Analytic algorithms for the moment polytope

Cole Franks
Rutgers University

Based on joint work with


Peter Bürgisser


Ankit Garg


Rafael Oliveira


Michael Walter


Avi Wigderson

Mainly from "Towards a theory of non-commutative optimization: geodesic 1st and 2nd order methods for moment maps and polytopes" FOCS 2019

Outline

1. Moment polytopes by example
2. Algorithms for the general problem

Moment polytopes

Motivating question

Horn's problem:

Are $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3 \in \mathbb{R}^n$ the spectra of three $n \times n$ matrices H_1, H_2, H_3 such that

$$H_1 + H_2 = H_3?$$

If so, can one find the matrices efficiently?

Motivating question

Horn's problem:

Are $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3 \in \mathbb{R}^n$ the spectra of three $n \times n$ matrices H_1, H_2, H_3 such that

$$H_1 + H_2 = H_3?$$

If so, can one find the matrices efficiently?

Horn set

Let $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$, define

$$\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$$

by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Note $\mathrm{eigs}(AA^\dagger) = \mathrm{eigs}(A^\dagger A)$, so

$$\mathrm{eigs}(A_1 A_1^\dagger), \quad \mathrm{eigs}(A_2 A_2^\dagger), \quad \mathrm{eigs}(A_1^\dagger A_1 + A_2^\dagger A_2)$$

is a "yes" instance to Horn's problem (in fact, all such instances take this form).

Horn set

Let $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$, define

$$\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$$

by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Note $\mathrm{eigs}(AA^\dagger) = \mathrm{eigs}(A^\dagger A)$, so

$$\mathrm{eigs}(A_1 A_1^\dagger), \quad \mathrm{eigs}(A_2 A_2^\dagger), \quad \mathrm{eigs}(A_1^\dagger A_1 + A_2^\dagger A_2)$$

is a “yes” instance to Horn’s problem (in fact, all such instances take this form).

Moment polytopes

- $G = \mathrm{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of G where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by G,

Moment map is the map $\mu : \mathcal{V} \to n \times n$ Hermitians $=: \mathrm{Herm}(n)$ given by

$$\mu : v \mapsto \nabla_{H \in \mathrm{Herm}(n)} \log \|e^H \cdot v\|$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

Theorem (Kirwan)

Image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as moment polytope, denoted $\Delta(\mathcal{V})$

Moment polytopes

- $G = \mathrm{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of G where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by G,

Moment map is the map $\mu : \mathcal{V} \to n \times n$ Hermitians $=: \mathrm{Herm}(n)$ given by

$$\mu : v \mapsto \nabla_{H \in \mathrm{Herm}(n)} \log \|e^H \cdot v\|$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

Theorem (Kirwan)

Image of

$$\mathcal{V} \xrightarrow{\mu} \mathrm{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as moment polytope, denoted $\Delta(\mathcal{V})$

Moment polytopes

- $G = \mathrm{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of G where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by G,

Moment map is the map $\mu : \mathcal{V} \to n \times n$ Hermitians $=: \mathrm{Herm}(n)$ given by

$$\boxed{\mu : v \mapsto \nabla_{H \in \mathrm{Herm}(n)} \log \|e^H \cdot v\|}$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

Theorem (Kirwan)

Image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as moment polytope, denoted $\Delta(\mathcal{V})$

Moment polytopes

- $G = \mathrm{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of G where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by G,

Moment map is the map $\mu : \mathcal{V} \to n \times n$ Hermitians $=: \mathrm{Herm}(n)$ given by

$$\boxed{\mu : v \mapsto \nabla_{H \in \mathrm{Herm}(n)} \log \|e^H \cdot v\|}$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

Theorem (Kirwan)

Image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as moment polytope, denoted $\Delta(\mathcal{V})$

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\mu} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Link to algebra

Why are moment polytopes interesting?

Encode asymptotic representation theory of coordinate ring of $\mathcal{V}$!

