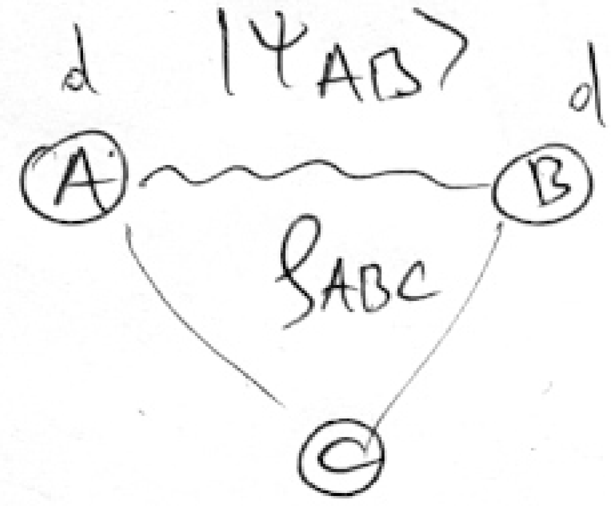
Lecture 13 Monogamy of entanglement

Can't share a maximally entangled state of (max dimension) with more than one party:

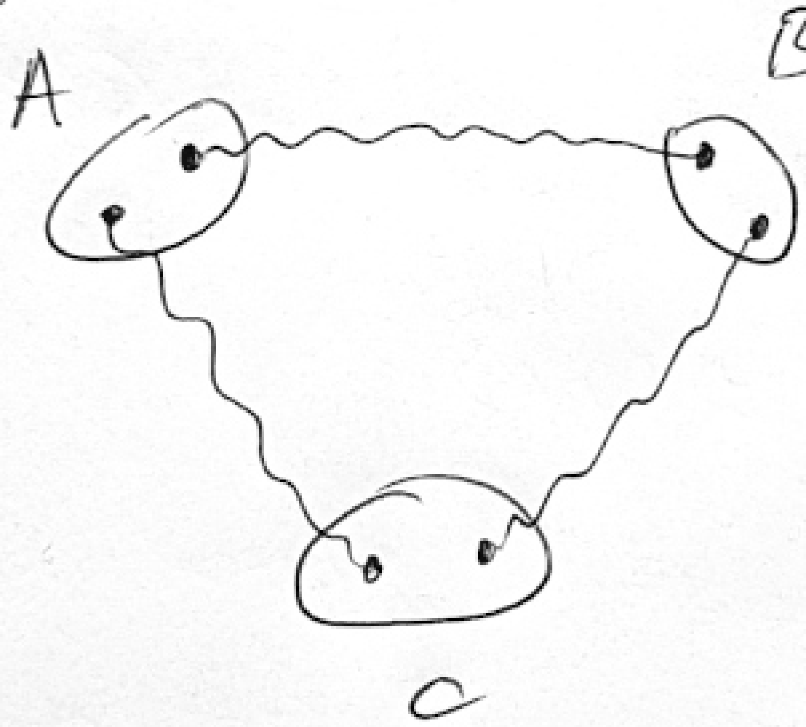


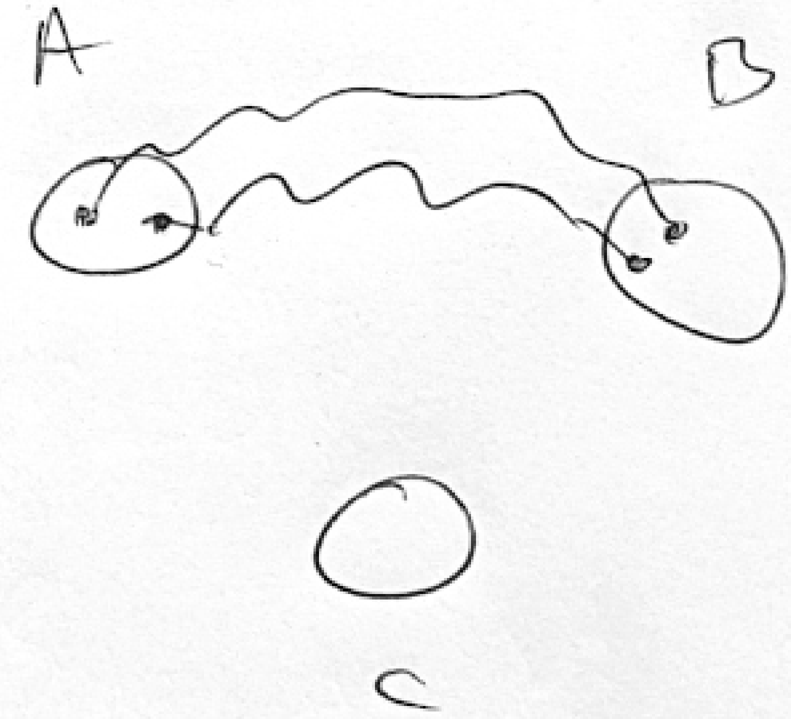
ρ_{AB} is pure $\Rightarrow \rho_{ABC} = \rho_{AB} \otimes \rho_C$

$$\rho_{AC} = \rho_A \otimes \rho_C$$

$$\rho_{BC} = \rho_B \otimes \rho_C$$

For any <u>pure</u> state $|\psi_{AB}\rangle$. E.g., maximally entangled.





In contrast:

$$\rho_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$$

$$\rho_{AB} = \rho_{BC} = \rho_{AC} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

