Quantum Information Theory, Spring 2020

Homework problem set #8 **due March 30, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Measurements and trace distance:** In this problem, you will revisit how to distinguish quantum states by using measurements. Given states $\rho, \sigma \in D(\mathcal{H})$ and a measurement $\mu\colon \Omega \to \mathrm{PSD}(\mathcal{H})$, let $p, q \in P(\Omega)$ denote the corresponding distributions of measurement outcomes.

   1. Prove that $\|p - q\|_1 \leqslant \|\rho - \sigma\|_1$.
   2. Show that, for every ρ and σ, there exists a measurement μ such that equality holds.

      Hint: Recall Helstrom's theorem.

2. *(4 points)* **Holevo χ-quantity:** Alice wants to communicate a classical message to Bob by sending a quantum state. She chooses one state $\rho_x \in D(\mathcal{H})$ for each possible message $x \in \Sigma$ that she may want to send, and Bob chooses a measurement $\mu\colon \Sigma \to \mathrm{PSD}(\mathcal{H})$ that he uses to decode.

   1. Write down a formula for the probability that Bob successfully decodes the message if the message is drawn according to an arbitrary probability distribution $p \in P(\Sigma)$.

   In class, we used the Holevo bound to prove that if this probability is 100% then, necessarily, the Holevo χ-quantity of the ensemble $\{p_x, \rho_x\}$ must be equal to $H(p)$.

   2. Show that this condition is also sufficient: If $\chi(\{p_x, \rho_x\}) = H(p)$ then there exists a measurement μ such that Bob decodes the message with 100% probability of success.

      Hint: In class we discussed when an ensemble satisfies $\chi(\{p_x, \rho_x\}) = H(p)$.

3. *(4 points)* **Applications of monotonicity:** Prove the following two inequalities by using the monotonicity of the quantum relative entropy:

   1. *Entropy increase:* $H(\Phi[\rho]) \geqslant H(\rho)$ for every $\rho \in D(\mathcal{H})$ and *unital* channel $\Phi \in C(\mathcal{H}, \mathcal{H}')$. Recall that a channel is *unital* if $\Phi[I_{\mathcal{H}}] = I_{\mathcal{H}'}$.
   2. *Joint convexity* of relative entropy: $D(\sum_{x\in\Sigma} p_x \rho_x \| \sum_{x\in\Sigma} p_x \sigma_x) \leqslant \sum_{x\in\Sigma} p_x D(\rho_x \| \sigma_x)$, where $(p_x)_{x\in\Sigma}$ is an arbitrary finite probability distribution and $(\rho_x)_{x\in\Sigma}$, $(\sigma_x)_{x\in\Sigma}$ families of states in $D(\mathcal{H})$. You may assume that the operators ρ_x and σ_x are positive definite.

      Hint: In the exercise class, we computed the logarithm of a cq-state.