Quantum Information Theory, Spring 2020

Homework problem set #2 **due February 17, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Nayak's bound:** Alice wants to communicate m bits to Bob by sending n qubits. She chooses one state $\rho(x) \in D(\mathcal{H})$, where $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, for each possible message $x \in \{0,1\}^m$ that she may want to send. Bob uses a measurement $\mu\colon \{0,1\}^m \to \mathrm{PSD}(\mathcal{H})$ to decode the message.
   (a) Write down a formula for the probability that Bob successfully decodes Alice's message, assuming the latter is drawn from a known probability distribution $p(x)$ on $\{0,1\}^m$.
   (b) Show that if the message is drawn *uniformly* at random, then the probability that Bob successfully decodes the bitstring is at most 2^{n-m}.

2. *(4 points)* **Trace distance and Helstrom's theorem:** The *(normalized) trace distance* between two quantum states ρ, σ on a Hilbert space $\mathcal{H}$ is defined as

$$T(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1,$$

   in terms of the *trace norm* $\|\cdot\|_1$, which you know from the lecture and the previous homework.
   (a) Show that $T(\rho, \sigma) \in [0, 1]$.
   (b) Show that $T(\rho, \sigma) = \max_{0 \leqslant Q \leqslant I} \mathrm{Tr}[Q(\rho - \sigma)]$, and that the maximum can be achieved by a projection Q. *Hint: Consider the spectral decomposition of* $\rho - \sigma$*.*

   Now suppose we want to distinguish ρ and σ by a measurement $\mu : \{0, 1\} \to \mathrm{PSD}(\mathcal{H})$. By convention, outcome '0' corresponds to state ρ, while outcome '1' corresponds to state σ. Assuming both states occur with 50% probability, the probability of success using μ is given by

$$p_{\text{success}} = \frac{1}{2}\,\mathrm{Tr}[\rho\mu(0)] + \frac{1}{2}\,\mathrm{Tr}[\sigma\mu(1)].$$

   (c) Use (b) to prove *Helstrom's theorem*, which states that the *maximal* probability of success (over all possible measurements) is $\frac{1}{2} + \frac{1}{2}T(\rho, \sigma)$ and can be achieved by a *projective* measurement.

3. *(4 points)* **Extensions of pure states:** Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be arbitrary Hilbert spaces.
   (a) Let $M_A \in L(\mathcal{H}_A)$ and $N_{BC} \in L(\mathcal{H}_B \otimes \mathcal{H}_C)$. Then, $\mathrm{Tr}_C[M_A \otimes N_{BC}] = M_A \otimes \mathrm{Tr}_C[N_{BC}]$.
   (b) Let $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that ρ_A is pure. Then, $\rho_{AB} = \rho_A \otimes \rho_B$.
   Hint: In class we proved this when ρ_{AB} *is pure. Use a purification to reduce to this case.*
   (c) Let $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ such that ρ_{AB} is pure. Then, $\rho_{AC} = \rho_A \otimes \rho_C$ and $\rho_{BC} = \rho_B \otimes \rho_C$.
   (d) Let $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ such that ρ_{AB} and ρ_{AC} are pure. Then, $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$.

   Notation: Just like in class, if ρ_{AB} *is a state then we write* ρ_A *and* ρ_B *for its reduced states obtained by taking suitable partial traces (likewise for* ρ_{ABC} *and its reduced states).*