So max classical correlations <u>can</u> be shared.

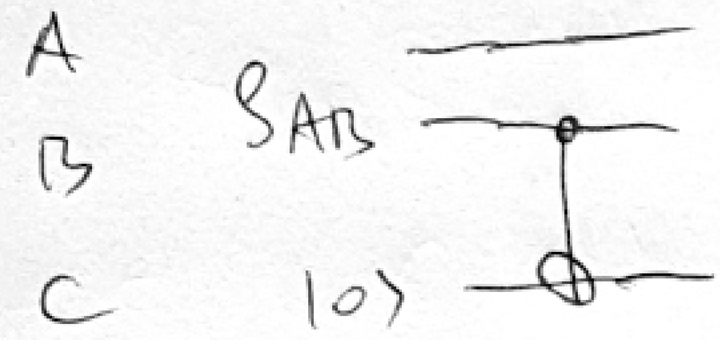
$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

Tr_C

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) \neq \phi = |\Phi^+_{AB}\rangle\langle \Phi^+_{AB}|$$

Intuition: $\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ can be shared.

$$\rho_{AB} \longmapsto \rho_{ABC} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)$$

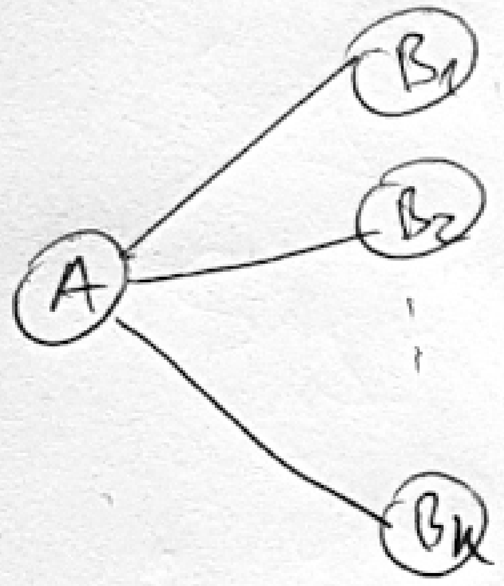


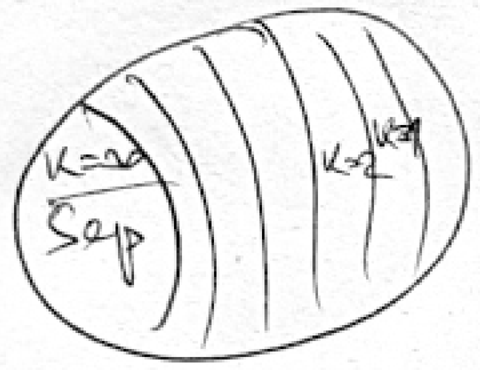
Def If you can do this

$$\rho_{AB} \longmapsto \rho_{AB_1 \ldots B_k}$$

s.t. $\rho_{AB} = \rho_{AB_i}$ $\forall i$

then ρ_{AB} is called <u>k-extendible</u>.

