Lecture 3: Quantum Channels (Chapter 2.2)

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces.

$L(\mathcal{X},\mathcal{Y}) = \{\text{linear maps } \mathcal{X}\to\mathcal{Y}\}$ "Superoperators"

$T(\mathcal{X},\mathcal{Y}) = \{\text{linear maps } L(\mathcal{X})\to L(\mathcal{Y})\}$

Positive semidefinite operators:

$\mathrm{Pos}(\mathcal{X}) = \{Y^*Y : Y\in L(\mathcal{X})\}$ (see p. 20 for alternative characterizations)

Quantum states: $D(\mathcal{X}) \subseteq L(\mathcal{X}) \equiv L(\mathcal{X},\mathcal{X})$

- unit trace: $\mathrm{Tr}\,\rho = 1$
- positive semi-definite: $\rho \geqslant 0$ $\left(\rho\in\mathrm{Pos}(\mathcal{X})\right)$

Quantum channels: $C(\mathcal{X},\mathcal{Y}) \subseteq T(\mathcal{X},\mathcal{Y})$

To preserve the property of being a quantum state, a superoperator $\Phi\in T(\mathcal{X},\mathcal{Y})$ must be

- trace-preserving:

$$\mathrm{Tr}\left[\Phi(X)\right] = \mathrm{Tr}(X), \ \forall X\in L(\mathcal{X})$$ identity channel

- completely positive: $\Phi\otimes \mathbb{I}_{\mathcal{Z}}$ is positive, $\forall \mathcal{Z}$

$$\left(\Phi\otimes\mathbb{I}_{\mathcal{Z}}\right)(P)\in\mathrm{Pos}(\mathcal{Y}\otimes\mathcal{Z}),\ \forall P\in\mathrm{Pos}(\mathcal{X}\otimes\mathcal{Z}),\ \forall\mathcal{Z}$$

Intuition: Φ preserves positivity even when acting on any part of a larger system.

CPTP = completely positive, trace-preserving.

Vectorization $\mathcal{X} = \mathbb{C}^\Sigma, \mathcal{Y} = \mathbb{C}^\Gamma$

Matrix	Bipartite pure state
$X \in L(\mathcal{Y}, \mathcal{X})$	$\lvert\psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$

$$X = \sum_{i\in\Sigma, j\in\Gamma} \alpha_{ij} \lvert i\rangle\langle j\rvert \xrightarrow{\text{vec}} \lvert\psi\rangle = \sum_{i\in\Sigma, j\in\Gamma} \alpha_{ij} \lvert i\rangle \otimes \lvert j\rangle$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Def vec : $L(\mathcal{Y}, \mathcal{X}) \to \mathcal{X} \otimes \mathcal{Y}$ is defined on the standard basis states as $\text{vec}(\lvert i\rangle\langle j\rvert) = \lvert i\rangle\lvert j\rangle$, for all $i \in \Sigma$, $j \in \Gamma$, and extended by linearity to complex linear combinations.

Key property: (homework) $\boxed{(A \otimes B)\, \text{vec}(X) = \text{vec}(A X B^T)}$

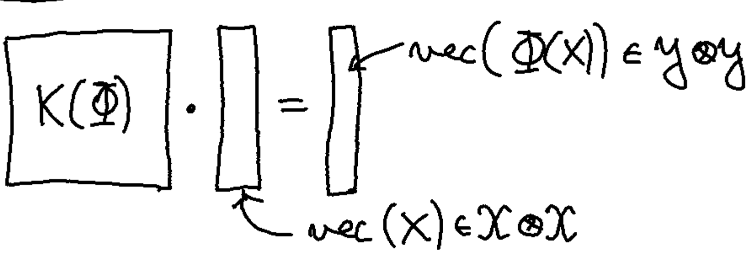
Representations of superoperators

Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$ be an arbitrary superoperoperoperator where $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Gamma$.

1. Natural representation

If we vectorize both the input and output, we get a linear map $\text{vec}(X) \mapsto \text{vec}(\Phi(X))$. This is the natural representation $K(\Phi) \in L(\mathcal{X} \otimes \mathcal{X}, \mathcal{Y} \otimes \mathcal{Y})$:

$$\boxed{K(\Phi) \cdot \text{vec}(X) = \text{vec}(\Phi(X))} \quad X \in L(\mathcal{X}),\ \Phi(X) \in L(\mathcal{Y}).$$



The entries of $K(\Phi)$ as a $|\Sigma|^2 \times |\Gamma|^2$ matrix are

$$K(\Phi) = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \langle c|\Phi(|a\rangle\langle b|)|d\rangle \cdot |c\rangle\langle a| \otimes |d\rangle\langle b|$$

