Distance Measures & Intro to Q Channels

invariant under $M \mapsto VMW^\dagger$ for isometries V, W

Operator norms: For arbitrary $M \in L(\mathcal{H}, \mathcal{K})$ w/ singular values $\{s_i\}$

* trace norm: $\|M\|_1 = \sum_i s_i = \mathrm{tr}\left[\sqrt{M^\dagger M}\right]$ — square root of PSD operator
* Frobenius norm: $\|M\|_2 = \sqrt{\sum_i s_i^2} = \sqrt{\mathrm{tr}[M^\dagger M]}$ — induced by Hilbert-Schmidt inner product $(M, N) := \mathrm{tr}[M^\dagger N]$
* operator norm: $\|M\|_\infty = \max_i s_i = \max_{\|\phi\|=1} \|M|\phi\rangle\|$

USEFUL: submultiplicative

$$\& \ |\mathrm{tr}[AB]| \leq \begin{cases} \|A\|_2 \cdot \|B\|_2 & \text{Cauchy-Schwarz ineq} \\ \|A\|_1 \cdot \|B\|_\infty & \text{Hölder ineq} \end{cases}$$

$$\boxed{\|A\|_1 = \max_{\|B\|_\infty \leq 1} |\mathrm{tr}[AB]| = \max_{U \text{ unitary}} |\mathrm{tr}[AU]| \text{ for } A \in L(\mathcal{H})} \quad (*)$$

$\geq$ Hölder $\qquad \geq$ clear $\qquad \geq \|A\|_1$ by choosing $U|e_i\rangle = |g_i\rangle$

(Normalized) trace distance between $\rho, \sigma \in D(\mathcal{H})$:

$$\boxed{T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1 \quad \in [0,1]}$$

* $T(\rho,\sigma) = \sqrt{1 - |\langle\Phi|\Psi\rangle|^2}$ for $\rho = |\Psi\rangle\langle\Psi|$, $\sigma = |\Phi\rangle\langle\Phi|$ pure — HW 1
* $T(\rho,\sigma) = \max_{0 \leq Q \leq I} \mathrm{tr}[Q(\rho - \sigma)]$ — HW 2
  * nice interpretation! (Helmstrom's theorem)

Now comes a strange definition...

Fidelity between $\rho, \sigma \in D(\mathcal{H})$:

$$\boxed{F(\rho,\sigma) := \|\underbrace{\sqrt{\rho}\sqrt{\sigma}}_{\text{not Herm!}}\|_1 = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}}$$

* If $\rho = |\Psi\rangle\langle\Psi|$ pure: $F(\rho,\sigma) \overset{\text{use } \sqrt{\rho} = \rho}{=} \sqrt{\langle\Psi|\sigma|\Psi\rangle} = |\langle\Psi|\Phi\rangle|$ ← if also $\sigma = |\Phi\rangle\langle\Phi|$ pure

* $F(\rho,\sigma) = F(\sigma,\rho)$
* $0 \le F(\rho,\sigma) \le 1$, $F(\rho,\sigma) = \begin{cases} 1 & \text{if } \rho=\sigma \\ 0 & \text{if } \rho\sigma=0 \end{cases}$ Similarity measure!

Uhlmann's Theorem: If ρ_A, σ_A have purifications on $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$F(\rho_A,\sigma_A) = \max\left\{ |\langle \Psi_{AB} | \Phi_{AB}\rangle| : \begin{array}{l} |\Psi_{AB}\rangle, |\Phi_{AB}\rangle \in \mathcal{H}_A\otimes\mathcal{H}_B \\ \text{purifications of } \rho_A, \sigma_A \end{array} \right\}$$

NB: $\text{RHS} = \max\left\{ |\langle \Psi^{\text{fixed}}_{AB} | I_A \otimes U_B | \Phi^{\text{fixed}}_{AB}\rangle| : U_B \text{ unitary on } \mathcal{H}_B \right\}$