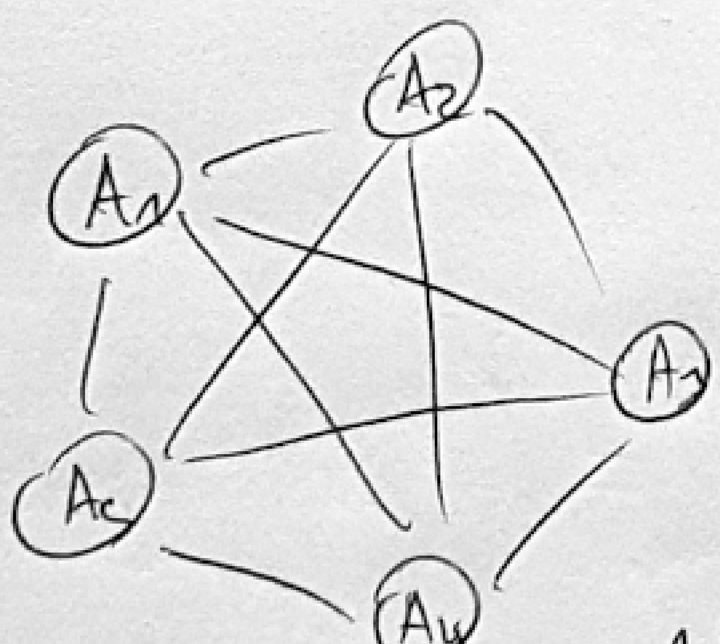




Then it turns out that

$$\rho_{AB} \simeq \sum_i p_i \, \rho_A^i \otimes \rho_B^i$$

In this class: Assume $\rho_{A_1 \ldots A_n}$ s.t.



$\rho_{A_i A_j}$ is the same for any $i \neq j$.

We will show that

$$\rho_{A_1 A_2} \simeq \int d\sigma \, p(\sigma) \, \sigma^{\otimes 2}$$

(a mixture of $\sigma^{\otimes 2}$) $\Rightarrow$ close to separable

Symmetric subspace

$|\psi_i\rangle \in \mathbb{C}^d = \mathcal{H}$

$\pi \in S_n$

$$R_\pi (|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle)$$
$$= |\psi_{\pi^{-1}(1)}\rangle \otimes \cdots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

Then $R_\pi R_\tau = R_{\pi\tau}$, $\forall \pi, \tau \in S_n$

$R_\pi \in \mathcal{U}(\mathbb{C}^{d^n})$

Def Symmetric subspace:

$$\mathrm{Sym}^n(\mathcal{H}) := \{ |\Phi\rangle \in \mathcal{H}^{\otimes n} : R_\pi |\Phi\rangle = |\Phi\rangle, \forall \pi \in S_n \}$$

Ex. • $|\psi\rangle^{\otimes n} \in \mathrm{Sym}^n(\mathcal{H})$ for any $|\psi\rangle \in \mathcal{H}$

• For any $\pi \in S_n$: $|\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H})$

$$\Updownarrow$$

$$R_\pi |\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H})$$

Ex ($n=2$, $d=2$ – two qubits)

$$\mathrm{Sym}^2(\mathbb{C}^2) = \mathrm{span}\left\{ |0,0\rangle, |1,1\rangle, \frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}} \right\}$$

$$\frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}} = \frac{1}{2}(I+F)|0,1\rangle\sqrt{2}$$

Remaining: $\frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}}$

The projection to the sym. subspace is

$$\Pi_n^{(d)} := \frac{1}{n!} \sum_{\pi \in S_n} R_\pi^{(d)}$$

Exercise:

- $\Pi_n^\dagger = \Pi_n$, • $\Pi_n^2 = \Pi_n$.

