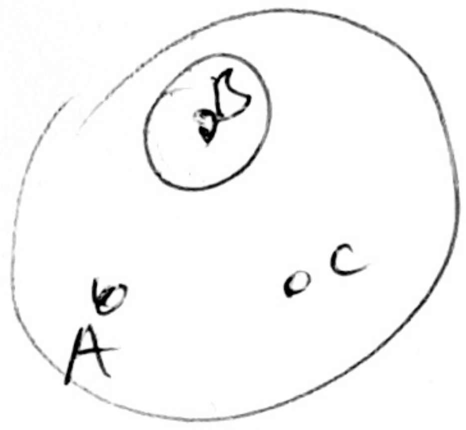
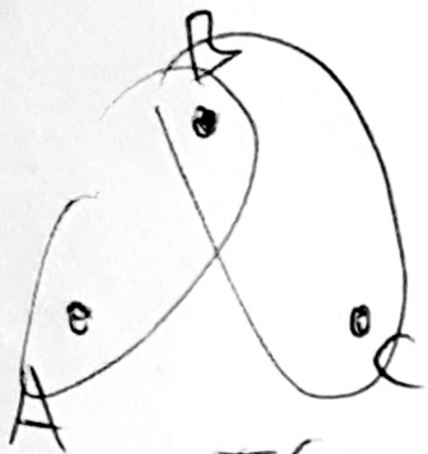
Lecture 8: Holevo quantity and relative entropy

ρ_{AB} $\quad \rho_A = \mathrm{Tr}_B\, \rho_{AB} \quad H(A) = H(\rho_A)$

ρ_{ABC}



Strong subadditivity:



$$H(AB) + H(BC) \geq H(ABC) + H(B)$$

↙ mutual information

$$I(A:B) = H(A) + H(B) - H(AB)$$

Monotonicity: $I(A:B) \leq I(A:BC)$

Data processing: $I(A:B)_\rho \geq I(A:C)_\omega$

Holevo bound $\quad \{p_x, \rho_x\} \quad x \in \Sigma \quad \rho_x \in D(\mathcal{H}_B)$

$$\rho_{XB} = \sum_{x \in \Sigma} p_x |x\rangle\langle x| \otimes \rho_x \in D(\mathcal{H}_X \otimes \mathcal{H}_B) \quad p \in P(\Sigma)$$

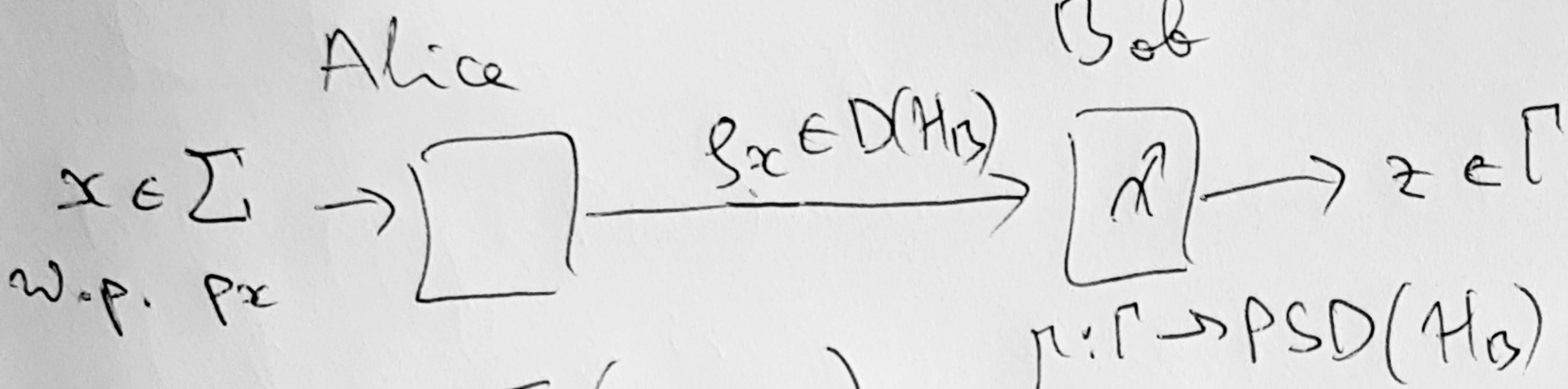
cq-state

Def Holevo χ-quantity of $\{p_x, \rho_x\}$ is

$$\chi(\{p_x, \rho_x\}) := I(X:B) = H\left(\sum_x p_x \rho_x\right) - \sum_x p_x H(\rho_x)$$

$$H(XB) = H(p) + \sum_x p_x H(\rho_x)$$

$$0 \leq \chi \leq H\left(\sum_x p_x \rho_x\right) \leq \log \dim \mathcal{H}_B$$



$p(x,z) = p_x \cdot \mathrm{Tr}(\rho_x \mu(z))$

$p \in P(\Sigma \times \Gamma)$ Want to know what is $I(X:Z)$.

