Analytic algorithms for null cone membership

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Overview

- *Null cone* membership: fundamental problem in *invariant theory*.
- *Connections* to several areas of computer science, mathematics and physics.

Geometric complexity theory – asymptotic vanishing of Kronecker coefficients.

Quantum information theory– one-body quantum marginal problem.

Functional analysis – Brascamp-Lieb inequalities.

Optimization– Geodesic convexity. Captures general linear programming.

Complexity theory and derandomization – Special cases of polynomial identity testing.

- *Analytic algorithms* for algebraic problems.
- *Non-convex* optimization problems but *geodesically convex*.

Outline

- Invariant theory and null cone
- Geometric invariant theory
- Algorithms: a sample
- Open problems

Invariant theory and null cone

Linear actions of groups

Group G acts *linearly* on vector space V $(= \boldsymbol{C}^d)$.

$\pi: G \to GL(V)$ ($d \times d$ matrices) group homomorphism.

$M_g: V \to V$ invertible linear map $\forall\, g \in G$.

$M_{g_1 g_2} = M_{g_1} M_{g_2}$ and $M_{id} = id$.

Example 1

$G = S_n$ acts on $V = \boldsymbol{C}^n$ by *permuting coordinates*.

$$M_\sigma\,(x_1, \dots, x_n) = \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right).$$

Example 2

$G = GL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})$ by *conjugation*.

$$M_A\, X = AXA^{-1}.$$

Objects of study

Group G acts *linearly* on vector space V.

- Invariant polynomials: Polynomial functions on V invariant under action of G. p s.t. $p(M_g v) = p(v)$ for all $g \in G, v \in V$.
- Orbits: Orbit of vector v, $O_v = \{M_g v : g \in G\}$.
- Orbit-closures: Orbits may not be closed. Take their closures.

  Orbit-closure of vector v, $\overline{O_v} = \text{cl}\,\{M_g v : g \in G\}$.

Null cone

Group G acts *linearly* on vector space V.

Null cone: Vectors v s.t. 0 lies in the orbit-closure of v.

$$\{v: 0 \in \overline{O_v}\}.$$

Sequence of group elements $g_1, \ldots, g_k, \ldots$ s.t. $\lim_{k \to \infty} M_{g_k} v = 0$.

Problem: Given $v \in V$, decide if it is in the null cone.

Captures many interesting questions.

[Hilbert 1893; Mumford 1965]: v in null cone iff $p(v) = 0$ for all homogeneous invariant polynomials p.

- One direction clear (polynomials are continuous).
- Other direction uses *Nullstellansatz* and some algebraic geometry.

Example 1

$G = S_n$ acts on $V = \boldsymbol{C}^n$ by permuting coordinates.

$$M_\sigma(x_1, \dots, x_n) \to \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right).$$

Null cone = $\{0\}$.

No closures.

Example 2

$G = GL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})$ by *conjugation*.

$$M_A\, X = AXA^{-1}.$$

- Invariants: generated by $\mathrm{tr}(X^i)$.
- Null cone: nilpotent matrices.

Example 3

$G = SL_n(\mathbf{C}) \times SL_n(\mathbf{C})$ acts on $V = M_n(\mathbf{C})$ by left-right multiplication.

$$M_{(A,B)}\, X = AXB.$$

- Invariants: generated by $\mathrm{Det}(X)$.
- Null cone: Singular matrices.

Example 4

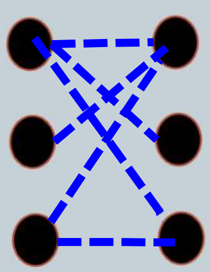
ST_n: group of $n \times n$ diagonal matrices with determinant 1.

$G = ST_n \times ST_n$ acts on $V = M_n(\boldsymbol{C})$ by left-right multiplication.

$$M_{(A,B)}\,X = AXB.$$

- Invariants: generated by $X_{1,\sigma(1)} \cdot X_{2,\sigma(2)} \cdots X_{n,\sigma(n)}$.
- Null cone: perfect matching.

A_H is in null cone iff H has no perfect matching.



