Quantum Information Theory, Spring 2019

Problem Set 6 **due March 18, 2019**

1. *(4 points)* **Compression and correlations:** In this problem, you will show that the definition of an (n, R, δ)-quantum code is more stringent than demanding that the average fidelity is high for any particular source. Consider a qubit source that emits states $\rho_0 = |0\rangle\langle 0|$ and $\rho_1 = |1\rangle\langle 1|$ with 50% probability each. Let $\rho \in D(\mathcal{X})$ denote the average output state of the source ($\mathcal{X} = \mathbb{C}^2$). Find channels $\mathcal{E}_n \in C(\mathcal{X}^{\otimes n}, (\mathbb{C}^2)^{\otimes n})$, $\mathcal{D}_n \in C((\mathbb{C}^2)^{\otimes n}, \mathcal{X}^{\otimes n})$ such that

$$\sum_{x_1,\dots,x_n} 2^{-n} F(\mathcal{D}_n[\mathcal{E}_n[\rho_{x_1} \otimes \ldots \otimes \rho_{x_n}]], \rho_{x_1} \otimes \ldots \otimes \rho_{x_n}) = 1,$$

but $F(\mathcal{D}_n \mathcal{E}_n, \rho^{\otimes n}) \to 0$ as $n \to \infty$.

Hint: Choose both $\mathcal{E}_n$ and $\mathcal{D}_n$ to be measure-and-prepare channels.

2. *(4 points)* **Monotonicity properties:** We already know that the trace distance and fidelity are monotone with respect to tracing out subsystems. In this problem, you will extend these properties to arbitrary quantum channels $\Phi \in C(\mathcal{X}, \mathcal{Y})$. Thus, show that:

   (a) $\|\Phi[A]\|_1 \leq \|A\|_1$ for all $A \in L(\mathcal{X})$, and hence $\|\Phi[\rho] - \Phi[\sigma]\|_1 \leq \|\rho - \sigma\|_1$ for all $\rho, \sigma \in D(\mathcal{X})$,
   (b) $F(\Phi[\rho], \Phi[\sigma]) \geq F(\rho, \sigma)$ for all $\rho, \sigma \in D(\mathcal{X})$.

   Hint: First prove that $\|VAV^*\|_1 = \|A\|_1$ for every $A \in L(\mathcal{X})$ and isometry $V \in L(\mathcal{X}, \mathcal{W})$.

3. *(4 points)* **Subadditivity:** Use Schumacher's theorem to prove the following inequality, which is known as the *subadditivity property* of the quantum entropy:

$$H(\rho_X) + H(\rho_Y) \geq H(\rho)$$

for every $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$ with reduced states $\rho_X = \mathrm{Tr}_Y[\rho]$ and $\rho_Y = \mathrm{Tr}_X[\rho]$.

Hint: You are allowed to use the following 'triangle inequality' for the fidelity (without proof): For any three states $\alpha, \beta, \gamma \in D(\mathcal{Z})$, if $F(\alpha, \beta) \geq 1 - \delta$ and $F(\beta, \gamma) \geq 1 - \delta$ then $F(\alpha, \gamma) \geq 1 - 4\delta$.

4. *(4 points)* **Practice:** In this problem, you can explore the behavior of the typical subspaces. Consider the state $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

   (a) Compute the largest eigenvalue p of ρ and the quantum entropy $H(\rho)$.
   (b) Plot the following functions for $n = 100$ and $n = 1000$:

$$d(k) = \binom{n}{k}, \quad r(k) = \frac{1}{n} \log \binom{n}{k}, \quad q(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

   for $k \in \{0, 1, \ldots, n\}$.
   (c) Plot the following functions for $\varepsilon = 0.1$ and $\varepsilon = 0.01$:

$$r(n) = \frac{1}{n} \log \dim S_{n,\varepsilon}, \quad p(n) = \mathrm{Tr}[\Pi_{n,\varepsilon} \rho^{\otimes n}],$$

   for $n \in \{1, \ldots, 1000\}$, where $\Pi_{n,\varepsilon}$ denotes the orthogonal projection onto the typical subspace $S_{n,\varepsilon}$ of ρ.