<u>Theorem</u> (Holevo) $I(X:Z) \leq \chi(\{p_x, \rho_x\})$

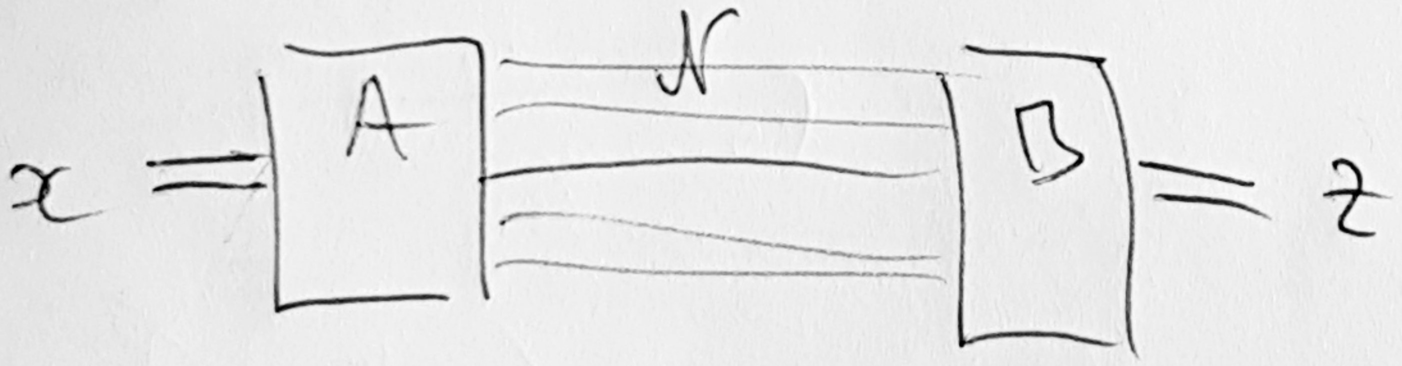
<u>Proof</u> $\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$

$$\chi(\{p_x, \rho_x\}) = I(X:B)_\rho \geq I(X:Z)_\omega$$

$$\omega_{XZ} = (I_X \otimes \Phi_{B\to Z})(\rho_{XB})$$

where $\Phi_{B\to Z}(\sigma) = \sum_{z\in\Gamma} \mathrm{Tr}[\sigma \mu(z)] \cdot |z\rangle\langle z|$

<u>Classical capacity of a quantum channel</u>



The classical capacity is

$$\lim_{n\to\infty} \frac{1}{n} \underbrace{\chi(\mathcal{N}^{\otimes n})}$$

is the supremum of $\chi(\{p_x, \sigma_x\})$ over all ensembles, $\sigma_x = \mathcal{N}(\rho_x)$

Relative entropy

Def $p, q \in P(\Sigma)$, the relative entropy of p with q is

$$D(p\|q) = \begin{cases} \sum_{x \in \Sigma} p(x) \log \frac{p(x)}{q(x)} & \text{if } * \\ \infty & \text{otherwise} \end{cases}$$

