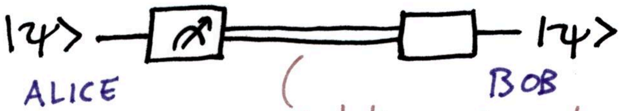
Entanglement

Watrous: lecture notes #11
book §6.1

Last time: entropy ~ measure of information
can communicate max n bits with n qubits

Teleportation

Can we send qubits over a classical channel?



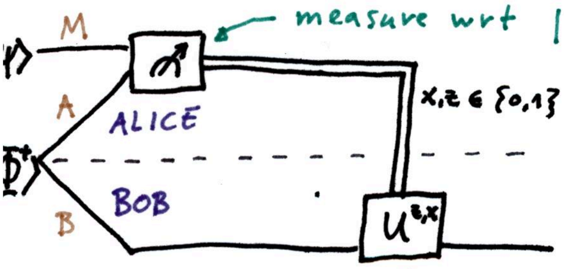
notation: classical communication = double lines

↯ infinitely many $|\psi\rangle$ ... cannot encode in finite number of bits...

But we can do it with an extra resource!

$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ = maximally entangled state = "ebit"

generalization: $\frac{1}{\sqrt{|\Sigma|}} \sum_{a\in\Sigma} |a\rangle \otimes |a\rangle$



measure wrt $|\Phi^{z,x}_{MA}\rangle = (Z^z X^x \otimes I_A)|\Phi^+_{MA}\rangle$ $\boxed{\text{ESET}}$

e.g. $|\Phi^{0,1}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

...

apply $U^{z,x} = Z^z X^x$

Prf (Sketch):

① $\mathrm{SWAP} = \frac{1}{2} \sum_{z,x\in\{0,1\}} Z^z X^x \otimes X^x Z^z$

$|a\rangle \otimes |b\rangle \mapsto |b\rangle \otimes |a\rangle$

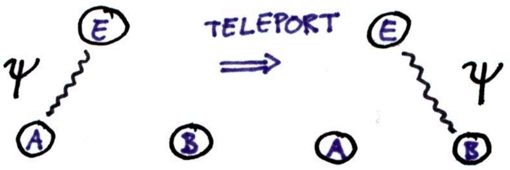
details in $\boxed{\text{ESET}}$ $\boxed{\text{PSET}}$

② $(X\otimes I)|\Phi^+\rangle = (I \otimes X^T)|\Phi^+\rangle$

①+② $\Rightarrow$ $|\psi_M\rangle \otimes |\Phi^+_{AB}\rangle = \frac{1}{2}\sum_{z,x} |\Phi^{z,x}_{MA}\rangle \otimes X^x Z^z |\psi_B\rangle$

exactly what we need! □

: preserves entanglement to other system: $\boxed{\text{PSET}}$



Resource inequality:

$e + 2[c \to c] \geq [q \to q]$

↳ teleportation

$e + [q \to q] \geq 2[c \to c]$

↳ superdense coding

Many questions...

* What is entanglement? — today!
* How to distinguish entanglement? — hard... but some results today
* How to manipulate entanglement? — next week: LOCC "Local Operations Classical Communication" e.g. teleportation
* How much entanglement in a given state $|\psi\rangle$?

↳ How many bits can I teleport with $|\psi\rangle$ as resource?
How many ebits do I need to construct $|\psi\rangle$ with LOCC?

Separable vs. Entangled

Def Separable

- pure state $|\psi_{XY}\rangle = |\psi_X\rangle \otimes |\psi_Y\rangle$
- general states $\rho_{XY} = \sum p_i \rho_X^i \otimes \rho_Y^i$
- general operators $X_{XY} = \sum P_X \otimes Q_Y$

positive

$\boxed{\text{ESET}}$

Entangled if not separable!

* Classical states $\sum p(x,y) |x\rangle\langle x| \otimes |y\rangle\langle y|$ are separable

↳ "entanglement = nonclassical correlations"

Convention: if $|\psi\rangle$ a pure state, write $\psi = |\psi\rangle\langle\psi|$.

Entanglement entropy

since ψ pure, Schmidt decomposition

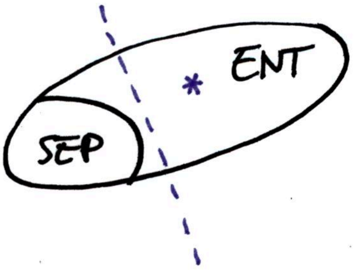
$|\psi_{XY}\rangle$ pure, $\quad S_E(\psi_{XY}) := H(\psi_X) = H(\psi_Y)$

* Pure state ψ_{XY} entangled $\iff S_E(\psi_{XY}) \neq 0$ $\boxed{\text{ESET}}$
* Later: $|\psi_{XY}\rangle^{\otimes n} \underset{\text{LOCC}}{\longleftrightarrow} |\Phi^+_{XY}\rangle^{\otimes \lfloor nR \rfloor}$ at rate $R = S_E(\psi_{XY})$.

