Quantum Information Theory, Spring 2020

Practice problem set #10

You do **not** have to hand in these exercises, they are for your practice only.

1. **Vectorization:**

   (a) Let $\mathcal{H}_A = \mathbb{C}^\Sigma$, $\mathcal{H}_B = \mathbb{C}^\Gamma$, and $M = |b\rangle\langle a| \in L(\mathcal{H}_A, \mathcal{H}_B)$, for some $a \in \Sigma$ and $b \in \Gamma$. Recall from Definition 10.2 that the vectorization of M is given by $|M_{AB}\rangle = |a, b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Prove the vectorization identity

   $$(A \otimes B)|M\rangle = |BMA^T\rangle$$

   when $A = |c\rangle\langle a|$ and $B = |d\rangle\langle b|$, for some standard basis states $|c\rangle$ and $|d\rangle$.

   (b) Argue why this implies the same identity for arbitrary $A \in L(\mathcal{H}_A, \mathcal{H}_C)$, $B \in L(\mathcal{H}_B, \mathcal{H}_D)$, $M \in L(\mathcal{H}_A, \mathcal{H}_B)$.

2. **Separable maps:**

   (a) Let $\Xi \in CP(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_C \otimes \mathcal{H}_D)$. Show that $\Xi \in SepCP(\mathcal{H}_A, \mathcal{H}_C : \mathcal{H}_B, \mathcal{H}_D)$ if and only if there exist $A_x \in L(\mathcal{H}_A, \mathcal{H}_C)$ and $B_x \in L(\mathcal{H}_B, \mathcal{H}_D)$ such that

   $$\Xi(X) = \sum_{x \in \Sigma} (A_x \otimes B_x) X (A_x \otimes B_x)^\dagger,$$

   for all $X \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$.

   (b) Let $\Xi_1 \in SepCP(\mathcal{H}_A, \mathcal{H}_C : \mathcal{H}_B, \mathcal{H}_D)$ and $\Xi_2 \in SepCP(\mathcal{H}_C, \mathcal{H}_E : \mathcal{H}_D, \mathcal{H}_F)$. Show that their composition is also separable:

   $$\Xi_2 \circ \Xi_1 \in SepCP(\mathcal{H}_A, \mathcal{H}_E : \mathcal{H}_B, \mathcal{H}_F).$$

3. **Examples of separable maps:** Show that the following maps jointly implemented by Alice and Bob are separable:

   (a) Alice and Bob share a random variable distributed according to $p \in P(\Sigma \times \Gamma)$, where Σ labels Alice's register and Γ labels Bob's register. Moreover, Alice has a quantum register A and Bob has a quantum register B. They both observe their halves of the random variable. If Alice's value is $x \in \Sigma$, she applies a channel Φ_x on her register A. Similarly, if Bob's value is $y \in \Gamma$, he applies a channel Ψ_y on his register B.

   (b) Alice has a register A that she measures in the standard basis. She sends the measurement outcome $x \in \Sigma$ to Bob who applies a channel Ψ_x on his register B.

   (c) Any LOCC channel.

4. **Instruments:** Recall from Definition 10.7 that an instrument is a collection $\{\Phi_\omega : \omega \in \Omega\} \subset CP(\mathcal{H}_A, \mathcal{H}_B)$ such that $\sum_{\omega \in \Omega} \Phi_\omega \in C(\mathcal{H}_A, \mathcal{H}_B)$. When applied to a state $\rho \in D(\mathcal{H}_A)$, it produces outcome $\omega \in \Omega$ with probability $\mathrm{Tr}[\Phi_\omega[\rho]]$ and changes ρ to $\rho_\omega = \Phi_\omega[\rho]/\mathrm{Tr}[\Phi_\omega[\rho]]$. Show that any instrument can be implemented by a quantum channel, followed by an orthonormal measurement.