Quantum Information Theory, Spring 2020

Practice problem set #7

You do **not** have to hand in these exercises, they are for your practice only.

1. **Mutual information upper bound:** From class we know that $I(A:B) \leqslant \log d_A + \log d_B$, where $d_A = \dim \mathcal{H}_A$ and $d_B = \dim \mathcal{H}_B$. Give a simple proof of this fact.

2. **Weak monotonicity:** Use a purification to deduce the weak monotonicity inequality $H(AC) + H(BC) \geqslant H(A) + H(B)$ from the strong subadditivity inequality, and vice versa.

3. **Strict concavity of the von Neumann entropy:** In Homework Problem 6.4 you proved that $H(\rho)$ is a concave function of $\rho \in D(\mathcal{H})$. Revisit your proof and show that it is strictly concave using the equality condition for the subadditivity inequality from today's lecture.

4. **Equality condition for monotonicity:** In Homework Problem 5.3, you proved that the Shannon entropy satisfies the following monotonicity inequality: $H(XY) \geqslant H(X)$ for any probability distribution p_{XY}. (Warning: This inequality is in general *false* for quantum states!) Show that equality holds if and only if $p_{XY}(x,y) = p_X(x)\delta_{f(x),y}$ for a function $f\colon \Sigma_X \to \Sigma_Y$.

   In terms of random variables, this means that $Y = f(X)$, *i.e., the second is a function of the first!*

5. **Binary entropy function:** The Shannon entropy of a probability distribution with two possible outcomes is given by the so-called *binary entropy function*:

$$h(p) := H(\{p, 1-p\}) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p},$$

   (a) Sketch this function.
   (b) Does there exists a constant $L > 0$ such that $|h(p) - h(q)| \leqslant L|p-q|$ for all $0 \leqslant p, q \leqslant 1$? That is, is h Lipschitz continuous?