H

1	1	1
1	0	0
1	0	1

A_H

Example 5: Linear programming

T_n: (Abelian!) group of $n \times n$ diagonal matrices.

V: (Laurent) polynomials.

G acts on V by scaling variables. $\boldsymbol{t} \in T_n$, $\boldsymbol{t} = \mathrm{diag}(t_1, \dots, t_n)$.

$$M_{\boldsymbol{t}}\, q(x_1, \dots, x_n) = q(t_1 x_1, \dots, t_n x_n).$$

$q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha$. $\mathrm{supp}(p) = \{\alpha \in \Omega : c_\alpha \neq 0\}$.

Null cone ↔ Linear Programming

q *not in* null cone ↔ $0 \in conv\{\alpha : \alpha \in \mathrm{supp}(q)\}$.

In non-Abelian groups, the null cone (membership) problem is a *non-commutative* analogue of *linear programming*.

Example 6

$G = SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ acts on $V = M_n(\mathbb{C})^{\oplus m}$ by *simultaneous left-right* multiplication.

$$M_{(B,C)}\,(X_1, \ldots, X_m) = (BX_1C, \ldots, BX_mC).$$

- Invariants [DW 00, DZ 01, SdB 01, ANS 10]: generated by $\mathrm{Det}(\sum_i D_i \otimes X_i)$.
- Null cone: Non-commutative singularity. Captures non-commutative rational identity testing.

[GGOW 16, DM 16, IQS 16]: Deterministic polynomial time algorithms.

Geometric invariant theory: certification of null cone

GIT: computational perspective

What is *complexity* of *null cone* membership?

GIT puts it in $NP \cap coNP$ (morally).

- Hilbert-Mumford criterion: how to certify membership in null cone.
- Kempf-Ness theorem: how to certify non-membership in null cone.

Kempf-Ness

Group G acts linearly on vector space V.

How to *certify* v not in null cone?

Exhibit *invariant* polynomial P s.t. $P(v) \neq 0$.

Not feasible in general.

Invariants hard to find, high degree, high complexity etc.

Kempf-Ness provides another (efficient) way.

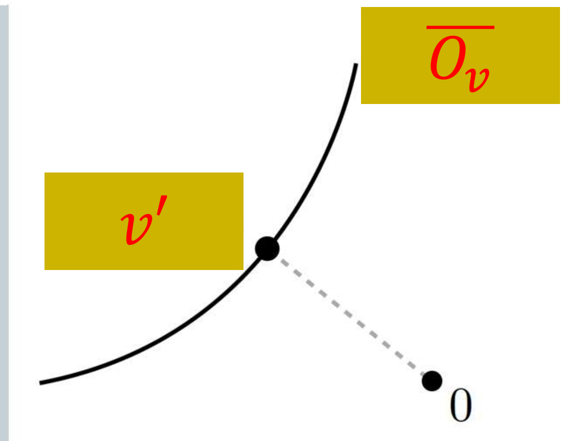
An optimization perspective

Finding *minimal norm* elements in orbit-closures!

Group G acts linearly on vector space V.

$$\mathrm{cap}(v) = \inf_{g \in G} \|M_g \, v\|_2^2.$$

Null cone: v s.t. $\mathrm{cap}(v) = 0$.



Moment map

Group G acts linearly on vector space V.

$f_v(g) = \|M_g\, v\|_2^2$.

Moment map $\mu_G(v)$: gradient of $f_v(g)$ at $g = id$.

How much *norm* of v decreases by *infinitesimal action* around id.

Much more general.

Moment → *momentum*.

Fundamental in *symplectic geometry* and *physics*.

Example 1

$G = \boldsymbol{C}^*$ acts on $V = \boldsymbol{C}$.

$\pi(t)\, v = t\, v$.

$f_v(t) = |t|^2 |v|^2$.

Moment map: consider action of $t = \exp(s)$, $s \approx 0$.

$$\mu_G(v) = \frac{d}{ds}\left[\exp(2s)\, |v|^2\right]\Big|_{s=0} = 2|v|^2.$$

Example 2

$G = ST_n \times ST_n$ acts on $V = M_n$.

