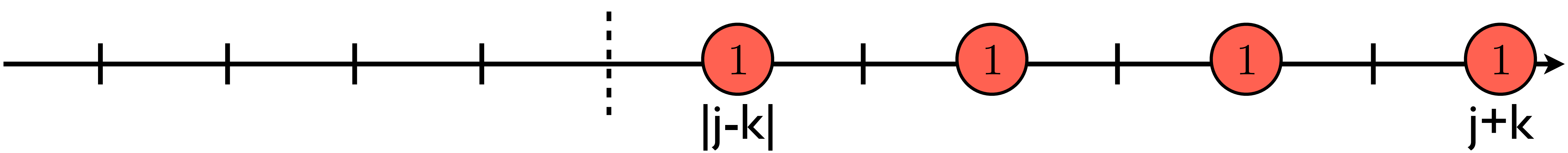
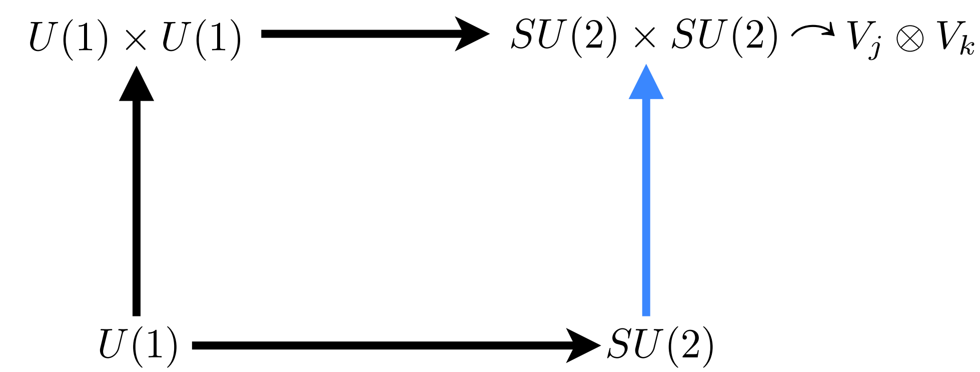




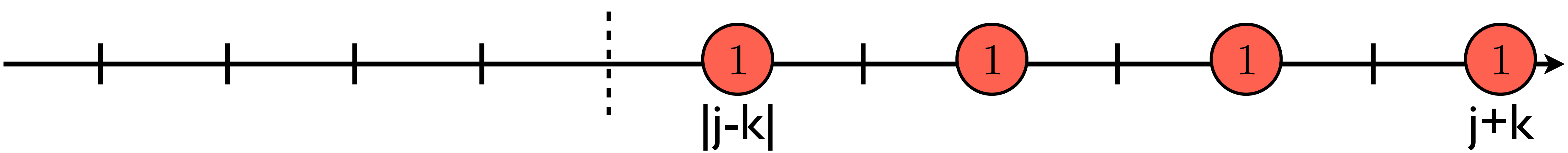
Result:

$$m_l^{j,k} = \begin{cases} 1 & \text{if } l = |j-k|, \ldots, j+k \\ 0 & \text{otherwise} \end{cases}$$

Example: Littlewood-Richardson for U(2)



3. Reconstruct decomposition from weight multiplicities:



Result:

$$m_l^{j,k} = \begin{cases} 1 & \text{if } l = |j-k|, \ldots, j+k \\ 0 & \text{otherwise} \end{cases}$$

piecewise periodic polynomial