Note that, if $|\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H})$ then $\Pi_n |\Phi\rangle = |\Phi\rangle$.

In fact, $(I_k \otimes \Pi_n)|\Phi\rangle = |\Phi\rangle$ if $|\Phi\rangle \in \mathrm{Sym}^{k+n}(\mathcal{H})$.

Ex $(n=2)$ ~~Then~~

Then $\Pi_2 = \frac{1}{2}(I + F)$ $F(|\alpha\rangle \otimes |\beta\rangle) = |\beta\rangle \otimes |\alpha\rangle$ $\forall |\alpha\rangle, |\beta\rangle \in \mathcal{H}$

What is the basis for $\mathrm{Sym}^n(\mathbb{C}^d)$?

$$\Lambda_{n,d} := \left\{ (t_1, \dots, t_d) \in \mathbb{Z}^d ; t_1, \dots, t_d \geq 0, \sum_{i=1}^d t_i = n \right\}$$

$t_i = \#$ of $|i\rangle$ in a basis vector

$$|T_{t_1,\dots,t_d}\rangle := |\overbrace{\underbrace{1,\dots,1}_{t_1}, \underbrace{2,\dots,2}_{t_2}, \dots, \underbrace{d,\dots,d}_{t_d}}^{n}\rangle$$

Lem $\mathrm{Sym}^n(\mathbb{C}^d) = \mathrm{span}\{ \Pi_n |T_{t_1,\dots,t_d}\rangle : (t_1,\dots,t_d) \in \Lambda_{n,d} \}$

Ex $(n=2, d=2)$

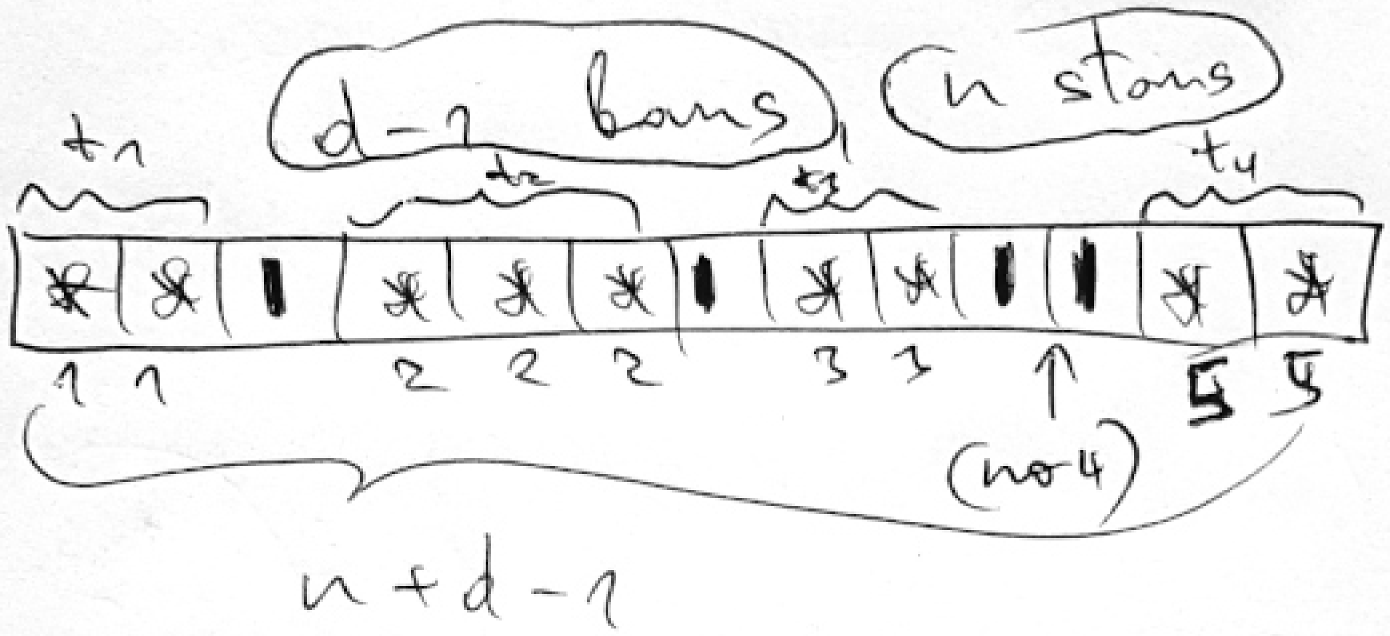
By symmetrizing by Π_2, you get the basis.