if * $\{x : q(x) = 0\}$ in $\{x : p(x) = 0\}$

$$\sum_x p(x) \log p(x) - \sum_x p(x) \log q(x)$$

$* \iff \forall x : q(x) = 0 \Rightarrow p(x) = 0$

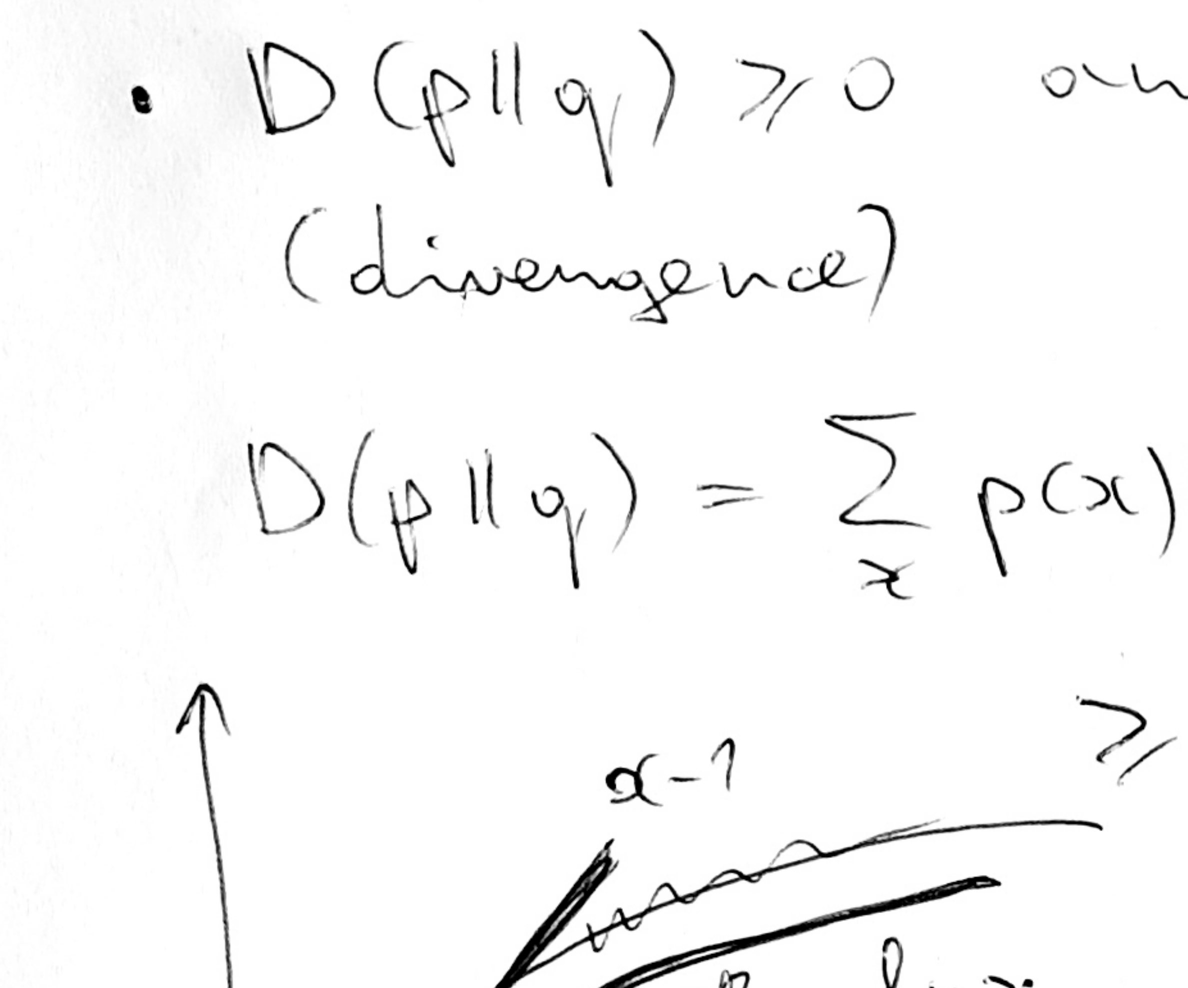
$\iff \forall x : p(x) > 0 \Rightarrow q(x) > 0$

$\cancel{\cos(\alpha + \beta) = \ldots}$

Properties:

- $D(p\|q) \geq 0$ and 0 iff $p = q$
  (divergence)

$$D(p\|q) = \sum_x p(x) \left(- \log \frac{q(x)}{p(x)} \right)$$



$$\geq \frac{1}{\ln 2} \sum_x p(x) \left(1 - \frac{q(x)}{p(x)} \right)$$

$$= \frac{1}{\ln 2} \left(\underbrace{\sum_x p(x)}_{1} - \underbrace{\sum_x q(x)}_{1} \right) = 0$$

$x - 1 \geq \ln x$

$$\log_2 x = \frac{\ln x}{\ln 2}$$

Applications:

- $p, q \in P(\Sigma)$ $\quad q = 1/|\Sigma|$

$$D(p\|q) = \sum_x p(x) \log p(x) - \sum_x p(x) \underbrace{\log q(x)}_{-\log|\Sigma|}$$

$$= -H(p) + \log|\Sigma| \geq 0$$

$$H(p) \leq \log|\Sigma|$$

- $p_{XY} \in P(\Sigma \times \Gamma)$ $\quad p_{XY}(x,y)$

q_{XY} $\quad q_{XY}(x,y) = p_X(x) \cdot p_Y(y)$

$$D(p\|q) = -H(p_{XY}) - \sum_{\substack{x\in\Sigma \\ y\in\Gamma}} p_{XY}(x,y) \log(p_X(x) \cdot p_Y(y))$$

$$= -H(p_{XY}) - \sum_{x\in\Sigma} p_X(x) \log p_X(x)$$

$$- \sum_{y\in\Gamma} p_Y(y) \log p_Y(y)$$

$$= H(p_X) + H(p_Y) - H(p_{XY}) = I(X:Y)_{p_{XY}}$$

$$\Rightarrow I(X:Y) \geq 0$$

$= 0$ iff $p_{XY}(x,y) = p_X(x) \cdot p_Y(y)$

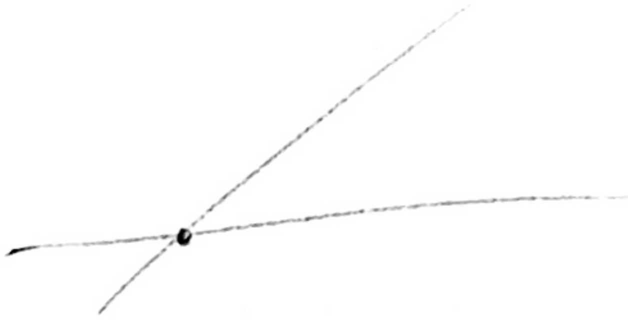
<u>Def</u> (q. rel. entropy)