This is consistent with the desired relation

$$K(\Phi) \cdot \mathrm{vec}(|a\rangle\langle b|) = \mathrm{vec}\Big(\Phi(|a\rangle\langle b|)\Big)$$

2. Choi-Jamiołkowski representation

For all standard basis matrix inputs, we stack the corresponding outputs in a big block matrix: $J(\Phi) \in L(\mathcal{Y}\otimes\mathcal{X})$ is defined as

$$J(\Phi) = \sum_{a,b\in\Sigma} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|$$

Equivalently, it is the output when Φ is applied to one half of the (unnormalized) maximally entangled state:

$$J(\Phi) = \left(\Phi \otimes I_X\right)\left(\sum_{a\in\Sigma} |a\rangle|a\rangle \sum_{b\in\Sigma} \langle b|\langle b|\right)$$

$$= \sum_{a,b\in\Sigma} \left(\Phi \otimes I_X\right)\left(|a\rangle\langle b| \otimes |a\rangle\langle b|\right)$$

The action of Φ on arbitrary input $X \in L(\mathcal{X})$ can be recovered as follows:

$$\Phi(X) = \mathrm{Tr}_X\left[J(\Phi)\cdot(I_Y \otimes X^T)\right]$$

the 2nd register

The matrix $J(\Phi)$ has similar properties as a bipartite quantum state (density matrix).

3. Kraus representations (not unique)

The superoperator $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is represented by a collection of operators

$$\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$$

such that

$$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*$$

Such representation always exists but is not unique. Usually $A_a = B_a$, for all $a \in \Gamma$. Then $\{A_a : a \in \Gamma\}$ are called Kraus operators of Φ.

4. Stinespring representations (not unique)

Another way to represent $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is by attaching another register Z with associated complex Euclidean space $\mathcal{Z}$, embeding the input $X \in L(\mathcal{X})$ into this enlarged space and then discarding the Z register. This involves operators $A, B \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$, for some space $\mathcal{Z}$, such that

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A X B^*)$$

Such representation always exists but is not unique. Again, usually one considers the case $A = B$.

These representations offer four different ways of looking at the same superoperator.

- How are they related?
- How can we tell if a superoperator is a quantum channel?

Relationship among representations

Proposition 2.20:

Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$ and $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$. Then the following represent the same Φ:

1. Natural: $K(\Phi) = \sum_{a\in\Gamma} A_a \otimes \overline{B_a}$
2. Choi: $J(\Phi) = \sum_{a\in\Gamma} \operatorname{vec}(A_a)\operatorname{vec}(B_a)^*$
3. Kraus: $\Phi(X) = \sum_{a\in\Gamma} A_a X B_a^*$
4. Stinespring: $\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(A X B^*)$ where

$$A = \sum_{a\in\Gamma} A_a \otimes \underset{\substack{\uparrow \\ \mathcal{Z} = \mathbb{C}^\Gamma}}{|a\rangle} \quad \text{and} \quad B = \sum_{b\in\Gamma} B_b \otimes |b\rangle$$

Proof: $3 \Leftrightarrow 4$ and $1 \Leftrightarrow 3$ (exercise)
$2 \Leftrightarrow 3$ (homework)

Note: It is not a priori clear how large the set Γ (and thus also the dimension of $\mathcal{Z}$) should be in these representations. One can show that it suffices to take $|\Gamma| = \operatorname{rank}(J(\Phi))$. Either way, $|\Gamma| \leq \dim(\mathcal{Y} \otimes \mathcal{X})$ since $J(\Phi) \in L(\mathcal{Y} \otimes \mathcal{X})$.

This concludes different ways of representing superoperators. But how can we tell if a superoperator is a quantum channel (i.e., maps quantum states to quantum states)?

Note: The natural representation is not very helpful for this, so we will not consider it.

Completely positive superoperators

Recall that $\Phi \in T(\mathcal{X},\mathcal{Y})$ is <u>positive</u> if $\Phi(P) \in \mathrm{Pos}(\mathcal{Y})$ for all $P \in \mathrm{Pos}(\mathcal{X})$, and <u>completely positive</u> if $\Phi \otimes \mathbb{1}_{\mathcal{Z}}$ is positive for every choice of a complex Euclidean space $\mathcal{Z} = \mathbb{C}^{\Gamma}$.