$$\Lambda_{2,2} = \{(2,0), (1,1), (0,2)\}$$

Lem $\dim\left(\mathrm{Sym}^n(\mathbb{C}^d)\right) = |\Lambda_{n,d}| = \binom{n+d-1}{n} = \binom{n+d-1}{d-1}$

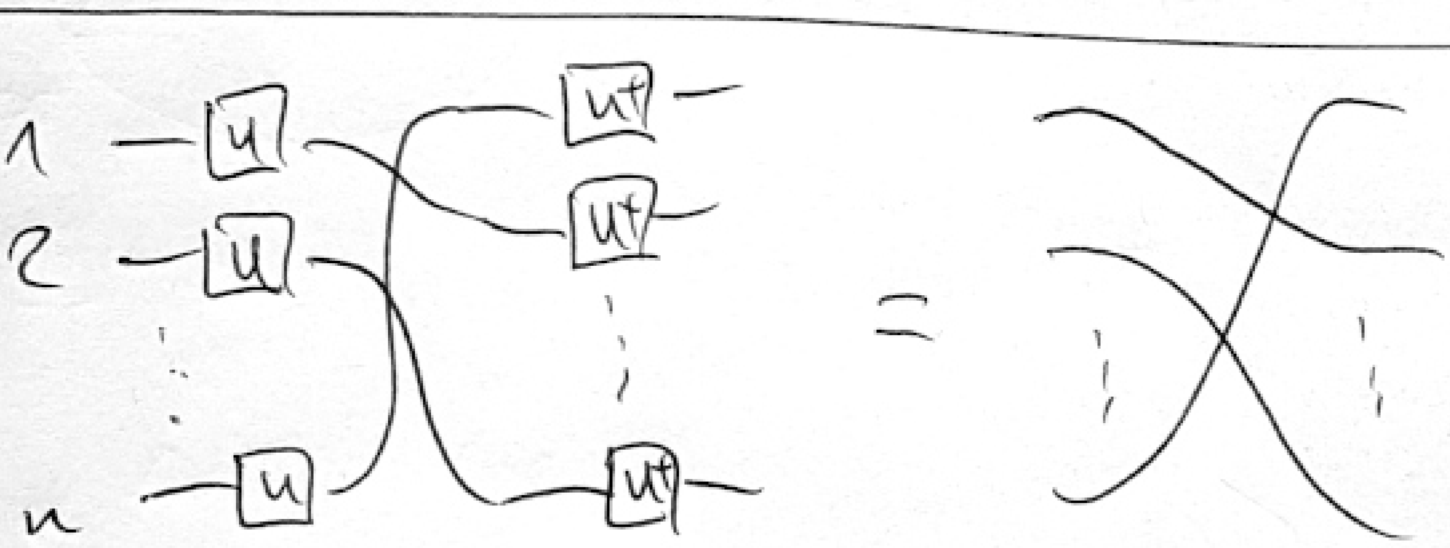
$$= \frac{(n+d-1)!}{n!\,(d-1)!}$$

Proof: "stars & bars"



