Quantum Information Theory, Spring 2020

Homework problem set #5 **due March 9, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(2 points)* **Fidelity between classical-quantum states:** Show that the fidelity between two classical-quantum states $\rho_{XB} = \sum_{x\in\Sigma} p(x)\,|x\rangle\langle x| \otimes \rho_{B,x}$ and $\sigma_{XB} = \sum_{x\in\Sigma} q(x)\,|x\rangle\langle x| \otimes \sigma_{B,x}$ is

$$F(\rho_{XB}, \sigma_{XB}) = \sum_{x\in\Sigma} \sqrt{p(x)q(x)}\, F(\rho_{B,x}, \sigma_{B,x}).$$

2. *(3 points)* **Gentle measurement lemma:** This useful technical result states that if $\rho \in D(\mathcal{H})$ is a state and $0 \leqslant Q \leqslant I$ an operator such that $\mathrm{Tr}[Q\rho] \geqslant 1-\varepsilon$, then the following inequalities hold:

$$F\Big(\rho, \frac{\sqrt{Q}\rho\sqrt{Q}}{\mathrm{Tr}[Q\rho]}\Big) \geqslant \sqrt{1-\varepsilon} \quad \text{and} \quad T\Big(\rho, \frac{\sqrt{Q}\rho\sqrt{Q}}{\mathrm{Tr}[Q\rho]}\Big) \leqslant \sqrt{\varepsilon} \tag{1}$$

   (a) Prove that $\mathrm{Tr}\sqrt{\sqrt{\rho}\sqrt{Q}\rho\sqrt{Q}\sqrt{\rho}} = \mathrm{Tr}[\sqrt{Q}\rho]$ and $\sqrt{Q} \geqslant Q$.
   Hint: The square root $\sqrt{A}$ of a PSD operator A is the unique PSD operator that squares to A.

   (b) Prove the first inequality in Eq. (1) using part (a), and deduce the second inequality from the first by using a result from the practice problems.

3. *(4 points)* **Properties of the Shannon entropy:** Given a joint distribution, we write $H(XY)$ for its Shannon entropy and $H(X)$, $H(Y)$ for the entropies of its marginal distributions.

   (a) *Monotonicity:* Show that $H(XY) \geqslant H(Y)$.

   (b) *Subadditivity:* Show that $H(X) + H(Y) \geqslant H(XY)$.

   (c) Can you interpret the two inequalities in the context of compression?

   Hint: For both (a) and (b), write the left-hand side minus the right-hand side of the inequality as a single expectation value. For (b), use Jensen's inequality.

4. *(3 points)* **Optimality of the Shannon entropy:** In this problem, you will prove the converse part of Shannon's source coding theorem which states that it is impossible to compress at rates below the entropy of the source. Given a probability distribution p on a finite set Σ, recall that an (n, R, δ)-*code* consists of functions $E\colon \Sigma^n \to \{0,1\}^{\lfloor nR \rfloor}$ and $D\colon \{0,1\}^{\lfloor nR \rfloor} \to \Sigma^n$ such that $\sum_{x^n\in\Sigma^n : D(E(x^n))=x^n} p(x_1)\cdots p(x_n) \geqslant 1-\delta$. Show that:

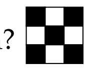
   (a) For any (n, R, δ)-code, there are at most 2^{nR} many strings x^n such that $D(E(x^n)) = x^n$.

   (b) For fixed $\delta \in (0,1)$ and $R < H(p)$, (n, R, δ)-codes can only exist for finitely many n.
   Hint: Distinguish between typical and atypical sequences.

5. *(2 bonus points)* **Practice:** A binary image of size $r \times s$ can be represented by a bitstring of length rs, where we list the pixel values (`0`=black pixel, 1=white pixel) row by row, starting with

the top row. We can thus compress the image in the following *lossless* fashion: First, compute the number k of ones in the bitstring. Next, compute the index $m \in \{0, 1, \ldots, \binom{rs}{k} - 1\}$ of the bitstring in the lexicographically sorted list of all bitstrings of length rs that contain k ones. The quadruple (r, s, k, m) defines the compression of the image.

For example, the 2×3-image  corresponds to the bitstring `000100`. There are six strings with $k = 1$ ones. In lexicographic order: `000001`, `000010`, `000100`, `001000`, `010000`, and `100000`. The index of our bitstring in this list is $m = 2$. Thus, we would compress this picture by $(2, 3, 1, 2)$.

(a) What is the bitstring corresponding to the following image? What is its compression? 

(b) Can you decompress the image given by $(r, s, k, m) = (7, 8, 8, 243185306)$?