<u>Theorem 2.22</u>: The following are equivalent for $\Phi \in T(\mathcal{X},\mathcal{Y})$:

1. Φ is completely positive.
2. $\Phi \otimes \mathbb{1}_{\mathcal{X}}$ is positive.
3. $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$.
4. There is a finite set Γ and $\{A_a : a \in \Gamma\} \subset L(\mathcal{X},\mathcal{Y})$ such that
$$\Phi(X) = \sum_{a \in \Gamma} A_a X A_a^*.$$
5. $|\Gamma| = \mathrm{rank}(J(\Phi))$ is enough in 4.
6. There is $A \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$, for some $\mathcal{Z}$, such that
$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A X A^*).$$
7. It is enough to have $\dim(\mathcal{Z}) = \mathrm{rank}(J(\Phi))$ in 6.

$$\begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[r, Rightarrow] & 3 \arrow[r, Rightarrow] & 5 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & 4 \arrow[r, Rightarrow] & 1 \\ & & & & 7 \arrow[r, Rightarrow] & 6 \arrow[u, Rightarrow] \end{tikzcd}$$

<u>Proof</u>: $1 \Rightarrow 2$ by definition, $5 \Rightarrow 4$ and $7 \Rightarrow 6$ are trivial.

$5 \Rightarrow 7$ Since $\dim(\mathcal{Z}) = |\Gamma|$ when constructing A out of the Kraus operators $\{A_a : a \in \Gamma\}$.

$2 \Rightarrow 3$ $J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathcal{X}})(\underbrace{|\Psi\rangle\langle\Psi|}_{\geq 0}) \geq 0$ where

$|\Psi\rangle = \sum_{a \in \Sigma} |a\rangle|a\rangle = \mathrm{vec}(\mathbb{1}_{\mathcal{X}})$ is the unnormalized maximally entangled state.

$3 \Rightarrow 5$ Let $\mathcal{J}(\Phi) = \sum_{a\in\Gamma} |v_a\rangle\langle v_a|$ be a spectral decomposition of $\mathcal{J}(\Phi)$ where the vectors $|v_a\rangle \in \mathcal{Y}\otimes\mathcal{X}$ are not necessarily normalized and the (positive) eigenvalues have been absorbed. Note that $|\Gamma| = \text{rank}(\mathcal{J}(\Phi))$. Taking $A_a \in L(\mathcal{X},\mathcal{Y})$ be such that $\text{vec}(A_a) = |v_a\rangle$, $\mathcal{J}(\Phi) = \sum_{a\in\Gamma} \text{vec}(A_a)\,\text{vec}(A_a)^*$, which translates to Kraus representation $\Phi(X) = \sum_{a\in\Gamma} A_a X A_a^*$ with $|\Gamma| = \text{rank}(\mathcal{J}(\Phi))$.

$4 \Rightarrow 1$ If $\Phi(X) = \sum_{a\in\Gamma} A_a X A_a^*$ and $P \in \text{Pos}(\mathcal{X}\otimes\mathcal{W})$ then $(\Phi\otimes I_W)(P) = \sum_{a\in\Gamma} (A_a\otimes I_W)P(A_a\otimes I_W) \geq 0$ since each term is positive and the sum of two positive operators is also positive.

$6 \Rightarrow 1$ Let $P \in \text{Pos}(\mathcal{X}\otimes\mathcal{W})$ and $\Phi(X) = \text{Tr}_Z(AXA^*)$, $A \in L(\mathcal{X},\mathcal{Y}\otimes\mathcal{Z})$.

$$(\Phi\otimes I_W)(P) = \text{Tr}_Z\left[(A\otimes I_W)P(A\otimes I_W)^*\right]$$

$$= \sum_{a\in\Gamma}\left(I_Y\otimes\langle a|_Z\otimes I_W\right)\left(A_{YZ}\otimes I_W\right)P\left(A^*_{YZ}\otimes I_W\right)\left(I_Y\otimes|a\rangle\otimes I_W\right) \geq 0.$$

This concludes the characterization of complete positivity. It remains to characterize trace-preserving superoperators.

Trace-preserving superoperators

Recall that a superoperator $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is trace-preserving if
$$\mathrm{Tr}\big(\Phi(X)\big) = \mathrm{Tr}\, X, \quad \forall X \in L(\mathcal{X}).$$

Theorem 2.26: The following are equivalent for $\Phi \in T(\mathcal{X}, \mathcal{Y})$:

1. Φ is trace-preserving.
2. $\mathrm{Tr}_{\mathcal{Y}}(J(\Phi)) = I_{\mathcal{X}}$
3. There exist $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$ such that $\Phi(X) = \sum_{a\in\Sigma} A_a X B_a^*$ and $\sum_{a\in\Sigma} A_a^* B_a = I_{\mathcal{X}}$.
4. $\sum_{a\in\Sigma} A_a^* B_a = I_{\mathcal{X}}$ holds for all Kraus decompositions of Φ.
5. There exist $A, B \in L(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$, for some $\mathcal{Z}$, such that $\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A X B^*)$ and $A^* B = I_{\mathcal{X}}$.
6. $A^* B = I_{\mathcal{X}}$ holds for all Stinespring representations.

$$\begin{tikzcd}
4 \arrow[rr, Rightarrow] & & 3 \arrow[dl, Rightarrow] \\
& 1 \arrow[ul, Rightarrow] \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow] & 2 \\
6 \arrow[rr, Rightarrow] & & 5 \arrow[ul, Rightarrow]
\end{tikzcd}$$

Implications we will prove.