Theorem (Mumford, Ness '84, Brion '87)

Let $V_{G,\lambda}$ denote irrep of G of type λ. Then

$$\bigcup_k \frac{1}{k}\{\boldsymbol{\lambda} : V_{G,\lambda} \subset \mathbb{C}[\mathcal{V}]_k\} = \Delta(\mathcal{V}) \cap \mathbb{Q}^n!$$

Additional math (Schur-Weyl duality, Saturation [KT00]) $\Longrightarrow$

Horn polytope $\cap\ (\mathbb{Z}^n)^3 = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) : V_{\mathrm{GL}(n),\boldsymbol{\lambda}_3} \in V_{\mathrm{GL}(n),\boldsymbol{\lambda}_1} \otimes V_{\mathrm{GL}(n),\boldsymbol{\lambda}_2}\}$

Link to algebra

Why are moment polytopes interesting?

Encode asymptotic representation theory of coordinate ring of $\mathcal{V}$!

Theorem (Mumford, Ness '84, Brion '87)

Let $V_{G,\lambda}$ denote irrep of G of type λ. Then

$$\bigcup_k \frac{1}{k}\{\boldsymbol{\lambda} : V_{G,\lambda} \subset \mathbb{C}[\mathcal{V}]_k\} = \Delta(\mathcal{V}) \cap \mathbb{Q}^n!$$

Additional math (Schur-Weyl duality, Saturation [KT00]) $\Longrightarrow$

Horn polytope $\cap (\mathbb{Z}^n)^3 = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) : V_{\mathrm{GL}(n),\boldsymbol{\lambda}_3} \in V_{\mathrm{GL}(n),\boldsymbol{\lambda}_1} \otimes V_{\mathrm{GL}(n),\boldsymbol{\lambda}_2}\}$

Link to algebra

Why are moment polytopes interesting?

Encode asymptotic representation theory of coordinate ring of $\mathcal{V}$!

Theorem (Mumford, Ness '84, Brion '87)

Let $V_{G,\lambda}$ denote irrep of G of type λ. Then

$$\bigcup_k \frac{1}{k}\{\boldsymbol{\lambda} : V_{G,\lambda} \subset \mathbb{C}[\mathcal{V}]_k\} = \Delta(\mathcal{V}) \cap \mathbb{Q}^n!$$

Additional math (Schur-Weyl duality, Saturation [KT00]) $\Longrightarrow$

Horn polytope $\cap\, (\mathbb{Z}^n)^3 = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) : V_{\mathrm{GL}(n),\boldsymbol{\lambda}_3} \in V_{\mathrm{GL}(n),\boldsymbol{\lambda}_1} \otimes V_{\mathrm{GL}(n),\boldsymbol{\lambda}_2}\}$

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
   - $\|\mu(v) - \mathrm{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
   - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp(\text{poly})$-search suffices for membership!

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
   - $\|\mu(v) - \text{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
   - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp(\text{poly})$-search suffices for membership!

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
   - $\|\mu(v) - \mathrm{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
   - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp(\text{poly})$-search suffices for membership!

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
    - $\|\mu(v) - \text{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
    - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp(\text{poly})$-search suffices for membership!

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
   - $\|\mu(v) - \mathrm{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
   - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp$(poly)-search suffices for membership!

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \mathrm{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \mathrm{diag}(\boldsymbol{\lambda}_3)$,
      Set $\boxed{A_i \leftarrow A_i B}$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$,
      Set $\boxed{A_i \leftarrow B_i^\dagger A_i.}$

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \mathrm{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \mathrm{diag}(\boldsymbol{\lambda}_3)$,
      Set $A_i \leftarrow A_i B$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$,
      Set $A_i \leftarrow B_i^\dagger A_i$.
3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \text{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \text{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \text{diag}(\boldsymbol{\lambda}_3)$,
      Set $\boxed{A_i \leftarrow A_i B}$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \text{diag}(\boldsymbol{\lambda}_i)$,
      Set $\boxed{A_i \leftarrow B_i^\dagger A_i.}$

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \mathrm{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \mathrm{diag}(\boldsymbol{\lambda}_3)$,
   Set $\boxed{A_i \leftarrow A_i B}$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$,
   Set $\boxed{A_i \leftarrow B_i^\dagger A_i.}$

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \mathrm{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \mathrm{diag}(\boldsymbol{\lambda}_3)$,
      Set $\boxed{A_i \leftarrow A_i B}$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \mathrm{diag}(\boldsymbol{\lambda}_i)$,
      Set $\boxed{A_i \leftarrow B_i^\dagger A_i.}$

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Complexity of moment polytope membership?

