Holevo Bound & Relative Entropy

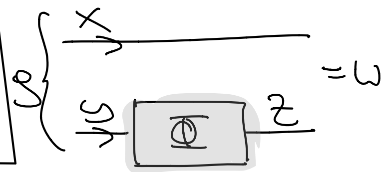
Last week: Entropy, Mutual info, properties

Recall: Strong subadditivity: $\forall \rho_{ABC} \in D(A \otimes B \otimes C)$:

$I(A:B) \leq I(A:BC)$ i.e. $H(AB) + H(BC) \geq H(B) + H(ABC)$

Data Processing Ineq (DPI): $\forall \rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y}),\ \Phi \in C(\mathcal{Y}, \mathcal{Z})$:

$I(X:Z)_\omega \leq I(X:Y)_\rho$ for $\omega_{XZ} = (I_X \otimes \Phi)[\rho_{XY}]$



* generalizes SSA: $X = A,\ Y = BC,\ Z = B,\ \Phi = \mathrm{tr}_C$
* implied by SSA: choose Stinespring isometry $V: \mathcal{Y} \to Z \otimes W$ for Φ:

$\omega_{XZW} = (I_X \otimes V)\rho_{XY}(I_X \otimes V^\dagger)$ extends ω_{XZ}

& $I(X:Y)_\rho \overset{\text{Isometric invariance}}{=} I(X:ZW)_\omega \overset{\text{SSA}}{\geq} I(X:Z)_\omega$ $\square$

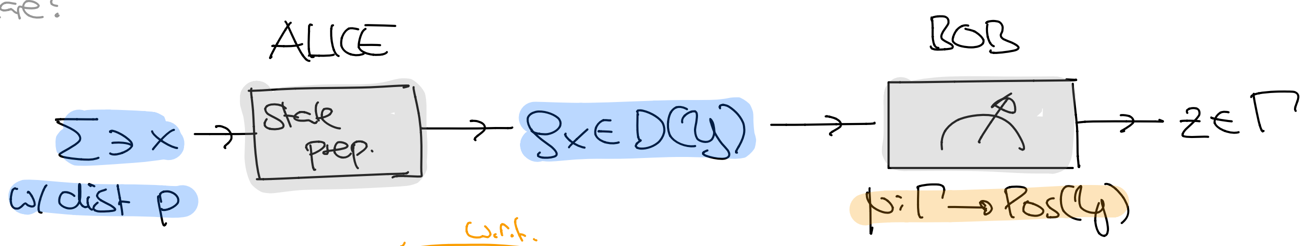
Recall: Ensembles $\{p_x, \rho_x\}$ can be represented by cq-states

$$\rho_{XY} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \in D(\mathcal{X} \otimes \mathcal{Y})$$

Holevo χ-quantity: $\chi(\{p_x, \rho_x\}) = I(X:Y) = H\left(\sum_x p_x \rho_x\right) - \sum_x p_x H(\rho_x)$

* $\chi \leq H(X)$, with "=" iff $\{\rho_x : p_x > 0\}$ pairwise orthogonal image $\rightsquigarrow$ $\boxed{\text{ESET}}$!

Why care?



How large can $I(X:Z)$ be? w.r.t. $p(x,z) = p_x \cdot \mathrm{tr}[\rho_x \mu(z)]$

Maximum over all μ is known as "accessible information" of ensemble.

Thm (Holevo): $I(X:Z) \leq \chi(\{p_x, \rho_x\}) = I(X:Y)$

Pf: Apply DPI to ρ_{XY} and $\Phi[\sigma] = \sum_z \mathrm{tr}[\sigma P(z)]\,|z\rangle\langle z|$ $\square$

$$\left(\omega_{XZ} = \sum_x p_x |x\rangle\langle x| \otimes \Phi[\rho_x] = \sum_{x,z} p(x,z)\, |x\rangle\langle x| \otimes |z\rangle\langle z| \right)$$

Interpretation?

can decode X from Z (i.e. $X = f(Z)$) $\overset{\text{ESET 7}}{\Leftrightarrow}$ $H(XZ) = H(Z)$ $\Leftrightarrow$ $I(X:Z) = H(X)$

$\Downarrow$ Holevo (PSET)

"To communicate n random bits (perfectly), need to send n qubits."

$\dim \mathcal{Y} \geq |\Sigma|$ $\overset{p_x > 0}{\underset{\{\rho_x\}\text{ orthog.}}{\Longleftarrow}}$ $\chi = H(X)$

* quantitative bounds via Fano's inequality
* Change of perspective: $x \longrightarrow$ [Channel] $\longrightarrow \rho_x$

Relative Entropy

Relative entropy: For $p, q \in \mathcal{P}(\Sigma)$, define

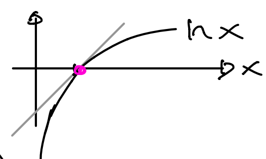
Consistent w/ $\log(\ldots + \varepsilon)$ and $\varepsilon \to 0$

$$D(p\|q) = \begin{cases} \sum_x p(x) \log p(x) - \sum_x p(x) \log q(x) & \text{if } \forall x: q(x) = 0 \Rightarrow p(x) = 0 \\ \infty & \text{otherwise} \end{cases}$$

NB: $p(x) \cdot \log q(x)$ clear if $p(x) > 0$, $= \begin{cases} 0 & \text{if } q(x) = p(x) = 0 \\ -\infty & \text{if } q(x) = 0,\ p(x) > 0 \end{cases}$

* $\boxed{D(p\|q) \geq 0, \quad = 0 \text{ iff } p = q}$ $\longleftarrow$ asymmetric distance measure

Pf: WLOG all $p(x) > 0$. Use $\ln(x) \leq x - 1$, "=" iff $x = 1$.



$$D(p\|q) = \sum_x p(x) \left(-\log \frac{q(x)}{p(x)} \right) \geq \frac{1}{\ln 2} \sum_x p(x) \left(1 - \frac{q(x)}{p(x)} \right) = 0$$

"=" iff $q(x) = p(x)$ $\forall x$ $\square$

Why do we care?