Pf: For simplicity, assume $\mathcal{H}_A = \mathcal{H}_B$. Take

$$|\Psi^{\text{std}}_{AB}\rangle = (\sqrt{\rho_A}\otimes I_B)\sum_x |x,x\rangle \quad \& \quad |\Phi^{\text{std}}_{AB}\rangle = (\sqrt{\sigma_A}\otimes I_B)\sum_x |x,x\rangle$$

$$\Rightarrow |\langle \Psi^{\text{std}}_{AB} | I_A \otimes U_B | \Phi^{\text{std}}_{AB}\rangle| = \sum_{x,y} (\langle x| \otimes \langle x|)(\sqrt{\rho_A}\sqrt{\sigma_A} \otimes U_B)(|y\rangle\otimes|y\rangle)$$

$$= \sum_{x,y} \langle x|\sqrt{\rho_A}\sqrt{\sigma_A}|y\rangle \underbrace{\langle x|U_B|y\rangle}_{=\langle y|U_B^T|x\rangle} = \operatorname{tr}\left[\sqrt{\rho_A}\sqrt{\sigma_A}\, \underset{\text{arbitrary unitary}}{U_B^T}\right]$$

$$\Rightarrow \max_{U_B} \overset{(*)\text{ above}}{=} \|\sqrt{\rho_A}\sqrt{\sigma_A}\|_1 \qquad \square$$

What if $\mathcal{H}_A \neq \mathcal{H}_B$? See lecture notes!

Properties of the fidelity:

* **Monotonicity:** $F(\rho_{AB},\sigma_{AB}) \le F(\rho_A,\sigma_A) \quad \forall \rho_{AB},\sigma_{AB}$
* **Joint concavity:** $F\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \ge \sum_i p_i F(\rho_i,\sigma_i)$

HW

How about the trace distance? Monotonous with $\ge$, jointly convex via triangle ineq.

Fuchs-van de Graaf inequality: $\boxed{1 - F \le T \le \sqrt{1-F^2}}$

pure states HW1

Channels in a classical world

$$\Gamma_X \ni x \longrightarrow \boxed{?} \longrightarrow y \in \Gamma_Y$$

How to describe mathematically? $y = f(x)$? In general, y can be random:

$$p(y|x) = \Pr(\text{output } y \,|\, \text{input } x) \quad \text{s.th.} \quad \begin{cases} p(y|x) \geq 0 \quad \forall x,y \quad \& \\ \sum_y p(y|x) = 1 \quad \forall x \end{cases}$$

* "Conditional prob distribution", "(memoryless) channel", "transition matrix", ...

* if input has distribution $p(x)$

$\Rightarrow$ output distribution $\underbrace{p(y) = \sum_x p(y|x)\, p(x)}_{\text{matrix-vector multiplication!}}$

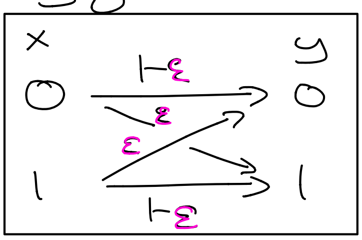
$P_X \mapsto P_Y$ is linear!

"$P_Y = P_{Y|X} \cdot P_X$"

joint distribution $p(x,y) = p(x)\, p(y|x)$

Ex:

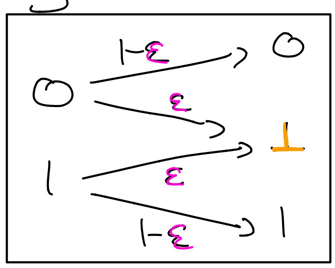
"binary symmetric channel"



$p(0|0) = p(1|1) = 1-\varepsilon$
$p(1|0) = p(0|1) = \varepsilon$

flip bit w/ probability ε

"binary erasure channel"



$p(0|0) = p(1|1) = 1-\varepsilon$
$p(\perp|0) = p(\perp|1) = \varepsilon$

bit lost w/ probability ε

$\Rightarrow$ just like probability distributions can describe "uncertain states", channels can describe "uncertain" or "noisy" evolutions

$\boxed{\text{GOAL of IT:}}$ Communicate reliably in the presence of uncertainty & noise!

How does the picture modify in quantum info theory?

Quantum Channels

$$D(\mathcal{H}_A) \ni \rho_A \longrightarrow \boxed{\;?\;} \longrightarrow \rho_B \in D(\mathcal{H}_B)$$

Since q. states are linear operators, this should be a map

$$\Phi : L(\mathcal{H}_A) \longrightarrow L(\mathcal{H}_B)$$

* Φ should be linear $\rightsquigarrow \Phi \in L(L(\mathcal{H}_A), L(\mathcal{H}_B))$, "superoperator"

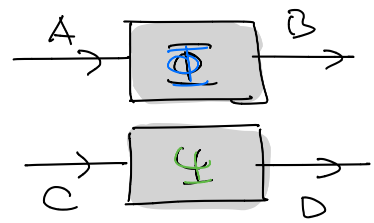
Why? So that $\Phi\left[\sum_i p_i \rho_i\right] \overset{!}{=} \sum p_i \Phi[\rho_i]$

* NOTATION: $\Phi_{A\to B}[\rho_A]$ etc.

