Quantum Information Theory, Spring 2020

Practice problem set #11

You do **not** have to hand in these exercises, they are for your practice only.

1. **Majorization examples:**

   (a) Let $p = (0.1, 0.7, 0.2)$ and $q = (0.3, 0.2, 0.5)$. Determine whether $p \prec q$ or $q \prec p$.
   (b) Find a sequence of Robin Hood transfers that converts one distribution into the other.
   (c) Express this sequence as a single stochastic matrix and verify that this matrix is in fact doubly stochastic.
   (d) Express this matrix as a convex combination of permutations.
   (e) Find a pair of probability distributions p and q such that neither $p \prec q$ nor $q \prec p$.

2. **Alternative definitions of majorization:** Let $u = (u_1, \ldots, u_n)$ be a vector and let r denote *reverse sorting* and s denote *sorting*:

$$r_1(u) \geqslant r_2(u) \geqslant \cdots \geqslant r_n(u),$$
$$s_1(u) \leqslant s_2(u) \leqslant \cdots \leqslant s_n(u),$$

   such that $\{r_i(u) : i = 1, \ldots, n\} = \{s_i(u) : i = 1, \ldots, n\} = \{u_i : i = 1, \ldots, n\}$ as multisets. Let u and v be two probability distributions over $\Sigma = \{1, \ldots, n\}$, i.e., $u_i \geqslant 0$, $v_i \geqslant 0$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i = 1$. Show that the following ways of expressing $v \prec u$ are equivalent:

   (a) $\sum_{i=1}^m r_i(v) \leqslant \sum_{i=1}^m r_i(u)$, for all $m \in \{1, \ldots, n-1\}$.

   (b) $\sum_{i=1}^m s_i(v) \geqslant \sum_{i=1}^m s_i(u)$, for all $m \in \{1, \ldots, n-1\}$.

   (c) $\forall t \in \mathbb{R} : \sum_{i=1}^n \max(v_i - t, 0) \leqslant \sum_{i=1}^n \max(u_i - t, 0)$.

3. **Vectorization and partial trace:**

   (a) Show that, for all $L, R \in L(\mathcal{H}_A, \mathcal{H}_B)$,

$$\mathrm{Tr}_A \left[|L\rangle\langle R| \right] = LR^\dagger.$$

   (b) Let $\Xi \in \mathrm{SepC}(A : B)$ be given by

$$\Xi(M) = \sum_{a \in \Sigma} (A_a \otimes B_a) M (A_a \otimes B_a)^\dagger,$$

   for all $M \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$. Show that, for all $X \in L(\mathcal{H}_A, \mathcal{H}_B)$,

$$\mathrm{Tr}_A \left[\Xi\big(|X\rangle\langle X|\big) \right] = \sum_{a \in \Sigma} B_a X A_a^\mathsf{T} \bar{A}_a X^\dagger B_a^\dagger.$$