Lecture 12

Last time: $\xleftarrow{\text{doubly stoch.}}$

- majorization: $\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \prec p \prec (1, 0, \dots, 0)$
- Nielsen's theorem: $|u_{AB}\rangle \xmapsto{LOCC?} |v_{AB}\rangle$
  iff $\mathrm{Tr}_A[|u\rangle\langle u|] \prec \mathrm{Tr}_A[|v\rangle\langle v|]$

$$|\Phi^+_{AB}\rangle \xmapsto{LOCC} |\Omega_{AB}\rangle \xmapsto{LOCC} |\alpha_A\rangle|\beta_B\rangle$$

$$I/\dim(\mathcal{H}_B) \qquad\qquad |\beta\rangle\langle\beta|_B$$

Issues: • not all pairs of states comparable

• maybe $|\Phi^+_{AB}\rangle \xmapsto{LOCC} |\Omega_{AB}\rangle^{\otimes 7}$

We want to measure the amount of entanglement by a single number.

Choose some "golden standard":

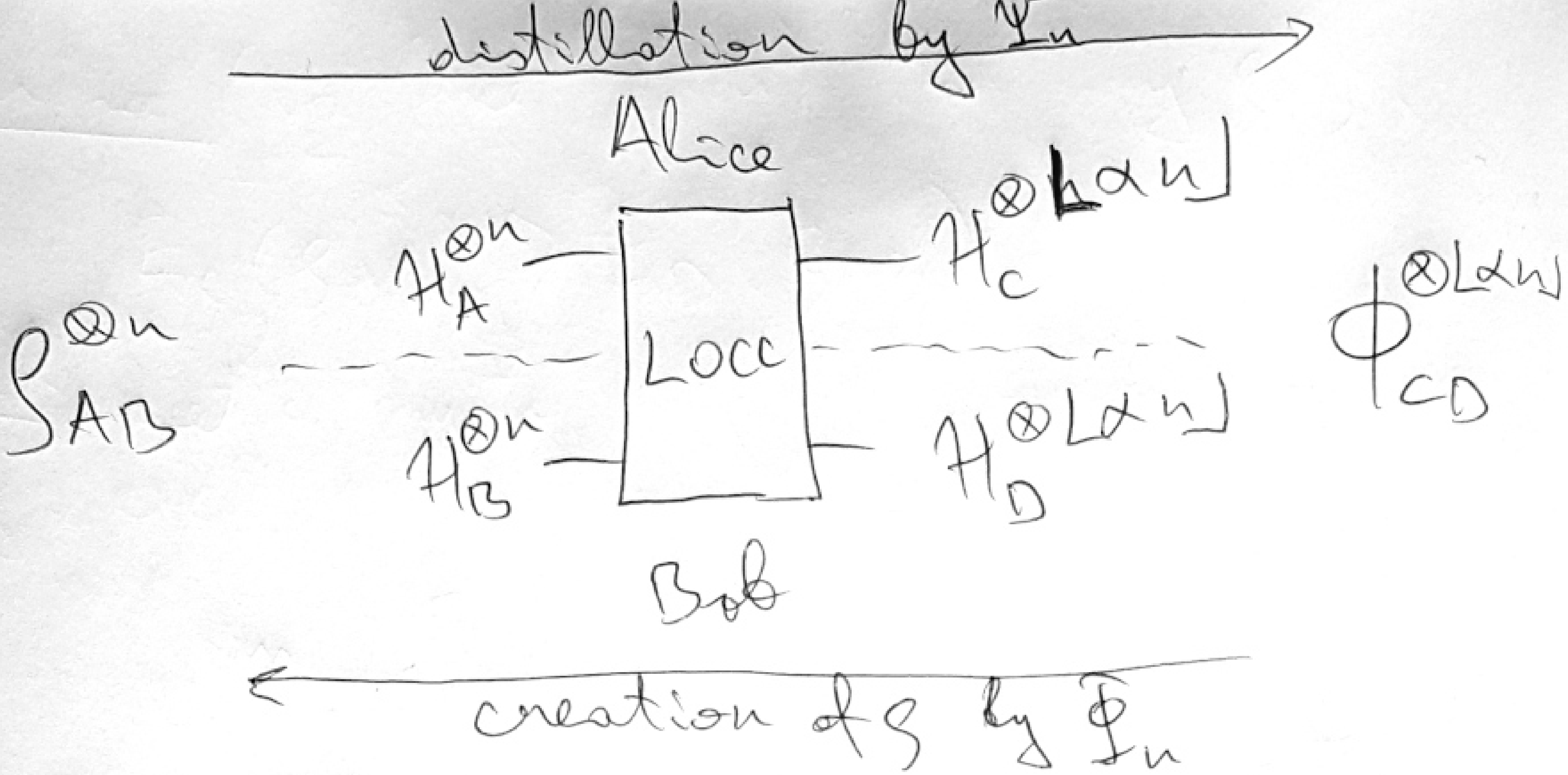
$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0_A, 0_B\rangle + |1_A, 1_B\rangle\right)$$

with density matrix

$$\phi = |\Phi^+\rangle\langle\Phi^+| \leftarrow \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note: $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} = (\mathcal{H}_A \otimes \mathcal{H}_B) \otimes \dots \otimes (\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n} = \underbrace{(\mathcal{H}_A \otimes \dots \otimes \mathcal{H}_A)}_{\text{Alice}} \otimes \underbrace{(\mathcal{H}_B \otimes \dots \otimes \mathcal{H}_B)}_{\text{Bob}}$$



