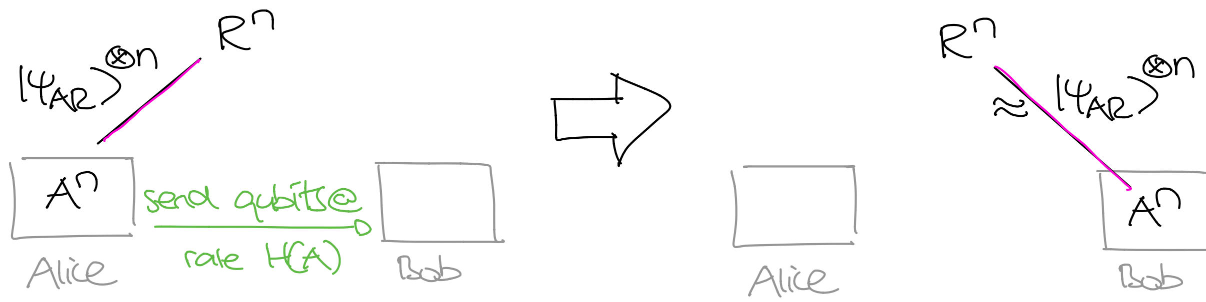
Quantum state merging

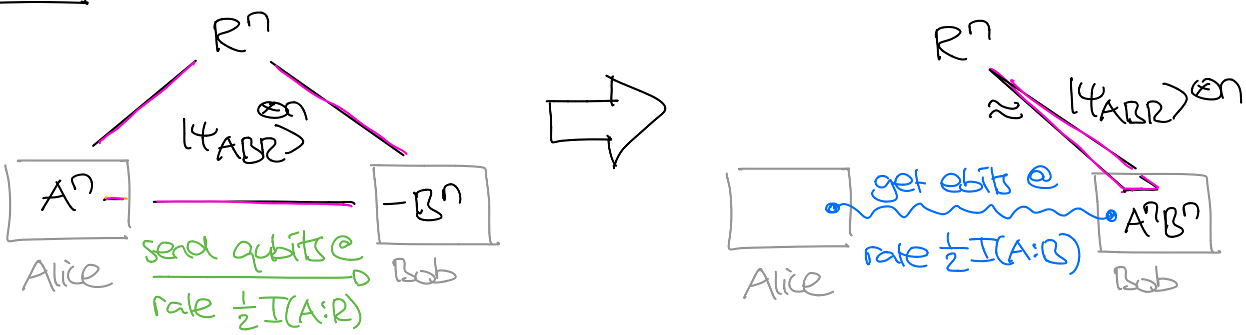
Recall Schumacher compression: Given $|\psi_{AR}\rangle$, want



$|\psi_{AR}\rangle^{\otimes n}$ $\approx$ $|\psi_{A^nR}\rangle$ $\approx$ σ_{A^nR}

What if Bob already has part of state? Rate $H(A)$ is still possible! But...

$\boxed{\text{GOAL:}}$



"merge Alice' state w/ Bob"

(Other variants: state splitting (Alice keeps part of state) or combination)

* No B (or ψ_{AR} pure): Schumacher compression at rate $\frac{1}{2}I(A:R) = H(A)$ ✓

  in general: $\frac{1}{2}I(A:R) \leq H(A)$ and may even get entanglement...

* No R (or ψ_{AB} pure): Entanglement distillation at rate $\frac{1}{2}I(A:B) = H(B)$ ✓

  w/o communication

  in general? given $\psi_{AB}^{\otimes n}$, can distill ebits at rate $\frac{1}{2}I(A:B) - \frac{1}{2}I(A:R)$

  using LOCC $= H(B) - H(AB)$ if ≥ 0

  AB pure

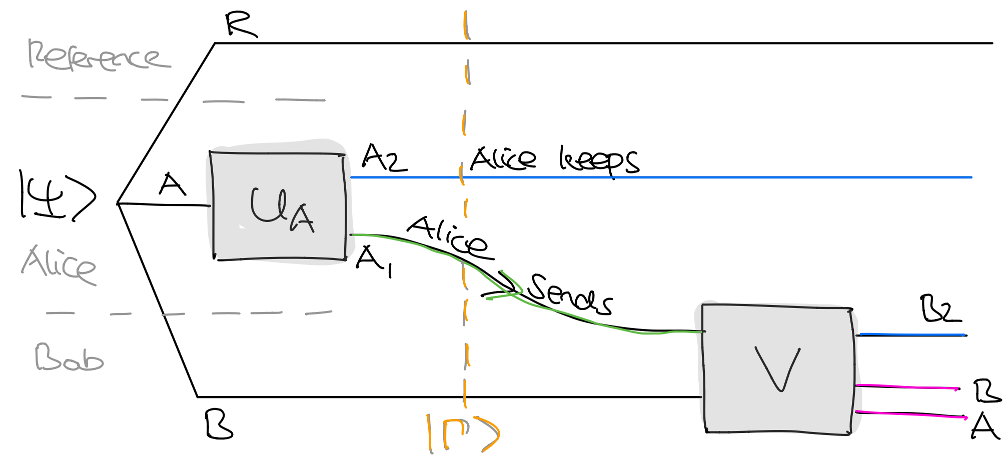
  Pf: Send qubits via teleportation. ($\square$)

Exercises: Noisy teleportation

How to solve it?

Consider first $|\psi_{ABR}\rangle^{\otimes n} \rightsquigarrow |\psi_{ABR}\rangle$ general pure state.

WANT: Unitary U_A and isometry V s.th.



$\approx |\psi_{ABR}\rangle \otimes |\Phi^+_{A_2B_2}\rangle$ (!)

and A_2 as large as possible ⇝ A_1 as small as possible.