* $q(x) \equiv \frac{1}{|\Sigma|}$: $D(p\|q) = \log|\Sigma| - H(p) \geq 0$, $= 0$ iff p uniform ☺
* $p_{XY} \in \mathcal{P}(\Sigma \times \Gamma)$, $q_{XY}(x,y) \equiv p_X(x)\, p_Y(y)$:

$D(p_{XY}\|q_{XY}) = \ldots = I(X:Y)_p \geq 0$, $= 0$ iff $p_{XY}(x,y) = p_X(x)\, p_Y(y)$ ☺

Quantum relative entropy: $\rho, \sigma \in D(\mathcal{H})$

Consistent w/ $\log(\cdot + \varepsilon I)$ and $\varepsilon \to 0$

$$D(\rho\|\sigma) = \begin{cases} \mathrm{tr}[\rho \cdot \log \rho] - \mathrm{tr}[\rho \cdot \log \sigma] & \text{if } \ker(\sigma) \subseteq \ker(\rho) \\ \infty & \text{otherwise} \end{cases}$$

equ: $\mathrm{im}(\rho) \subseteq \mathrm{im}(\sigma)$

* $\rho \cdot \log(\sigma)$ clear on $\ker(\sigma)^\perp$, define as zero on $\ker(\sigma)$ if $\ker(\sigma) \subseteq \ker(\rho)$

$\leadsto$ $D(\rho\|\sigma) < \infty$ iff $\ker(\sigma) \subseteq \ker(\rho)$ e.g. $D(|0\rangle\langle 0| \,\|\, |+\rangle\langle +|) = \infty$

* $\rho = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$, $\sigma = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$: $D(\rho\|\sigma) = D(p\|q)$

Monotonicity: $\rho, \sigma \in D(\mathcal{H})$, $\Phi \in C(\mathcal{H}, \mathcal{Y})$

$\Longrightarrow$ $D(\rho\|\sigma) \geq D(\Phi[\rho] \,\|\, \Phi[\sigma])$

"FUNDAMENTAL THEOREM OF QIT"

Pf: HARD ↯

Klein's inequality: $D(\rho\|\sigma) \geq 0$, $= 0$ iff $\rho = \sigma$

Pf: Let $\mu: \Gamma \to \mathrm{Pos}(\mathcal{H})$ measurement & $\Phi \in C(\mathcal{H}, \mathbb{C}^\Gamma)$ corresp. channel
$\Phi[\omega] = \sum_{y \in \Gamma} \mathrm{tr}[\omega\, \mu(y)]\, |y\rangle\langle y|$

$$\begin{cases} p(y) = \mathrm{tr}[\rho\, \mu(y)] \\ q(y) = \mathrm{tr}[\sigma\, \mu(y)] \end{cases} \qquad \begin{cases} \Phi[\rho] = \sum_y p(y) |y\rangle\langle y| \\ \Phi[\sigma] = \sum_y q(y) |y\rangle\langle y| \end{cases}$$

a) $D(\rho\|\sigma) \geq D(\Phi[\rho] \,\|\, \Phi[\sigma]) = D(p\|q) \geq 0$

b) $\rho = \sigma$: $D(\rho\|\sigma) = 0$

c) $\rho \neq \sigma$: $\exists \mu$: $\|p - q\|_1 = \|\rho - \sigma\|_1 > 0$ $\Longrightarrow$ $D(\rho\|\sigma) \geq D(p\|q) \overset{p\neq q}{>} 0$ □

(PSET)

$\leadsto$ PSET / ESET for further properties

Applications: Quick proofs of things we proved last time

* $D(\rho\| \frac{I_X}{d}) \overset{\text{ESET}}{=} \log d - H(\rho)$ $\leadsto$ $H(\rho) \underset{\text{klein}}{\leq} \log d$, $=$ iff $\rho = \frac{I_X}{d}$ ☺

* Subadditivity: $D(\rho_{XY} \| \rho_X \otimes \rho_Y) \overset{\text{ESET}}{=} I(X:Y)_{\rho_{XY}} \overset{\text{klein}}{\geq} 0$, $= 0$ if $\rho_{XY} = \rho_X \otimes \rho_Y$ ☺

* Strong subadditivity:

$$I(X:YZ) = D(\rho_{XYZ} \| \rho_X \otimes \rho_{YZ}) \underset{\Phi = \mathrm{tr}_Z}{\overset{\text{monotonicity}}{\geq}} D(\rho_{XY} \| \rho_X \otimes \rho_Y) = I(X:Y)$$

* Basis measurement: $\rho \in D(\mathbb{C}^\Sigma)$, $p(x) = \langle x|\rho|x\rangle$, $\sigma = \sum_x p(x)\, |x\rangle\langle x|$

$$\Longrightarrow \boxed{H(p) = H(\sigma) \geq H(\rho), \quad = \text{ iff } \rho = \sigma}$$

Pf: $D(\rho \| \sigma) = -H(\rho) - \mathrm{tr}[\rho \cdot \log(\sigma)] = -H(\rho) + H(p) \rightsquigarrow$ KLEIN. $\square$

$\log(\sigma) = \sum_x \log(\langle x|\rho|x\rangle) |x\rangle\langle x|$