Quantum Information Theory, Spring 2019

Problem Set 3 **due February 25, 2019**

1. *(4 points)* **Vectorization:** Recall that if $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Gamma$ then $\mathrm{vec} : \mathrm{L}(\mathcal{Y}, \mathcal{X}) \to \mathcal{X} \otimes \mathcal{Y}$ is defined as $\mathrm{vec}(|i\rangle\langle j|) = |i\rangle \otimes |j\rangle$, for all $i \in \Sigma$ and $j \in \Gamma$, and then extended by linearity.

   (a) Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ be complex Euclidean spaces and let $A \in \mathrm{L}(\mathcal{X}, \mathcal{X}')$, $B \in \mathrm{L}(\mathcal{Y}, \mathcal{Y}')$, and $X \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$. Show that

   $$(A \otimes B)\,\mathrm{vec}(X) = \mathrm{vec}(AXB^{\mathsf{T}}).$$

   (b) Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\sigma \in \mathrm{D}(\mathcal{X})$. Recall that the standard purification of σ is given by

   $$|\psi\rangle = (\sqrt{\sigma} \otimes I_{\mathsf{X}}) \cdot \sum_{x \in \Sigma} |x\rangle \otimes |x\rangle.$$

   Show that $|\psi\rangle = \mathrm{vec}(\sqrt{\sigma})$.

2. *(2 points)* **Quantum channels:** Show that the following maps Φ are quantum channels by directly verifying that they are trace-preserving and completely positive.

   (a) (Mixed unitary): Let $(p_1, \ldots, p_n)$ be a probaility distribution, let $U_1, \ldots, U_n \in \mathrm{U}(\mathcal{X})$ be a set of unitary matrices, and let $\Phi \in \mathrm{T}(\mathcal{X})$ be defined as follows:

   $$\Phi(X) = \sum_{i=1}^{n} p_i U_i X U_i^*.$$

   (b) (State preparation): Let $\sigma \in \mathrm{D}(\mathcal{X})$ and let $\Phi \in \mathrm{T}(\mathcal{X})$ be defined as follows:

   $$\Phi(X) = \mathrm{Tr}[X]\,\sigma. \tag{1}$$

3. *(2 points)* **Kraus vs Choi:** Recall that the Kraus representation of a superoperator $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is given by

   $$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*,$$

   for some operators $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset \mathrm{L}(\mathcal{X}, \mathcal{Y})$. Show that the Choi representation of the same superoperator is given by

   $$J(\Phi) = \sum_{a \in \Gamma} \mathrm{vec}(A_a)\,\mathrm{vec}(B_a)^*.$$

4. *(4 points)* **Kraus operators:** Derive a Kraus representation for the following quantum channels:

   (a) (Discarding the input): Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\Phi \in \mathrm{C}(\mathcal{X}, \mathbb{C})$ be the quantum channel that discards the input. That is, for all $X \in \mathrm{L}(\mathcal{X})$,

   $$\Phi(X) = \mathrm{Tr}[X].$$

   (b) (State preparation): Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{X})$ be the quantum channel that discards the input and replaces it by some fixed state $\sigma \in \mathrm{D}(\mathcal{X})$, see Eq. (1).

5. *(4 points)* **Practice:** The files `pset3-A1.txt`, `pset3-A2.txt`, `pset3-A3.txt` contain the entries of three 5×5 matrices $A_1, A_2, A_3 \in \mathrm{L}(\mathcal{X})$ where $\mathcal{X} = \mathbb{C}^5$. These matrices define a superoperator $\Phi \in \mathrm{T}(\mathcal{X})$ that acts as

$$\Phi(X) = \sum_{i=1}^{3} A_i X A_i^*.$$

   (a) Compute the eigenvalues of the Choi matrix $J(\Phi)$.
   (b) Verify that Φ is a quantum channel (explain what you did).

   Hint: On the course homepage you can find instructions for loading a complex matrix.