Quantum Information Theory, Spring 2020

Practice problem set #12

You do **not** have to hand in these exercises, they are for your practice only.

1. **Maximally entangled states:** A pure state $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is *maximally entangled* if

$$\mathrm{Tr}_B\left[|\Psi\rangle\langle\Psi|\right] = \frac{I_A}{\dim(\mathcal{H}_A)} \qquad \text{and} \qquad \mathrm{Tr}_A\left[|\Psi\rangle\langle\Psi|\right] = \frac{I_B}{\dim(\mathcal{H}_B)}.$$

   (a) Show that it must be the case that $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.
   (b) Let $|\Psi_{AB}\rangle, |\Psi'_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be two maximally entangled states. Show that there exist local unitaries $U_A \in U(\mathcal{H}_A)$ and $V_B \in U(\mathcal{H}_B)$ such that $(U_A \otimes V_B)|\Psi_{AB}\rangle = |\Psi'_{AB}\rangle$.
   (c) Let $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\Phi_{A'B'}\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$ be maximally entangled. Show that $|\Psi_{AB}\rangle \otimes |\Phi_{A'B'}\rangle$ is also maximally entangled with respect to the partition $\mathcal{H}_A \otimes \mathcal{H}_{A'} : \mathcal{H}_B \otimes \mathcal{H}_{B'}$.
   (d) Let $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a maximally entangled state with $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$, and let

$$|\Phi^+_{AB}\rangle = \frac{|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}}$$

   be the canonical two-qubit maximally entangled state. Show that an exact copy of $|\Psi_{AB}\rangle$ can be obtained by LOCC from $|\Phi^+_{AB}\rangle^{\otimes n}$, for some large enough n. What is the smallest value of n for which this holds?

2. **Fidelity and composition of channels:** Let $\tau_1 \in D(\mathcal{H})$, $\sigma \in D(\mathcal{K})$, $\tau_2 \in D(\mathcal{L})$ be quantum states and let $\Phi \in C(\mathcal{H}, \mathcal{K})$ and $\Psi \in C(\mathcal{K}, \mathcal{L})$ be quantum channels. Assuming that

$$F\big(\Phi(\tau_1), \sigma\big) > 1 - \varepsilon, \qquad\qquad F\big(\Psi(\sigma), \tau_2\big) > 1 - \varepsilon, \tag{1}$$

   for some $\varepsilon > 0$, show that

$$F\big((\Psi \circ \Phi)(\tau_1), \tau_2\big) > 1 - 4\varepsilon, \tag{2}$$

   where $\Psi \circ \Phi$ denotes the composition of the two channels.

   Hint: Recall from Homework Problem 4.1 that fidelity is monotonic under any quantum channel. Moreover, you will show in Homework Problem 12.1 that $F(\rho_1, \sigma)^2 + F(\rho_2, \sigma)^2 \leqslant 1 + F(\rho_1, \rho_2)$, *for any states* $\rho_1, \rho_2, \sigma \in D(\mathcal{H})$.

3. **From any state to any other:** Let $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\sigma \in D(\mathcal{H}_{A'} \otimes \mathcal{H}_{B'})$ be two arbitrary pure states. How many copies of the state σ can be distilled per copy of ρ by LOCC?