* $\{$separable states $\rho_{XY}\}$ = convex & compact

by def. — convex hull of states $|\psi_x\rangle\langle\psi_x| \otimes |\phi_y\rangle\langle\phi_y|$

$\Downarrow$



Can separate any entangled state from separable set by a hyperplane

"Entanglement witness"

i.e. given entangled ρ_{XY}, $\exists H$, $H = H^*$ s.t.

$\langle \rho, H \rangle < 0$

$\langle \sigma, H \rangle > 0$ for all σ_{XY} separable

Thm (Horedecki)

ρ_{XY} separable $\iff (\Psi_X \otimes I_Y)(\rho) \geq 0$ for all unital positive superop's $\Psi_X : L(X) \to L(Y)$

↳ not completely positive!

Pf "$\Rightarrow$" $(\Psi \otimes I)(\sum p_i \rho_x^i \otimes \rho_y^i) = \sum p_i \Psi(\rho_x^i) \otimes \rho_y^i \geq 0$

"$\Leftarrow$" Suppose ρ_{XY} entangled

① $\langle \Phi^+ | (\Psi \otimes I)\rho | \Phi^+ \rangle = \langle (\Psi^* \otimes I)(\Phi^+), \rho \rangle$

max ent state on Y — def of Ψ^*

$= n_Y \langle J(\Psi^*), \rho \rangle$ — $|\Phi^+\rangle = \frac{1}{\sqrt{n_Y}} \operatorname{vec}(I_Y)$, check with def. J in Watrous!

$n_Y = \dim(Y)$; J: Choi matrix

② P, Q positive, $\langle P, \Psi(Q) \rangle = n_Y \langle \Phi^+, (\Psi \otimes I)(Q \otimes P^T) \rangle$

$= \langle J(\Psi^*), Q \otimes P^T \rangle$

① + ② $\Longrightarrow$ Choose Ψ s.t. $J(\Psi^*)$ is entanglement witness for ρ!

① $\rightsquigarrow$ $(\Psi\otimes I)\rho \not\geq 0$

② $\rightsquigarrow$ Ψ positive

Sketch of getting unital Ψ:

$\Psi \xrightarrow{\text{slightly disturb}} \tilde{\Psi}$ s.t. $\tilde{\Psi}(I)$ has full rank (and still $(\tilde{\Psi}\otimes I)\rho \not\geq 0$)

unital positive superop $X \to \tilde{\Psi}(I)^{-\frac{1}{2}}\,\tilde{\Psi}(X)\,\tilde{\Psi}(I)^{-\frac{1}{2}}$ □

Ex $\Psi(X) = X^T \longrightarrow$ "partial transpose test"

ρ is entangled if $(\Psi\otimes I)(\rho) \not\geq 0$ — shows $|\Phi^+\rangle$ entangled! $\boxed{\text{ESET}}$ week 3!

(& conversely for 2×2 or 2×3 systems!)

$\boxed{\textbf{Thm}: \text{There is a ball of separable states around } \frac{I_{XY}}{n_{XY}}}$

$\longrightarrow$ too much noise destroys entanglement!

Prf (Sketch) Idea: use Horodecki criterion!

Need: ① Write $X_{XY} = \sum \underbrace{X_{a,b}}_{X} \otimes \underbrace{|a\rangle\langle b|}_{Y} \Rightarrow \|X_{XY}\|^2 \leq \sum_{a,b} \|X_{a,b}\|^2$

similar to $\|A\| \leq \|A\|_2$

② Φ positive unital superop $\Rightarrow \|\Phi(X)\| \leq \|X\|$

see Watrous

So $\|(\Phi\otimes I)(X)\|^2 \overset{①}{\leq} \sum_{a,b} \|\Phi(X_{a,b})\|^2 \overset{②}{\leq} \sum_{a,b} \|X_{a,b}\|^2 \leq \|X\|_2^2$

positive, unital

$\rightarrow$ if $\|X\|_2^2 \leq \frac{1}{n_{XY}}$ then $(\Phi\otimes I)(X) < \frac{I_{XY}}{n_{XY}}$ for all pos. unital Φ

$\underset{\text{Horodecki}}{\Longrightarrow}$ $\frac{I_{XY}}{n_{XY}} - X$ separable □