Quantum Information Theory, Spring 2020

Practice problem set #3

You do **not** have to hand in these exercises, they are for your practice only.

1. **Positive semidefinite operators:** For all $Q \in \mathrm{PSD}(\mathcal{H})$, show that:

   (a) $A^\dagger Q A$ is positive semidefinite for all $A \in L(\mathcal{H})$.
   (b) If Q is invertible, its inverse Q^{-1} is again positive semidefinite.

   A positive semidefinite operator that is invertible is often called *positive definite*.

2. **Properties of the trace distance:** Show that the trace distance $T(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$ satisfies the following properties:

   (a) *Invariance:* $T(\rho, \sigma) = T(V\rho V^\dagger, V\sigma V^\dagger)$ for all states ρ, σ and any isometry V.
   (b) *Monotonicity:* $T(\rho_A, \sigma_A) \leqslant T(\rho_{AB}, \sigma_{AB})$ for all states ρ_{AB}, σ_{AB}.

   The fidelity $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is likewise invariant under isometries (can you see why?). However, its monotonicity goes the opposite way (see the homework). Why is this intuitive?

3. **Quantum channels:** Show that the following maps Φ are quantum channels by directly verifying that they are trace-preserving and completely positive.

   (a) *Basis change:* $\Phi[M] = UMU^\dagger$ for a unitary U.
   (b) *Add state:* $\Phi[M] = M \otimes \sigma$ for a state σ.
   (c) *Partial trace:* $\Phi[M_{AB}] = \mathrm{Tr}_B[M_{AB}]$.
   (d) *Classical channel:* $\Phi[M] = \sum_{x,y} p(y|x) \langle x|M|x\rangle |y\rangle\langle y|$, where $p(y|x)$ is a conditional probability distribution (i.e., $p(y|x)$ is a probability distribution in y for each fixed x).

4. **Composing channels:** If $\Phi_{A\to B}$, $\Psi_{B\to C}$ are quantum channels, then so is $\Psi_{B\to C} \circ \Phi_{A\to B}$. If $\Phi_{A\to B}$, $\Xi_{C\to D}$ are quantum channels, then so is $\Phi_{A\to B} \otimes \Xi_{C\to D}$.

5. **Schmidt decomposition:** Let $\rho_A = \sum_{i=1}^r p_i |e_i\rangle\langle e_i|$ be an arbitrary eigendecomposition, where $p_1, \dots, p_r$ are the nonzero eigenvalues of ρ_A and the $|e_i\rangle$ corresponding eigenvectors. If some eigenvalue appears more than once then this decomposition is *not* unique.

   (a) Show that, nevertheless, *any* purification $|\Psi_{AB}\rangle$ of ρ_A has a Schmidt decomposition of the form $|\Psi_{AB}\rangle = \sum_{i=1}^r s_i |e_i\rangle \otimes |f_i\rangle$, with the same $|e_i\rangle$ as above.
   Hint: Start with an arbitrary Schmidt decomposition and rewrite it in the desired form.
   (b) Conclude that any two purifications $|\Psi_{AB}\rangle$ and $|\Phi_{AB}\rangle$ are related by a unitary U_B – as claimed in Lecture 2.

   How about if we consider purifications on different Hilbert spaces?