Quantum Information Theory, Spring 2020

Homework problem set #9 **due April 6, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

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1. *(4 points)* **Teleportation and entanglement swapping:**

   (a) Let $|\psi\rangle \in \mathbb{C}^2$ be an arbitrary pure single-qubit state. Verify the teleportation identity

   $$|\psi\rangle \otimes |\Phi^{00}\rangle = \frac{1}{2} \sum_{z,x\in\{0,1\}} |\Phi^{zx}\rangle \otimes X^x Z^z |\psi\rangle,$$

   where $|\Phi^{zx}\rangle$ are the four Bell states and X and Z are the Pauli matrices.
   Hint: Some identities from the exercise class might be handy.

   (b) Consider the following generalization of the teleportation identity involving three friends Alice, Bob and Charlie who share four qubits: Alice has two qubits in systems A and A', Bob has a qubit in system B, while Charlie has a qubit in system C. Their joint state is

   $$|\Psi\rangle_{CA} \otimes |\Phi^{00}\rangle_{A'B}$$
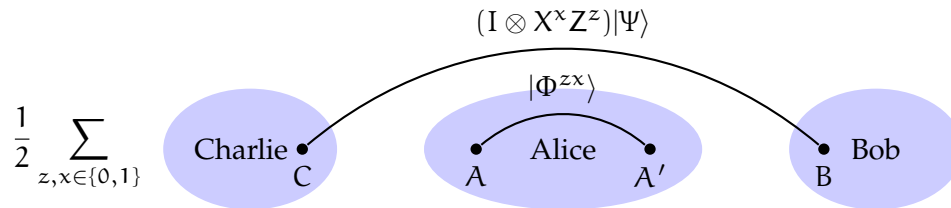
   where $|\Psi\rangle_{CA} \in \mathbb{C}^4$ is an arbitrary pure two-qubit state. This can be depicted as follows:

   Show that

   $$|\Psi\rangle_{CA} \otimes |\Phi^{00}\rangle_{A'B} = \frac{1}{2} \sum_{z,x\in\{0,1\}} |\Phi^{zx}\rangle_{AA'} \otimes (I \otimes X^x Z^z)|\Psi\rangle_{CB}.$$

   Note that pictorially the right-hand side looks as follows:

   (c) Using the equation from the previous part, explain what happens if Alice and Bob perform the usual teleportation protocol. More specifically, what are the probabilities of outcomes if Alice measures her two qubits in the Bell basis? For each outcome, what is the corresponding post-measurement state for Bob and Charlie? What Pauli correction operation should Bob apply at the end of the protocol?

   (d) What could be a potential application of the protocol performed by Alice, Bob and Charlie?

2. *(3 points)* **Superdense coding:** Assume that Alice and Bob share the Bell state $|\Phi^{00}\rangle_{AB}$. Assume that Alice wants to send two bits $z, x \in \{0, 1\}$ to Bob. They perform the following protocol: (i) Alice applies some unitary operation on her qubit, (ii) sends her qubit to Bob, and (iii) Bob performs an orthogonal measurement to recover z and x.

   (a) What operation should Alice apply?
   (b) What measurement should Bob perform?
   (c) Formulate this procedure as a resource trade-off.

3. *(5 points)* **Partial transpose test:** Let Σ be an alphabet, $n = |\Sigma|$, and $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^\Sigma$. Let

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{n}} \sum_{a \in \Sigma} |a\rangle_A \otimes |a\rangle_B$$

be an n-dimensional *maximally entangled state* on registers AB. Let $t \in [0, 1]$ and define $\rho_0, \rho_1, \rho(t) \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$ as follows:

$$\rho_0 = |\Phi^+\rangle\langle\Phi^+|, \qquad \rho_1 = \frac{I \otimes I - \rho_0}{n^2 - 1}, \qquad \rho(t) = (1 - t)\rho_0 + t\rho_1.$$

   (a) Show that $\rho(t)$ is a quantum state when $t \in [0, 1]$.
   (b) Let $\mathcal{T} \in L(L(\mathcal{H}_A), L(\mathcal{H}_A))$ be the *transpose* map defined as $\mathcal{T}[X] = X^T$, for all $X \in L(\mathcal{H}_A)$. Show that

$$(\mathcal{T}_A \otimes \mathcal{I}_B)[\rho_0] = \frac{1}{n}W,$$

   where $W \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the *swap operator* defined as

$$W(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle,$$

   for all $x, y \in \Sigma$.
   (c) Compute $(\mathcal{T}_A \otimes \mathcal{I}_B)[\rho(t)]$.
   (d) For what range of t can we say that $\rho(t)$ is entangled?

4. *(2 bonus points)* ▦ **Practice** In this problem, you will play around with the partial transpose test (part a) and the entanglement entropy (part b).

   (a) In the files `A.txt`, `B.txt` and `C.txt`, you will find density matrices of the following dimensions:

$$A \in D(\mathbb{C}^2 \otimes \mathbb{C}^2), \qquad B \in D(\mathbb{C}^2 \otimes \mathbb{C}^3), \qquad \text{and} \qquad C \in D(\mathbb{C}^2 \otimes \mathbb{C}^4).$$

   For each of the states A, B and C, compute its partial transpose (where the transpose is applied to the second register) and output the smallest eigenvalue of the resulting matrix. For each state, output whether the state is entangled or separable, or whether the partial transpose test was inconclusive.
   Hint: Recall that the partial transpose test is always conclusive when the total dimension is at most 6.
   (b) In the file `D.txt`, you will find the *pure* state $D \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Calculate its entanglement entropy.