__Decoupling approach:__ Enough to consider

$$|\Gamma_{ABR}\rangle := (U_A \otimes I_B \otimes I_R)\, |\psi_{ABR}\rangle$$

"decoupled"

* If V exists: $\boxed{\Gamma_{A_2R} \approx \frac{I_{A_2}}{d_{A_2}} \otimes \psi_R =: \tilde{\Gamma}_{A_2R}}$ (*)

* This is also __sufficient__, i.e. (*) $\Rightarrow$ (!) for suitable V! Get existence of V for free! ☺

__Idea:__ Γ_{A_2R} has purification $|\Gamma_{ABR}\rangle$

$\tilde{\Gamma}_{A_2R}$ —— " —— $|\Phi^+_{A_2B_2}\rangle \otimes |\psi_{ABR}\rangle$ **HW**

$F \approx 1$ by (*) $\xRightarrow[\text{theorem}]{\text{Uhlmann}}$ $\approx$ up to isometry $A_1B \xrightarrow{V} BRB_2$ (□)

How to achieve (*)?

$$\boxed{\begin{array}{l}\textbf{Decoupling theorem: } \text{If } \psi_{AR} \in D(A \otimes R) \text{ where } A = A_1 \otimes A_2: \\ \text{Haar meas.} \displaystyle\int dU_A \, \Big\| \underbrace{\operatorname{tr}_{A_1}\left[(U_A^\dagger \otimes I_R)\psi_{AR}(U_A \otimes I_R)\right]}_{\Gamma_{A_2R}} - \frac{I_{A_2}}{d_{A_2}} \otimes \psi_R \Big\|_1^2 \\ \displaystyle\leq \frac{d_A d_R}{d_{A_1}^2} \operatorname{tr}[\psi_{AR}^2] \quad \leftarrow \text{the smaller the more we trace out!}\end{array}}$$

Proof? $\boxed{\text{HW!}}$

Let's return to $|\psi_{ABR}\rangle^{\otimes n}$ and solve the state merging problem. How to choose $|\psi\rangle$? Final trick: Typical subspaces! Just like for compression...

Recall: Typical subspaces $S_{n,\varepsilon} \subseteq \mathcal{H}^{\otimes n}$ for $\rho^{\otimes n}$ satisfy

① $\mathrm{tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}] \to 1$ as $n \to \infty$ where $\Pi_{n,\varepsilon}$ orthog. projection

② eigenvalues of $\Pi_{n,\varepsilon}\rho^{\otimes n}\Pi_{n,\varepsilon}$ in $2^{-n(H(\rho)\pm\varepsilon)}$

③ $\dim S_{n,\varepsilon} \leq 2^{n(H(\rho)+\varepsilon)}$ and $\geq 2^{n(H(\rho)-\varepsilon)}\,\mathrm{tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}]$

$\geq 2^{n(H(\rho)-2\varepsilon)}$ for large n

Define

$$\boxed{|\psi_{\tilde{A}\tilde{B}\tilde{R}}\rangle = (\Pi^A_{n,\varepsilon}\otimes\Pi^B_{n,\varepsilon}\otimes\Pi^R_{n,\varepsilon})\,|\psi_{ABR}\rangle^{\otimes n}} \in \tilde{\mathcal{A}}\otimes\tilde{\mathcal{B}}\otimes\tilde{\mathcal{R}} \subseteq (\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{R})^{\otimes n}$$

where $\Pi^A_{n,\varepsilon}$ orthogonal projection onto typical subspace $\tilde{\mathcal{A}} := S_{n,\varepsilon}(\psi_A) \subseteq \mathcal{A}^{\otimes n}$ etc

$$\Rightarrow d_{\tilde{A}} d_{\tilde{R}} \overset{③}{\leq} 2^{n(H(A)+H(R)+2\varepsilon)}$$

$\mathrm{tr}[\psi^2_{\tilde{B}}]$

$$\mathrm{tr}[\psi^2_{\tilde{A}\tilde{R}}] = \mathrm{tr}[\psi^2_{\tilde{B}}] \leq \mathrm{tr}[(\underbrace{\Pi^B_{n,\varepsilon}\psi_B^{\otimes n}\Pi^B_{n,\varepsilon}}_{\substack{\text{rank} \leq 2^{n(H(B)+\varepsilon)} \\ \text{eigenvalues} \leq 2^{-n(H(B)-\varepsilon)}}})^2] \overset{②③}{\leq} 2^{-n(H(B)-3\varepsilon)}$$

$\boxed{\text{ESET}}$ $\mathrm{tr}_X[(P\otimes I)\rho_{XY}(P\otimes I)] \leq \rho_Y$

$$\Rightarrow \text{RHS in Decoupling Thm} \leq \frac{2^{n(I(A:R)+5\varepsilon)}}{d^2_{\tilde{A}_1}}$$

$\boxed{\text{RESULT:}}$ There exists U_A s.th. ⊛ holds if we choose $d_{\tilde{A}_1} = 2^{n\left(\frac{I(A:R)}{2}+3\varepsilon\right)}$

i.e. send qubits at rate $\approx \frac{I(A:R)}{2}$ ☺

And in this case also obtain ebits at rate $\approx \frac{I(A:B)}{2}$ ☺ □

Pf: $\frac{1}{n}\log d_{\tilde{A}_2} = \frac{1}{n}\log d_{\tilde{A}} - \left(\frac{I(A:R)}{2}+3\varepsilon\right)$

$\overset{③}{\geq} H(A) - \frac{I(A:R)}{2} - 5\varepsilon = \frac{I(A:B)}{2} - 5\varepsilon$ □

What we did **NOT** cover:

Noisy q. Communication **channels** + their **capacities**

to send bits, qubits, ...

GOOD LUCK FOR THE EXAM ☺