The case $\lambda = 0$ is the null-cone problem from Ankit's talk!

1. Is membership in **P**?
   - For tori ($G = \mathbb{C}^n_\times$) Folklore,[SV17]
   - For Horn polytope, by saturation conjecture[MNS12]
2. Is it in **RP**?
   - We think so in general, but no proof yet!
3. Is it in **NP** or **coNP**?
   - In **NP** $\cap$ **coNP** for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCMW17]
   - Not known in general!

Complexity of moment polytope membership?

The case $\lambda = 0$ is the null-cone problem from Ankit's talk!

1. Is membership in **P**?
   - For tori ($G = \mathbb{C}^n_\times$) Folklore,[SV17]
   - For Horn polytope, by saturation conjecture[MNS12]
2. Is it in **RP**?
   - We think so in general, but no proof yet!
3. Is it in **NP** or **coNP**?
   - In **NP** $\cap$ **coNP** for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCMW17]
   - Not known in general!

Complexity of moment polytope membership?

The case $\lambda = 0$ is the null-cone problem from Ankit's talk!

1. Is membership in **P**?
   - For tori ($G = \mathbb{C}_\times^n$) Folklore,[SV17]
   - For Horn polytope, by saturation conjecture[MNS12]
2. Is it in **RP**?
   - We think so in general, but no proof yet!
3. Is it in **NP** or **coNP**?
   - In $\mathbf{NP} \cap \mathbf{coNP}$ for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCMW17]
   - Not known in general!

General algorithms

Convert ε-search to an optimization problem

For $b \in B :=$ upper triangular matrices, define

$$\mathrm{cap}_\lambda(v) := \inf_{b \in B} \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Kempf-Ness Theorem

$$\lambda \in \Delta(\mathcal{V}) \iff \mathrm{cap}_\lambda(v) > 0 \text{ for } \textbf{generic } v \in \mathcal{V}$$

ε-search reduces to finding algorithm for the following:

- Given b with $\|\mu(b \cdot v) - \mathrm{diag}(\lambda)\| > \varepsilon$,
- Output b' with

$$\frac{\|b' \cdot v\|}{\prod_i |b'_{ii}|^{\lambda_i}} < (1 - \delta) \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Convert ε-search to an optimization problem

For $b \in B :=$ upper triangular matrices, define

$$\mathrm{cap}_\lambda(v) := \inf_{b \in B} \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Kempf-Ness Theorem

$$\lambda \in \Delta(\mathcal{V}) \iff \mathrm{cap}_\lambda(v) > 0 \text{ for } \textbf{generic } v \in \mathcal{V}$$

ε-search reduces to finding algorithm for the following:

- Given b with $\|\mu(b \cdot v) - \mathrm{diag}(\lambda)\| > \varepsilon$,
- Output b' with

$$\frac{\|b' \cdot v\|}{\prod_i |b'_{ii}|^{\lambda_i}} < (1-\delta) \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Convert ε-search to an optimization problem

For $b \in B :=$ upper triangular matrices, define

$$\text{cap}_\lambda(v) := \inf_{b \in B} \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Kempf-Ness Theorem

$$\boxed{\lambda \in \Delta(\mathcal{V}) \iff \text{cap}_\lambda(v) > 0 \text{ for } \textbf{generic } v \in \mathcal{V}}$$

ε-search reduces to finding algorithm for the following:

- Given b with $\|\mu(b \cdot v) - \text{diag}(\lambda)\| > \varepsilon$,
- Output b' with

$$\frac{\|b' \cdot v\|}{\prod_i |b'_{ii}|^{\lambda_i}} < (1 - \delta) \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Convert ε-search to an optimization problem

For $b \in B :=$ upper triangular matrices, define

$$\mathrm{cap}_{\lambda}(v) := \inf_{b \in B} \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Kempf-Ness Theorem

$$\boxed{\lambda \in \Delta(\mathcal{V}) \iff \mathrm{cap}_{\lambda}(v) > 0 \text{ for } \textbf{generic } v \in \mathcal{V}}$$

ε-search reduces to finding algorithm for the following:

- Given b with $\|\mu(b \cdot v) - \mathrm{diag}(\lambda)\| > \boldsymbol{\varepsilon}$,
- Output b' with

$$\frac{\|b' \cdot v\|}{\prod_i |b'_{ii}|^{\lambda_i}} < (1 - \delta) \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$

Optimization algorithms

Alternating minimization: poly($1/\varepsilon$) time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, k-tensors

$\log \mathrm{cap}_\lambda(v)$ **can be cast as a geodesically convex program!**

Domain is positive-semidefinite matrices; geodesics through P take the form $\sqrt{P}e^{Ht}\sqrt{P}$

Geodesic gradient descent: poly($1/\varepsilon$) time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \mathrm{Sym}^k\mathbb{C}^n$, arbitrary quivers

Geodesic trust-regions: poly($\log(1/\varepsilon), \log\kappa$) time [BFGOWW19]

- κ is smallest condition-number of an ε-optimizer for $\mathrm{cap}_\lambda(v)$
- polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$

Optimization algorithms

Alternating minimization: poly$(1/\varepsilon)$ time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, k-tensors

$\log \text{cap}_\lambda(v)$ can be cast as a geodesically convex program!

Domain is positive-semidefinite matrices; geodesics through P take the form $\sqrt{P}e^{Ht}\sqrt{P}$

Geodesic gradient descent: poly$(1/\varepsilon)$ time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \text{Sym}^k\mathbb{C}^n$, arbitrary quivers

Geodesic trust-regions: poly$(\log(1/\varepsilon), \log \kappa)$ time [BFGOWW19]

- κ is smallest condition-number of an ε-optimizer for $\text{cap}_\lambda(v)$
- polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$

Optimization algorithms

Alternating minimization: poly$(1/\varepsilon)$ time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, k-tensors

$\log \operatorname{cap}_\lambda(v)$ can be cast as a geodesically convex program!

Domain is positive-semidefinite matrices; geodesics through P take the form $\sqrt{P}e^{Ht}\sqrt{P}$

Geodesic gradient descent: poly$(1/\varepsilon)$ time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \operatorname{Sym}^k \mathbb{C}^n$, arbitrary quivers

Geodesic trust-regions: poly$(\log(1/\varepsilon), \log \kappa)$ time [BFGOWW19]

- κ is smallest condition-number of an ε-optimizer for $\operatorname{cap}_\lambda(v)$
- polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$

Optimization algorithms

Alternating minimization: poly$(1/\varepsilon)$ time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, k-tensors

$\log \mathrm{cap}_\lambda(v)$ can be cast as a geodesically convex program!

Domain is positive-semidefinite matrices; geodesics through P take the form $\sqrt{P}e^{Ht}\sqrt{P}$

Geodesic gradient descent: poly$(1/\varepsilon)$ time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \mathrm{Sym}^k \mathbb{C}^n$, arbitrary quivers

Geodesic trust-regions: poly$(\log(1/\varepsilon), \log \kappa)$ time [BFGOWW19]

- κ is smallest condition-number of an ε-optimizer for $\mathrm{cap}_\lambda(v)$
- polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$

Open problems

1. Is moment polytope membership in **NP** $\cap$ **coNP**, or even **RP** or **P**?
2. Membership is in **P** for Horn's problem. But how about $\exp(-\operatorname{poly})$-search?
3. If (A_1, A_2) a random pair of matrices, does $\operatorname{cap}_\lambda(A_1, A_2)$ have an ε-minimizer with condition number at most

$$\exp(\operatorname{poly}(\log(1/\varepsilon), \langle\lambda\rangle))?$$

Open problems

1. Is moment polytope membership in $\mathbf{NP} \cap \mathbf{coNP}$, or even $\mathbf{RP}$ or $\mathbf{P}$?
2. Membership is in $\mathbf{P}$ for Horn's problem. But how about $\exp(-\operatorname{poly})$-search?
3. If (A_1, A_2) a random pair of matrices, does $\operatorname{cap}_\lambda(A_1, A_2)$ have an ε-minimizer with condition number at most

$$\exp(\operatorname{poly}(\log(1/\varepsilon), \langle\lambda\rangle))?$$

Merci!