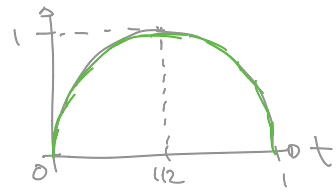
Entropy, Subsystems, Holevo bound

Recall: $H(p) = -\sum_i p_i \cdot \log p_i$ Shannon entropy

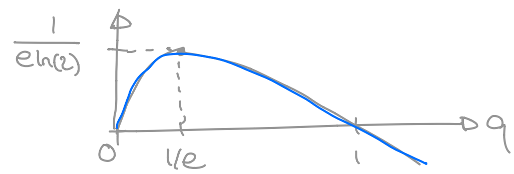
$H(\rho) = -\mathrm{tr}[\rho \cdot \log \rho] = H(p)$ where p = eigenvalues of ρ q. entropy

Properties:



* $h(t) := H(\{t, 1-t\}) \in [0,1]$ binary Shannon entropy
* $H(\rho) = H(V\rho V^*)$ $\forall$ isometries $V: \mathcal{H} \to \mathcal{Y}$
* $0 \leq H(\rho) \leq \log d$, where $d := \dim \mathcal{H}$

$\rho = |\psi\rangle\langle\psi|$ pure $\qquad \rho = \frac{I}{d}$



Pf: ≥ 0: since $f(q) := -q \cdot \log q \geq 0$ for $q \in [0,1]$

$\leq \log d$: assume $p_1, \ldots, p_r > 0$, $p_{r+1} = \ldots = p_d = 0$. Jensen's ineq for concave log:

$$H(\rho) = \sum_{i=1}^r p_i \cdot \log\left(\frac{1}{p_i}\right) \leq \log\left(\sum_{i=1}^r \frac{p_i}{p_i}\right) = \log r \leq \log d. \qquad \square$$

* Continuous:

Pf: f continuous; $\left\{ \begin{array}{l} p_1 \geq \ldots \geq p_d \text{ eigenvalues of } \rho \\ q_1 \geq \ldots \geq q_d \text{ eigenvalues of } \sigma \end{array} \right\} \Longrightarrow \|p-q\|_1 \leq \|\rho - \sigma\|_1$ $\square$

* Fannes-Audenaert inequality:

$$\boxed{|H(\rho) - H(\sigma)| \leq t \cdot \log(\dim \mathcal{H} - 1) + h(t)}$$ where $t = \frac{1}{2}\|\rho - \sigma\|_1 \in [0,1]$

NB: NOT Lipschitz continuous (not even $h(t)$)

* $H: D(\mathcal{H}) \to [0, \infty)$ concave, i.e. $H\left(\sum_x p_x \rho_x\right) \geq \sum_x p_x H(\rho_x)$ proof below

* Asymptotic equipartition property: $\forall \rho, \varepsilon > 0, n$ $\exists$ projections $\Pi_{n,\varepsilon}$ on $(\mathbb{C}^d)^{\otimes n}$

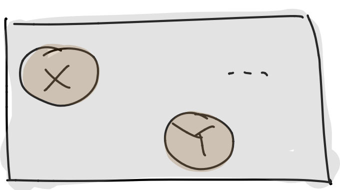
① $\mathrm{tr}[\Pi_{n,\varepsilon} \rho^{\otimes n}] \longrightarrow 1$ as $n \to \infty$

② $\mathrm{rk}(\Pi_{n,\varepsilon}) \leq 2^{n(H(\rho)+\varepsilon)}$

③ eigenvalues of $\Pi_{n,\varepsilon} \rho^{\otimes n} \Pi_{n,\varepsilon}$ in $2^{-n(H(\rho) \pm \varepsilon)}$

↳ $H(\rho)$ is optimal rate for Schumacher compression $\Rightarrow$ LECTURE 6

Entropies of subsystems



NOTATION: $p_{XY} \in \mathcal{P}(\Sigma_X \times \Sigma_Y) \rightsquigarrow p_X(x) = \sum_y p_{XY}(x,y), \ldots$

$H(XY)_p = H(p_{XY}),\quad H(X) = H(p_X), \ldots$

diagonal density ops usually omitted

Likewise for q. states: $\rho_{XY} \in D(\mathcal{H} \otimes \mathcal{K}) \rightsquigarrow \rho_X = \mathrm{tr}_Y[\rho_{XY}],\quad \rho_Y \in \mathrm{tr}_X[\rho_{XY}]$

$H(XY)_\rho := H(\rho_{XY}),\quad H(X) = H(\rho_X),\quad H(Y) = H(\rho_Y)$

usually omitted

Similarly if more than two subsystems.

* If ρ_{XY} pure: $H(XY) = 0$ & $H(X) = H(Y) =: S_E$ entanglement entropy

  Pf: $\rho_{XY} = |\Psi\rangle\langle\Psi|$ has eigenvalues $\{1, 0, \ldots, 0\}$

  Schmidt decomposition $|\Psi\rangle = \sum_i s_i |e_i\rangle \otimes |f_i\rangle$

  ↳ s_i^2 are nonzero eigenvalues of ρ_X & of ρ_Y. □

* If $\rho_{XY} = \rho_X \otimes \rho_Y$: $H(XY) = H(X) + H(Y)$

  NB: Notation consistent!

