Quantum Information Theory, Spring 2019

Problem Set 11 **due April 29, 2019**

1. *(4 points)* **Entanglement entropy and separable maps:** Let $|\Psi_1\rangle_{\mathsf{AB}}$ be a pure state on registers A and B, and assume that it can be perfectly transformed to another state $|\Psi_2\rangle_{\mathsf{AB}}$ by a separable operation. Show that such transformation cannot make the state more entangled in the sense of increasing its entanglement entropy. That is, show that

$$\mathrm{H}(\rho_1) \geq \mathrm{H}(\rho_2),$$

where $\rho_i = \mathrm{Tr}_{\mathsf{B}}\left[|\Psi_i\rangle\langle\Psi_i|_{\mathsf{AB}}\right]$ denotes the reduced state of $|\Psi_i\rangle_{\mathsf{AB}}$ on Alice and $\mathrm{H}(\rho)$ denotes the von Neumann entropy of ρ. *Hint: Entropy is a concave function.*

2. *(4 points)* **Local conversion with *no* communication:** Show that a pure state $|\Psi_1\rangle_{\mathsf{AB}}$ shared by Alice and Bob can be converted to another pure state $|\Psi_2\rangle_{\mathsf{AB}}$ using *only* local operations (and *no* communication) if and only if

$$\rho_1 \prec \rho_2 \qquad \text{and} \qquad \rho_2 \prec \rho_1$$

where $\rho_i = \mathrm{Tr}_{\mathsf{B}}\left[|\Psi_i\rangle\langle\Psi_i|_{\mathsf{AB}}\right]$ denotes the reduced state of $|\Psi_i\rangle_{\mathsf{AB}}$ on Alice. Show both directions of the implication.

3. *(4 points)* **Nielsen's theorem in action:** According to Nielsen's theorem, a maximally entangled state $|\Psi_1\rangle_{\mathsf{AB}}$ shared between Alice and Bob can be transformed to any other shared pure state $|\Psi_2\rangle_{\mathsf{AB}}$ of the same local dimensions by a one-way LOCC protocol from Bob to Alice. For each case below, devise an explicit one-way LOCC protocol that transforms $|\Psi_1\rangle_{\mathsf{AB}}$ to $|\Psi_2\rangle_{\mathsf{AB}}$ and succeeds with 100% probability. Write down the Kraus operators of Bob's instrument and the unitary corrections that Alice must apply after she receives Bob's measurement outcome.

   (a) Let $p \in [0,1]$ and

$$|\Psi_1\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}\,|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |1\rangle,$$
$$|\Psi_2\rangle_{\mathsf{AB}} = \sqrt{p}\,|0\rangle \otimes |0\rangle + \sqrt{1-p}\,|1\rangle \otimes |1\rangle.$$

   (b) Let $p \in \mathcal{P}(\mathbb{Z}_d)$ be an arbitrary probability distribution over $\mathbb{Z}_d = \{0, \ldots, d-1\}$ and

$$|\Psi_1\rangle_{\mathsf{AB}} = \sum_{i \in \mathbb{Z}_d} \frac{1}{\sqrt{d}}\,|i\rangle \otimes |i\rangle,$$
$$|\Psi_2\rangle_{\mathsf{AB}} = \sum_{i \in \mathbb{Z}_d} \sqrt{p(i)}\,|i\rangle \otimes |i\rangle.$$

   Hint: Let $S : \mathbb{C}^{\mathbb{Z}_d} \to \mathbb{C}^{\mathbb{Z}_d}$ denote the cyclic shift operator that acts as $S|i\rangle = |i+1\rangle$ where "+" denotes addition modulo d. Notice that $\frac{1}{d}\sum_{a\in\mathbb{Z}_d} S^a p = u$ where p is the original probability distribution and $u = (1, \ldots, 1)/d$ is the uniform distribution on $\mathbb{Z}_d$.

4. *(4 points)* **Practice:** Implement a subroutine that, given two probability distributions p and q (not necessarily of the same length) determines whether $p \prec q$.

   (a) The file `abc.txt` contains three probability distributions: a, b, and c. Compare the distributions a and b using your subroutine and output `"a < b"`, `"b < a"`, or `"incomparable"`.
   (b) Use your subroutine to compare the distributions $a \otimes c$ and $b \otimes c$.
   Output `"a*c < b*c"`, `"b*c < a*c"`, or `"incomparable"`.
   (c) How can you interpret this outcome?
   (d) The files `psi1.txt` and `psi2.txt` contain bipartite pure states

$$|\Psi_1\rangle_{\mathsf{AB}} \in \mathbb{C}^5 \otimes \mathbb{C}^7 \qquad \text{and} \qquad |\Psi_2\rangle_{\mathsf{AB}} \in \mathbb{C}^5 \otimes \mathbb{C}^9,$$

   where Alice's dimension is 5 and Bob's dimensions are 7 and 9, respectively. Output the eigenvalues of the reduced states on Alice's system A and determine whether $|\Psi_1\rangle_{\mathsf{AB}}$ can be perfectly transformed into $|\Psi_2\rangle_{\mathsf{AB}}$ by LOCC.

Mathematica hints: You can input the `abc.txt` file as follows:

```
{a,b,c} = Import[NotebookDirectory[]<>"abc.txt","Table"]
```

You can import the `psi*.txt` files using `GetPsi["psi1.txt"]` and `GetPsi["psi2.txt"]` where

```
GetPsi[f_] := Transpose[ToExpression[Import[NotebookDirectory[]<>f,"Table"]]/.j->I]
```