* Identity: $\mathcal{I}_A : L(\mathcal{H}_A) \longrightarrow L(\mathcal{H}_A)$, $\mathcal{I}_A[M_A] = M_A$ $\quad A \longrightarrow A$

* Tensor product of superoperators: $\quad L(\mathcal{H}_A \otimes \mathcal{H}_C) \longrightarrow L(\mathcal{H}_B \otimes \mathcal{H}_D)$

$$(\Phi_{A\to B} \otimes \Psi_{C\to D})[M_A \otimes N_C]$$
$$:= \Phi_{A\to B}[M_A] \otimes \Psi_{C\to D}[N_C]$$



& extend by linearity

What conditions should Φ satisfy to be a channel? 1st attempt:

① "Positivity": $M \geq 0 \Rightarrow \Phi[M] \geq 0 \quad \forall M$

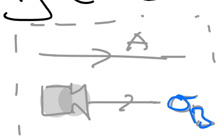
② Trace-preserving: $\mathrm{tr}[M] = \mathrm{tr}[\Phi[M]] \quad \forall M$

Together: ρ state $\Rightarrow \Phi[\rho]$ state

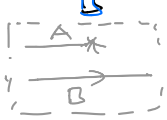
I write "ρ" since we will usually apply channels to states – but can plug in any linear operator!

Ex: * Base change: $\Phi[\rho] = U\rho U^\dagger$ for unitary (or isometry) U

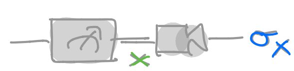
* Add state: $\Phi[\rho_A] = \rho_A \otimes \sigma_B$ for state σ



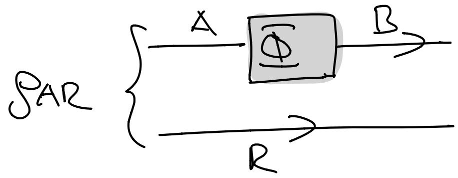
* Partial trace: $\Phi = \mathrm{tr}_A$



* Measure-and-prepare: $\Phi[\rho] = \sum_x \langle x|\rho|x\rangle \cdot \sigma_x$



PROBLEM: There are superoperators s.th. ① hold for Φ, but NOT for some $\Phi \otimes \mathcal{I}_R$ ↯



Transpose map: $\mathcal{T}[M] := M^T$ ← w.r.t. fixed basis

* Satisfies ① + ②
* $\rho_{AR} = |\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|\right)$ max ent state

$= |1\rangle\langle 0| \otimes |1\rangle\langle 0| \mapsto |1\rangle\langle 0|^T \otimes |1\rangle\langle 0| =$

$$\Longrightarrow (\mathcal{T}\otimes\mathcal{I}_R)[\rho_{AR}] = \frac{1}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\right)$$

$$= \underbrace{\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathrm{tr}=1\ :)} \text{ has eigenvalue } -1 \ ↯ \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

↳ ① is not good enough! The following defn fixes this:

Quantum channel: "superoperator" $\Phi_{A\to B} \in L(L(\mathcal{H}_A), L(\mathcal{H}_B))$ s. th.

① **Completely positive:** $\forall \mathcal{H}_R$: $M_{AR} \geq 0 \Longrightarrow (\Phi_{A\to B}\otimes\mathcal{I}_R)[M_{AR}] \geq 0$

② **Trace-preserving:** $\mathrm{tr}[M_A] = \mathrm{tr}[\Phi_{A\to B}[M_A]] \ \forall M_A$

$C(\mathcal{H}_A, \mathcal{H}_B) := \{\text{channels } \Phi_{A\to B}\}$, $C(\mathcal{H}_A) := C(\mathcal{H}_A, \mathcal{H}_A)$

* all examples above are channels — except the transpose → $\boxed{\text{EX \& HW}}$
* "old from new": $\Phi_{A\to B} \otimes \Phi_{C\to D}$ & $\Phi_{B\to C} \circ \Phi_{A\to B}$ are again channels
* any channel is composition of example channels? → next week

NB: "complete positivity" not an issue for probability theory

If $p(y|x)$ channel then so is $p(yz|xz') = p(y|x)\,\delta_{zz'}$