$d = 5$

$n = \sum_{i=1}^{d} t_i$



$$U^{\otimes n} R_\pi U^{\dagger \otimes n} = R_\pi \quad , \forall \pi \in S_n, U \in U(\mathbb{C}^d)$$

$$U^{\otimes n}\left(\sum_{\pi \in S_n} c_\pi R_\pi\right) U^{\dagger \otimes n} = \sum_{\pi \in S_n} c_\pi R_\pi$$

Lem* $U^{\otimes n} A U^{\dagger \otimes n} = A \quad , \forall U \in U(\mathcal{H})$

$$\Longleftrightarrow \quad A = \sum_{\pi \in S} c_\pi R_\pi .$$

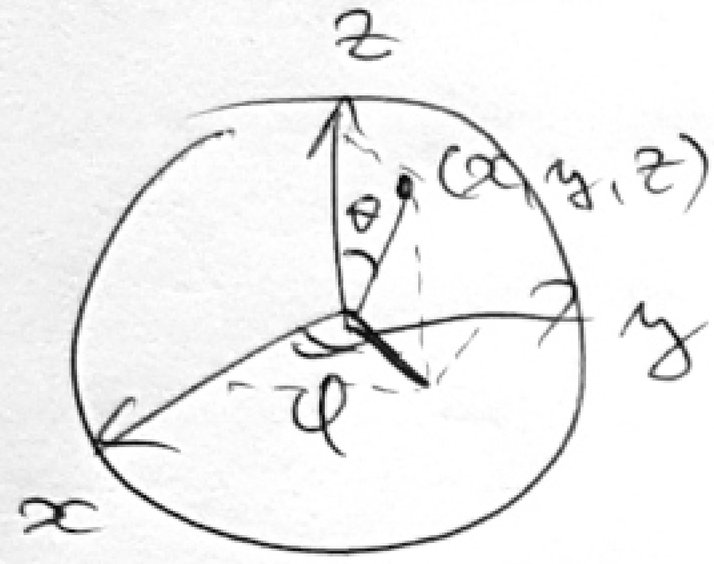
for some $c_\pi \in \mathbb{C}$

Lem $(n \geq 1,\ d \geq 2)$

$$\Pi_n = \binom{n+d-1}{n} \int d\psi \, \left(|\psi\rangle\langle\psi|\right)^{\otimes n}$$

where $d\psi$ is the uniform measure on pure states on $\mathbb{C}^d$.

Ex $(n=2,\ d=2)$



$$(x,y,z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{R}^3 \quad \text{– spherical coordinates}$$

$$\rho(x,y,z) = \frac{1}{2}\left(\mathbb{I} + xX + yY + zZ\right)$$

$$= |\psi(\theta,\varphi)\rangle\langle\psi(\theta,\varphi)| = \rho(\theta,\varphi)$$

where
$$|\psi(\theta,\varphi)\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \in \mathbb{C}^2$$

$$\theta \in [0,\pi]$$
$$\varphi \in [0,2\pi)$$

$$d\psi = \frac{1}{4\pi} \sin\theta \, d\theta \, d\varphi$$

$$\Pi_2 = \underbrace{\binom{2+2-1}{2}}_{3} \cdot \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho(\theta,\varphi)^{\otimes 2} \sin\theta \, d\theta \, d\varphi$$

$$\rho(\theta,\varphi) = \begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Proof:

$$|\psi\rangle^{\otimes n} \langle\psi|^{\otimes n}$$

$$\Pi_n = \frac{1}{n!}\sum_{\pi\in S_n} R_\pi \overset{?}{=} \binom{n+d-1}{n}\underbrace{\int d\psi\, \overbrace{(|\psi\rangle\langle\psi|)^{\otimes n}}^{\sim}}_{\overset{!}{=}\tilde{\Pi}_n}$$

Notice that if $|\psi\rangle$ is uniform, then so is $U|\psi\rangle$, for any $|\psi\rangle \in U(\mathcal{H})$. Hence

$$U^{\otimes n}\tilde{\Pi}_n U^{\dagger\otimes n} = \tilde{\Pi}_n$$

By Lem*, $\tilde{\Pi}_n = \sum_{\pi\in S_n} c_\pi R_\pi$, for some $c_\pi \in \mathbb{C}$.