Def The <u>distillable entanglement</u> $E_D(\rho_{AB})$ of $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the supremum (max) of all $\alpha \geq 0$ for which there exists a sequence of LOCC channels

$$\Psi_n \in LOCC\left(\mathcal{H}_A^{\otimes n}, \mathcal{H}_C^{\otimes \lfloor \alpha n \rfloor} : \mathcal{H}_B^{\otimes n}, \mathcal{H}_D^{\otimes \lfloor \alpha n \rfloor}\right)$$

such that

$$\lim_{n \to \infty} F\left(\Psi_n\left(\rho_{AB}^{\otimes n}\right), \phi_{CD}^{\otimes \lfloor \alpha n \rfloor}\right) = 1.$$

Def The <u>entanglement cost</u> $E_C(\rho_{AB})$ is the infimum (min)

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$$\Phi_n \in LOCC\left(\mathcal{H}_C^{\otimes \lfloor \alpha n \rfloor}, \mathcal{H}_A^{\otimes n} : \mathcal{H}_D^{\otimes \lfloor \alpha n \rfloor}, \mathcal{H}_B^{\otimes n}\right)$$

such that

$$\lim_{n \to \infty} F\left(\Phi_n\left(\phi_{CD}^{\otimes \lfloor \alpha n \rfloor}\right), \rho_{AB}^{\otimes n}\right) = 1.$$

Lemma (No free lunch) For any ρ_{AB},

$$E_C(\rho_{AB}) \geq E_D(\rho_{AB})$$

Proof Let's try to approximately implement by LOCC the map $\Psi_n \circ \Phi_n$:

$$\phi^{\otimes m} \xmapsto{\Phi_n} \rho^{\otimes n} \xmapsto{\Psi_n} \phi^{\otimes k}$$

Note that $\phi^{\otimes m}$ is the same as max ent. in dimension 2^m. (Its Schmidt coefficients are $1/2^m$.) The Schmidt number or entanglement rank of $\phi^{\otimes m}$ is 2^m. By Thm 10.5, $(\Psi_n \circ \Phi_n)(\phi^{\otimes m})$ has ent. rank $\leq 2^m$. You will show that for σ of ent. rank $\leq r$,

$$F(\sigma, \phi^{\otimes k})^2 \leq r/2^k \longleftarrow *$$

By def. of E_C & E_D:

$$F(\Phi_n(\phi^{\otimes m}), \rho^{\otimes n}) > 1 - \varepsilon,$$

$$F(\Psi_n(\rho^{\otimes n}), \phi^{\otimes k}) > 1 - \varepsilon.$$

Then (exercise)

$$F((\Psi_n \circ \Phi_n)(\phi^{\otimes m}), \phi^{\otimes k}) > 1 - 4\varepsilon.$$

So $F^2 > \left(1 - \frac{1}{4}\right)^2 = \frac{9}{16} > \frac{1}{2}$.

Let $\varepsilon < \frac{1}{16}$

On the other hand,

$$F^2 \leq 2^m / 2^k = 2^{m-k} \quad \text{by } (*)$$

We get $2^{m-k} > 2^{-1} \Rightarrow m > k-1 \Leftrightarrow m \geq k$.

Since $m = \lfloor \alpha n \rfloor$ and $k = \lfloor \beta n \rfloor$,

$\alpha \geq \beta$, So $E_C \geq E_D$. $\square$

Turns out, for <u>pure</u> states $E_C = E_D$.

<u>Thm</u> For any pure state $\rho_{AB} = |u\rangle\langle u|_{AB}$,

$$E_D(\rho_{AB}) = H(\rho_A) = H(\rho_B) = E_C(\rho_{AB}).$$

<u>Proof</u> Recall $H(\rho_A) = H(\rho_B) = H(p)$ where the distrib. p is such that $\sqrt{p}$ are the Schmidt coeff. of $|u_{AB}\rangle$:

$$|u_{AB}\rangle = \sum_{x \in \Sigma} \sqrt{p(x)}\, |a_x\rangle_A \otimes |b_x\rangle_B.$$

Strategy of proof: $\underbrace{E_C(\rho_{AB}) \leq H(p)}$ $\leq E_D(\rho_{AB})$.

Main tool: typical sequences.