$M_{(D_1,D_2)}\, X = D_1\, X\, D_2$.

$q_1, q_2 \in \boldsymbol{R}^n$: $\sum_i q_1(i) = \sum_j q_2(j) = 0$.

Directional derivative: action of $(\exp(s\, q_1)\,, \exp(s\, q_2))$, $s \approx 0$.

- $\mu_G(X) = (p_1, p_2)$, $\sum_i p_1(i) = \sum_j p_2(j) = 0$ s.t.
- $\langle p_1, q_1 \rangle + \langle p_2, q_2 \rangle = \frac{d}{ds}\left[\|\pi(\exp(s\, q_1)\,, \exp(s\, q_2))\, X\|_F^2\right]\Big|_{s=0}$

$$= 2\,\langle r_X, q_1 \rangle + 2\,\langle c_X, q_2 \rangle$$

$$= 2\,\langle r_X - \alpha \mathbf{1}, q_1 \rangle + 2\,\langle c_X - \alpha \mathbf{1}, q_2 \rangle$$

r_X, c_X vectors of *row* and *column* ℓ_2^2 norms of X.

$\mu_G(X) = 2(r_X - \alpha \mathbf{1}, c_X - \alpha \mathbf{1})$.

Kempf-Ness

Group G acts linearly on vector space V.

[Kempf, Ness 79]: v not in null cone iff *non-zero* w in *orbit-closure* of v s.t. $\mu_G(w) = 0$.

w *certifies* v not in null cone.

One direction easy.

- v not in null cone. Take w vector of *minimal norm* in orbit-closure of v. w non-zero.
- w minimal norm in its orbit. ⇒ Norm does not decrease by infinitesimal action around id. ⇒ $\mu_G(w) = 0$.
- *Global* minimum ⇒ *local* minimum.

Kempf-Ness

Other direction: *local* minimum ⇒ *global*. Some "*convexity*".

- *Commutative* group actions – *Euclidean convexity* (change of variables) [exercise].
- *Non-commutative* group actions: *geodesic convexity*.

Algorithms

Algorithms

Group G acts linearly on vector space V.

$\mathrm{cap}(v) = \inf_{g \in G} \|M_g\, v\|_2^2$.

Lots of work on Euclidean convex optimization.

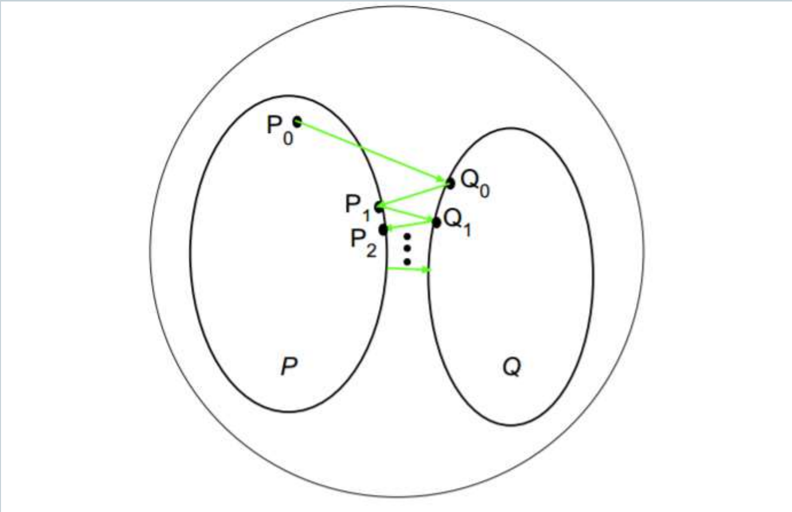
Few algorithms for geodesically convex case.

- First order: gradient descent, alternating minimization.
- Second order: Box constrained Newton's method.

No known generalization of interior point methods, ellipsoid method.

Alternating minimization

Widely used heuristic in machine learning and optimization.
Optimizing over *several* variables or constraints.
Optimizing/satisfying over an *individual* variable or constraint *easy*.
Alternately optimize/satisfy over variables/constraints.