Since $\Pi_n|\psi\rangle^{\otimes n} = |\psi\rangle^{\otimes n}$,

$$\tilde{\Pi}_n \overset{*}{=} \Pi_n\tilde{\Pi}_n = \frac{1}{n!}\sum_{\pi\in S_n} R_\pi \sum_{\tau\in S_n} c_\tau R_\tau = \frac{1}{n!}\sum_{\pi,\tau\in S_n} c_\tau R_{\underbrace{\pi\tau}_{\gamma}}$$

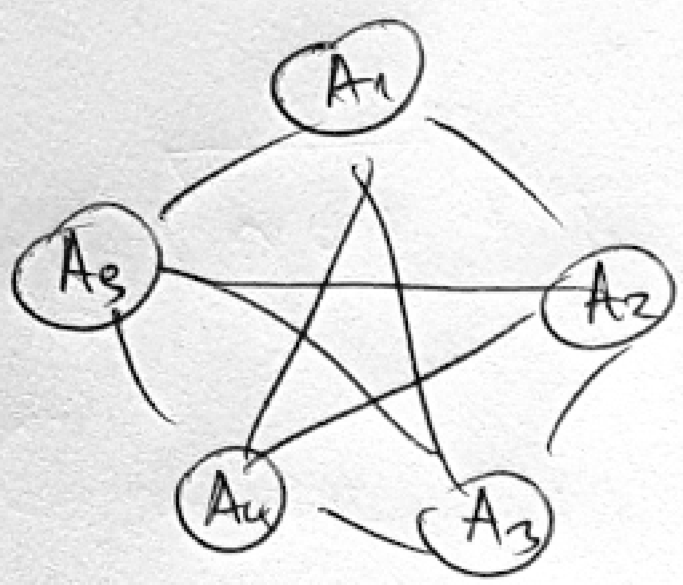
$$= \frac{1}{n!}\left(\sum_{\tau\in S_n} c_\tau\right)\sum_{\gamma\in S_n} R_\gamma = \left(\sum_{\tau\in S_n} c_\tau\right)\Pi_n$$

I.e., $\tilde{\Pi}_n = c\,\Pi_n$, $c = \sum_\tau c_\tau$. $\int d\psi = 1$

$$\mathrm{Tr}\left[\tilde{\Pi}_n\right] = \binom{n+d-1}{n} = \mathrm{Tr}\left[c\Pi_n\right] \Rightarrow c = 1$$

$$\Rightarrow \tilde{\Pi}_n = \Pi_n. \square$$

The quantum de Finetti theorem



$$|\Phi_{A_1\dots A_n}\rangle \in \mathcal{H}_{A_1}\otimes\dots\otimes\mathcal{H}_{A_n}$$

$$\Phi_{A_iA_j} = \Phi_{A_{i'}A_{j'}}$$

If $|\Phi_{A_1\dots A_n}\rangle \in \mathrm{Sym}^n(\mathcal{H})$ then

If $\Phi_{AA'}$ admits such extension $|\Phi_{A_1\dots A_n}\rangle$ then $\Phi_{AA'}$ should be close to separable.

$(k=2)$

<u>Thm</u> (Quantum de Finetti) We have $k+n$ systems $A_1\dots A_{k+n}$, each of dim $d\geq 2$. For any $|\Phi\rangle\in\mathrm{Sym}^{k+n}(\mathbb{C}^d)$, there exist a prob. density p on pure states in $\mathbb{C}^d$ s.t.

$$\frac{1}{2}\left\| \Phi_{A_1\dots A_k} - \int d\psi\, p(\psi)\, (|\psi\rangle\langle\psi|)^{\otimes k}\right\|_1 \leq \sqrt{\frac{dk}{k+n}}$$

$$\Phi_{A_1\dots A_k} = \mathrm{Tr}_{A_{k+1}\dots A_{k+n}}\left[|\Phi\rangle\langle\Phi|\right]. \quad \begin{pmatrix}\to 0 \text{ as} \\ n\to\infty\end{pmatrix}$$