  Pf: $\rho_X : \{p_i\},\ \rho_Y : \{q_j\} \rightsquigarrow \rho_X \otimes \rho_Y : \{p_i q_j\}$

  $$H(XY) = -\sum_{ij} p_i q_j \log(p_i q_j) = -\sum_{ij} p_i q_j \log(p_i) - \sum_{ij} p_i q_j \log(q_j) = H(X) + H(Y) \quad \square$$

In general:

* $\boxed{H(XY) \leq H(X) + H(Y)}$ Subadditivity (SA)

  you proved this on PSET 6.
  next week: another proof

* $H(XY) \geq |H(X) - H(Y)|$ Araki-Lieb inequality (AL)

  Pf: $\rho_{XY} \rightsquigarrow |\Psi_{XYZ}\rangle \in \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{Z}$ purification useful!

  $\Rightarrow H(XY) = H(Z) \overset{SA}{\geq} H(XZ) - H(X) = H(Y) - H(X)$ □

* $\boxed{H(XY) + H(YZ) \geq H(Y) + H(XYZ)}$ strong subadditivity (SSA)

  NONTRIVIAL! Proof next week...

  Why stronger? SSA ⇒ SA if no Y

* equivalent: $H(XY) + H(YZ) \geq H(X) + H(Z)$ weak monotonicity

Pf: $\rho_{XYZ} \mapsto |\psi_{XYZW}\rangle$: $LHS = H(ZW) + H(YZ) \geq H(Z) + H(YZW) = RHS$ $\square$

Mutual information

$$I(X:Y) = H(X) + H(Y) - H(XY)$$

$\widehat{=}$ information that we lose when treating X, Y independently ($\to$ compression)

* defined for q. states ρ_{XY} + prob. distributions p_{XY}

  relation: if $\rho_{XY} = \sum_{x,y} p(x,y)\, |x\rangle\langle x| \otimes |y\rangle\langle y|$: $\boxed{I(X:Y)_\rho = I(X:Y)_p}$

* invariant under isometries $\mathcal{X} \to \tilde{\mathcal{X}}$ or $\mathcal{Y} \to \tilde{\mathcal{Y}}$.

* $I(X:Y) \overset{SA}{\geq} 0$, $= 0$ iff $\rho_{XY} = \rho_X \otimes \rho_Y$ "only if"? next week!

* ρ_{XY} pure: $\frac{1}{2} I(X:Y) = H(X) = H(Y) = S_E$ "="? $\to$ $\boxed{\text{PSET}}$

* $I(X:Y) \overset{AL}{\leq} 2 \min\{H(X), H(Y)\} \leq 2 \cdot \log \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$

  no factor 2 for prob. dist.

Cf.

e.g. $\rho_{XY} = |\Phi\rangle\langle\Phi|$, $|\Phi\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$: $I(X:Y) = 1 + 1 - 0 = 2$

$\rho_{XY} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$: $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ $I(X:Y) = 1 + 1 - 1 = 1$

* $I(X:Y) \overset{SSA}{\leq} I(X:YZ)$ Why useful?

If $\{p_x, \rho_x\}$ ensemble, consider cq-state $\rho_{XY} = \sum_{x \in \Sigma} p_x |x\rangle\langle x| \otimes \rho_x$.

Holevo χ-quantity of ensemble:

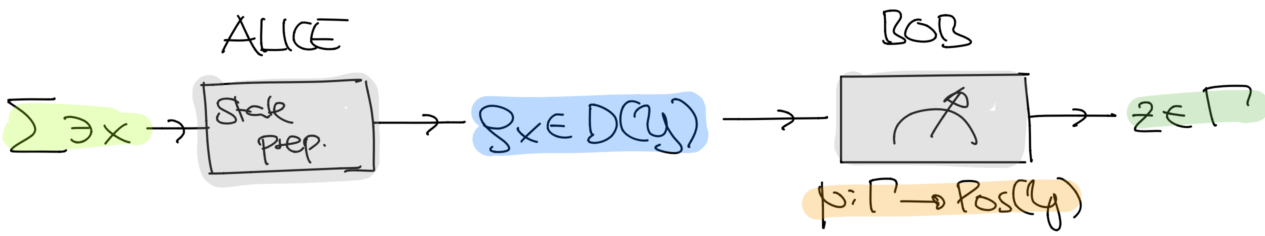
$$\boxed{\chi(\{p_x, \rho_x\}) = I(X:Y) = H\left(\sum_x p_x \rho_x\right) - \sum_x p_x H(\rho_x) \leq \log \dim \mathcal{Y}}$$

* used $H(XY) \overset{ESET}{=} H(p) + \sum_x p_x H(\rho_x)$, $H(X) = H(p)$, $H(Y) = H\left(\sum_x p_x \rho_x\right)$

* $I \geq 0$ implies concavity of H

Holevo's bound

How many bits can Alice reliably commun. to Bob by sending a q. state?



PSET 2: $\Sigma = \{0,1\}^m = \Gamma$, $\mathcal{Y} = (\mathbb{C}^2)^{\otimes n}$, $X \in \Sigma$ uniformly at random

$\boxed{\Pr(X=Z) \leq 2^{n-m}}$ i.e. need to send $n \geq m$ qubits to communicate m bits reliably

How large can $I(X:Z)$ be? w.r.t. $p(x,z) = p_x \cdot \mathrm{tr}[\mu(z)\rho_x]$.

$\boxed{\text{Thm (Holevo): } I(X:Z) \leq \chi \text{ for every ensemble } \{p_x, \rho_x\} \text{ \& meas. } \mu}$

Proof? Next time...