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\sigma] & \text{if } \ker\sigma \subseteq \ker\rho \quad * \\ \infty & \text{otherwise} \end{cases}$$

$* \iff \mathrm{im}\,\rho \subseteq \mathrm{im}\,\sigma$

<u>Ex.</u> $D(\underbrace{|0\rangle\langle 0|}_{\rho} \,\|\, \underbrace{|+\rangle\langle +|}_{\sigma})$

$\mathrm{im}\,\rho = \mathrm{span}\{|0\rangle\}$



<u>Properties:</u>

- if $\rho = \mathrm{diag}(p_1, \dots, p_d)$, $\sigma = \mathrm{diag}(q_1, \dots, q_d)$ then $D(\rho\|\sigma) = D(p\|q)$

$$\rho = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_d \end{pmatrix} \qquad \sigma = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_d \end{pmatrix}$$

- $D(U\rho U^\dagger \,\|\, U\cancel{\rho}\sigma U^\dagger) = D(\rho\|\sigma)$

  U - isometry

- Monotonicity:

$$D(\rho\|\sigma) \geq D(\Phi[\rho] \,\|\, \Phi[\sigma])$$

$\rho, \sigma \in D(\mathcal{H}) \qquad \Phi \in C(\mathcal{H}, \mathcal{H}')$

• Nonnegativity (Klein's ineq.): $D(\rho\|\sigma) \geq 0$ $\quad \rho, \sigma \in D(\mathcal{H})$

$= 0$ iff $\rho = \sigma$

<u>Proof:</u> $\mu : \Omega \to PSD(\mathcal{H})$

$$p(x) = \mathrm{Tr}[\mu(x)\rho] \qquad q(x) = \mathrm{Tr}[\mu(x)\sigma]$$

$$\Phi[\omega] = \sum_{x\in\Omega} \mathrm{Tr}[\mu(x)\omega] \cdot |x\rangle\langle x|$$

$$\Phi[\rho] = \sum_{x\in\Omega} p(x)\,|x\rangle\langle x| \qquad \Phi[\sigma] = \sum_{x\in\Omega} q(x)\,|x\rangle\langle x|$$

By monotonicity:

$$D(\rho\|\sigma) \geq D(\Phi[\rho]\|\Phi[\sigma])$$
$$= D(p\|q) \geq 0$$

$$\rho \neq \sigma \overset{?}{\Rightarrow} D(\rho\|\sigma) \neq 0$$

There exists a measurement μ s.t.

$0 \neq \|\rho - \sigma\|_1 = \|p - q\|_1$ where p & q are the outcome probabilities

$\Downarrow$

$p \neq q$

• Joint convexity:

$$D\Big(\sum_x p_x \rho_x \Big\| \sum_x p_x \sigma_x\Big) \leq \sum_x p_x D(\rho_x\|\sigma_x)$$

Applications:

- $\rho \in D(\mathcal{H})$, $\sigma = I/d$, $\mathcal{H} = \mathbb{C}^d$

$$D(\rho\|\sigma) = +\operatorname{Tr}[\rho\log\rho] - \operatorname{Tr}[\rho\log\sigma]$$
$$= -H(\rho) + \log d \geq 0$$
$$\Rightarrow H(\rho) \leq \log d$$

- $\rho_{AB} \in D(\mathcal{H}_{AB})$ $\quad \sigma_{AB} = \rho_A \otimes \rho_B$

$$\rho_A = \operatorname{Tr}_B \rho_{AB}$$

$$D(\rho_{AB}\|\sigma_{AB}) = D(\rho_{AB}\|\rho_A\otimes\rho_B)$$
$$= I(A:B)_{\rho_{AB}}$$

- ρ_{ABE} $\quad$ SSA:

$$I(A:BE)_{\rho_{ABE}} = D(\rho_{ABE}\|\rho_A\otimes\rho_{BE})$$
$$\overset{\operatorname{Tr}_E}{\geq} D(\rho_{AB}\|\rho_A\otimes\rho_B)$$
$$= I(A:B)_{\rho_{AB}}$$
$$\|$$
$$H(A)+H(B)-H(AB)$$