<u>Proof</u>
$$\Phi_{A_1\dots A_k} = \mathrm{Tr}_{k+1,\dots,k+n}\Big[\overbrace{(I_k\otimes\Pi_n)|\Phi\rangle}^{|\Phi\rangle}\langle\Phi|\Big]$$

$$= \binom{n+d-1}{n}\int d\psi\, \mathrm{Tr}_{k+1,\dots,k+n}\left[\left(I_k\otimes\underbrace{(|\psi\rangle\langle\psi|)^{\otimes n}}\right)|\Phi\rangle\langle\Phi|\right]$$

$$= \binom{n+d-1}{n}\int d\psi\, \left(I_k\otimes\langle\psi|^{\otimes n}\right)|\Phi\rangle\langle\Phi|\left(I_k\otimes|\psi\rangle^{\otimes n}\right)$$

$$\Phi_{A_1\cdots A_k} = \int d\psi \; p(\psi) \, |\Phi_\psi\rangle\langle\Phi_\psi|$$

$$\sqrt{p(\psi)}\, |\Phi_\psi\rangle = \sqrt{\binom{n+d-1}{n}} \left(I_k \otimes \langle\psi|^{\otimes n}\right) \cdot |\Phi\rangle$$

Choose $|\Phi_\psi\rangle$ to be unit vectors. Then p become a prob. density.

$k+n$

$$p(\psi) := \binom{n+d-1}{n} \underbrace{\left(I_k \otimes \langle\psi|^{\otimes n}\right) |\Phi\rangle}_{\text{state on } k \text{ systems}}$$

We want to show that ~~[illegible]~~ $\Phi_{A_1\cdots A_k}$ to the tensor power state

$$\tilde{\Phi}_{A_1\cdots A_k} = \int d\psi \; p(\psi) \left(|\psi\rangle\langle\psi| \right)^{\otimes k}$$

Then

$$\frac{1}{2}\left\| \Phi_{A_1\cdots A_k} - \tilde{\Phi}_{A_1\cdots A_k} \right\|_1 = \frac{1}{2}\left\| \int p(\psi)\, d\psi \left(|\Phi_\psi\rangle\langle\Phi_\psi| - \left(|\psi\rangle\langle\psi|\right)^{\otimes k}\right)\right\|_1$$

triangle ineq.

$$\leq \int d\psi \; p(\psi) \, \frac{1}{2} \left\| |\Phi_\psi\rangle\langle\Phi_\psi| - |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k} \right\|_1$$

$$= \int d\psi \; p(\psi) \sqrt{1 - |\langle\psi|^{\otimes k} |\Phi_\psi\rangle|^2}$$

Jensen's ineq.

$$\leq \sqrt{\int d\psi \, p(\psi) \left(1 - |\langle\psi|^{\otimes k} |\Phi_\psi\rangle|^2 \right)}$$

$$= \sqrt{1 - \int d\psi \; p(\psi) \, |\langle\psi|^{\otimes k} |\Phi_\psi\rangle|^2}$$

$$\int d\psi \, p(\psi) \left| \langle\psi|^{\otimes k} |\Phi_\psi\rangle \right|^2 = \binom{n+d-1}{n} \Big/ \binom{k+n+d-1}{k+n}$$

$$\geq 1 - \frac{dk}{k+n} .$$

$$\frac{1}{2} \| \Phi_{A_1 \dots A_k} - \tilde{\Phi}_{A_1 \dots A_k} \|_1 \leq \sqrt{1 - \left(1 - \frac{dk}{k+n}\right)} = \sqrt{\frac{dk}{k+n}} . \square$$

$\|M\|_1 =$ sum of singular values of M

$\|\rho\|_1 = 1$ $\qquad$ $\|M\|_1 = \mathrm{Tr} \sqrt{M^\dagger M}$