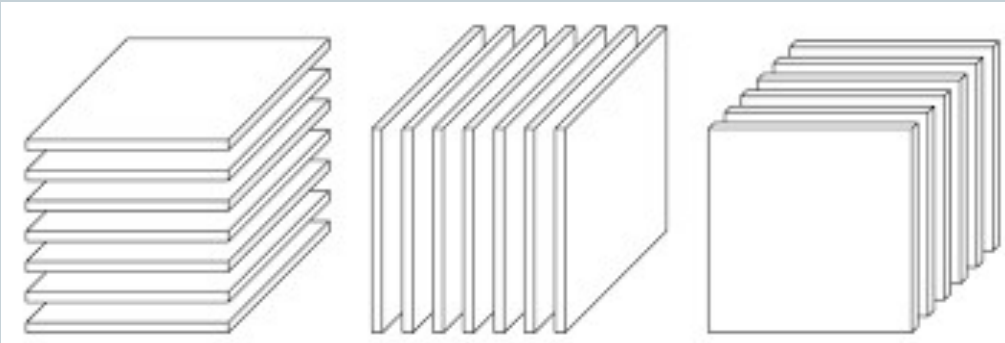
Lot of work on understanding conditions for convergence and convergence rates.
Very few cases in which *provably* converge in small number of iterations.

Tensor scaling

$G = SL_n(\boldsymbol{C}) \times SL_n(\boldsymbol{C}) \times SL_n(\boldsymbol{C})$ acts on $V = \boldsymbol{C}^n \otimes \boldsymbol{C}^n \otimes \boldsymbol{C}^n$ naturally.

Tensor $T \in \boldsymbol{C}^n \otimes \boldsymbol{C}^n \otimes \boldsymbol{C}^n$.

$\mu_G(T) = 0$: require tristochasticity. Slices orthogonal (in all 3 directions).



Tensor scaling: alternating minimization

Operations: Group action i.e. basis change (in all 3 directions).

Algorithm: Alternate basis change for N steps.

[BGOWW 17]: T not in null cone ⇒ ϵ-convergence in $N = \text{poly}(n, b, 1/\epsilon)$ steps.

GIT: T in null cone ⇒ no convergence.

- $\text{poly}(n, b, \log(1/\epsilon))$ time algorithm solves null cone membership.
- For some problems (e.g. *left-right action*) $\text{poly}(n, b, 1/\epsilon)$ suffices [GGOW 16].
- Second order algorithm gets $\text{poly}(n, b, \log(1/\epsilon))$ run time for such cases (e.g. *left-right action*) [AGLOW 18, BFGOWW 19].

Analysis using invariants

Potential function: *invariant* polynomial P.

- Homogeneous degree d.
- $P\big((g_1 \otimes g_2 \otimes g_3)\, T\big) = \mathrm{Det}(g_1)^{d/n}\mathrm{Det}(g_2)^{d/n}\mathrm{Det}(g_3)^{d/n}\, P(T)$.

T not in null cone ⇒ invariant P s.t. $P(T) \neq 0$.

$\Phi(T) = |P(T)|^{1/d}$.

3-step Analysis

- [Lower bound]: Initially $\Phi \geq L$.
- [Progress per step]: If ϵ-far from tristochasticity, one step increases Φ by a factor of $\exp(n\,\epsilon)$. Consequence of a robust *AM-GM* inequality and invariance.
- [Upper bound]: $\Phi \leq U$.

$N = \frac{\log\left(\frac{U}{L}\right)}{n\epsilon}$ steps suffice. [BGOWW 17]: $U = n^3$. $L = 2^{-b}$.

Open problems

Open problems

- *Null cone membership* in $NP \cap coNP$?
- Polynomial time algorithms for *null cone membership*?
  - Ellipsoid/interior point methods for geodesically convex problems. poly($\log(1/\epsilon)$) running time.
- More *applications*?

Lectures and videos

- IAS workshop videos:

https://www.math.ias.edu/ocit2018

- Avi's CCC 2017 tutorial:

http://computationalcomplexity.org/Archive/2017/tutorial.php

Thank You