$T_{n,\varepsilon}(p)$ = the set of ε-typical strings

$n \geq 1$, $\varepsilon > 0$

$$= \left\{ x_1 \dots x_n \in \Sigma^n : 2^{-n(H(p)+\varepsilon)} < p(x_1) \cdots p(x_n) < 2^{-n(H(p)-\varepsilon)} \right\}$$

Define

$$|v_{n,\varepsilon}\rangle := \sum_{x_1 \dots x_n \in T_{n,\varepsilon}(p)} \sqrt{\overbrace{p(x_1) \cdots p(x_n)}^{p(x)}}\, \left(|a_{x_1}\rangle \otimes \dots \otimes |a_{x_n}\rangle\right) \otimes \left(|b_{x_1}\rangle \otimes \dots \otimes |b_{x_n}\rangle\right) \quad \sim |u\rangle^{\otimes n}$$

$$p_{n,\varepsilon} := \| |v_{n,\varepsilon}\rangle \|^2 = \sum_{x_1 \dots x_n \in T_{n,\varepsilon}(p)} p(x_1) \cdots p(x_n) = P_n\left(X^n \in T_{n,\varepsilon}(p)\right).$$

where X is a random variable on Σ_1 w. distr. p.

From AEP

$$p_{n,\varepsilon} \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

σ – a constant that depends on p

So $p_{n,\varepsilon} \to 1$ as $n \to \infty$.

Let's normalize $|v_{n,\varepsilon}\rangle$:

$$|w_{n,\varepsilon}\rangle := \frac{|v_{n,\varepsilon}\rangle}{\sqrt{p_{n,\varepsilon}}}$$

Given $\phi^{\otimes m}$, we will try to create $|w_{n,\varepsilon}\rangle$ and show it is close to $|u\rangle^{\otimes n}$.

Take $\alpha > H(p)$ and let $\varepsilon > 0$ be small so that $\alpha > H(p) + 2\varepsilon$, and let $n > 1/\varepsilon$ so that $n\varepsilon > 1$.

$$m := \lfloor \alpha n \rfloor \geq \lfloor n(H(p) + \varepsilon) + n\varepsilon \rfloor > n(H(p) + \varepsilon).$$

Want to create as many copies of $|u\rangle$ as possible from $\phi_{CD}^{\otimes m}$.

$$\lambda_j\left(\mathrm{Tr}_{D_1 \dots D_m}[\phi^{\otimes m}]\right) = 2^{-m} = \frac{1}{2^m}$$

$\mathrm{Tr}_D[\phi_{CD}] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to$ eigenvalues are $\frac{1}{2}, \frac{1}{2}$

$\mathrm{Tr}_{D_1 D_2} \phi_{CD}^2 = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to$ — " — $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$

The eigenvalues of $\underbrace{\quad\quad}_{\text{the reduced state of}}$ $|w_{n,\varepsilon}\rangle_{A^nB^n}$ on $A_1 \dots A_n$ are $p(x)/p_{n,\varepsilon}$ because $\sqrt{\frac{p(x)}{p_{n,\varepsilon}}}$ are the Schmidt coefficients.

~~[illegible]~~ That is

$$\lambda_j\left(\mathrm{Tr}_{B_1\dots B_n}\left[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right]\right) = \frac{p(x)}{p_{n,\varepsilon}},$$

for some $x \in T_{n,\varepsilon}(p)$

Recall from AEP:

$$\frac{2^{-n(H(p)+\varepsilon)}}{p_{n,\varepsilon}} < \lambda_j(w) < \frac{2^{-n(H(p)-\varepsilon)}}{p_{n,\varepsilon}}$$

Note that

$$2^{-m} \leq 2^{-n(H(p)+\varepsilon)} \leq \frac{2^{-n(H(p)+\varepsilon)}}{p_{n,\varepsilon}}$$

$(p_{n,\varepsilon} \leq 1)$

$$\boxed{2^{-m} < \lambda_j(w)}$$

$$\sum_{j=1}^{k} \lambda_j\left(\mathrm{Tr}_{A_1\dots A_m}\left[\phi^{\otimes m}\right]\right) \leq \sum_{j=1}^{k} \lambda_j\left(\mathrm{Tr}_{B_1\dots B_n}\left[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right]\right)$$

This implies the desired majorization, so by Nielsen's thm, $\exists\, \Phi_n \in LOCC$,

$$\Phi_n(\phi^{\otimes m}) = |w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|$$

Why is $|w_{n,\varepsilon}\rangle$ close to $|u\rangle^{\otimes n}$?

$$F(|u\rangle\langle u|^{\otimes n}, |w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|)^2$$

$$= |\langle u|^{\otimes n} \cdot |w_{n,\varepsilon}\rangle|^2$$

$$= \frac{1}{p_{n,\varepsilon}} |\langle u|^{\otimes n} \cdot |v_{n,\varepsilon}\rangle|^2$$

$$= \frac{1}{p_{n,\varepsilon}} \left(\sum_{x \in T_{n,\varepsilon}(p)}' p(x) \right)^2 = \frac{p_{n,\varepsilon}^2}{p_{n,\varepsilon}} = p_{n,\varepsilon} \to 1$$

as $n \to \infty$

$$\left(p_{n,\varepsilon} \geq 1 - \frac{\sigma^2}{n\varepsilon^2} \right)$$

The other inequality $H(p) \leq E_D(\rho_{AB})$. $\square$

Together they prove $E_C \leq E_D$.