Proof: $4 \Rightarrow 3$ and $6 \Rightarrow 5$ since $\forall$ is more powerful than $\exists$.

$1 \Rightarrow 2$ Recall that $J(\Phi) \in L(\mathcal{Y}\otimes\mathcal{X})$, $J(\Phi) = \sum_{a,b\in\Sigma} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|$.
$$\mathrm{Tr}_{\mathcal{Y}}\big[J(\Phi)\big] = \sum_{a,b\in\Sigma} \underbrace{\mathrm{Tr}\big[\Phi(|a\rangle\langle b|)\big]}_{\mathrm{Tr}(|a\rangle\langle b|) = \delta_{a,b}} |a\rangle\langle b| = \sum_{a\in\Sigma} |a\rangle\langle a| = I_{\mathcal{X}}.$$

$2 \Rightarrow 1$ $$I_{\mathcal{X}} = \mathrm{Tr}_{\mathcal{Y}}\big[J(\Phi)\big] = \sum_{a,b\in\Sigma} \underbrace{\mathrm{Tr}\big[\Phi(|a\rangle\langle b|)\big]}_{\delta_{a,b} = \mathrm{Tr}(|a\rangle\langle b|)} |a\rangle\langle b|.$$

$3 \Rightarrow 1$ $\mathrm{Tr}[\Phi(X)] = \sum_{a\in\Sigma} \mathrm{Tr}[A_a X B_a^*] = \mathrm{Tr}\left[\left(\sum_{a\in\Gamma} B_a^* A_a\right) X\right]$

$$= \mathrm{Tr}\Big[\underbrace{\Big(\sum_{a\in\Sigma} A_a^* B_a\Big)^*}_{I_X} X\Big] = \mathrm{Tr}[X].$$

$1 \Rightarrow 4$ $\mathrm{Tr}[X] = \mathrm{Tr}[\Phi(X)] = \mathrm{Tr}[M^* X]$ where $M = \sum_{a\in\Sigma} A_a^* B_a$.
Since this holds for all $X \in L(\mathcal{X})$, $M = I_X$.
(Exercise.)

$5 \Rightarrow 1$ $\mathrm{Tr}[\Phi(X)] = \mathrm{Tr}[\mathrm{Tr}_Z(AXB^*)] = \mathrm{Tr}[AXB^*] =$

$$= \mathrm{Tr}[B^* A X] = \mathrm{Tr}[\underbrace{(A^* B)^*}_{I_X} X] = \mathrm{Tr}[X].$$

$1 \Rightarrow 6$ $\mathrm{Tr}[X] = \mathrm{Tr}[\Phi(X)] = \mathrm{Tr}[M^* X]$ where $M = A^* B$.
Since this holds for all $X \in L(\mathcal{X})$, $M = I_X$.

<u>Corollary 2.23</u>: (Freedom in Kraus representation)
If $\{A_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$ and $\{B_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$ are such that

$$\sum_{a\in\Gamma} A_a X A_a^* = \sum_{a\in\Gamma} B_a X B_a^*,$$

for all $X \in L(\mathcal{X})$, then there exists a unitary $U \in U(\mathbb{C}^\Gamma)$ such that $B_a = \sum_{b\in\Gamma} U(a,b) A_b$, for all $a \in \Gamma$.

<u>Corollary 2.24</u>: (Freedom in Stinespring representation)
If $A, B \in L(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$ satisfy $\mathrm{Tr}_Z(AXA^*) = \mathrm{Tr}_Z(BXB^*)$, for all $X \in L(\mathcal{X})$ then there exists a unitary $U \in U(\mathcal{Z})$ such that $B = (I_Y \otimes U) A$.

Characterization of quantum channels

By combining Theorems 2.22 and 2.26 we get:

Corollary 2.27: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$. The following are equivalent:

1. Φ is a quantum channel.
2. $J(\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ and $\mathrm{Tr}_{\mathcal{Y}}[J(\Phi)] = I_{\mathcal{X}}$.
3. There exists $\{A_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$ such that:

$$\sum_{a \in \Gamma} A_a^* A_a = I_{\mathcal{X}} \quad \text{and} \quad \Phi(X) = \sum_{a \in \Gamma} A_a X A_a^*, \quad \forall X \in L(\mathcal{X}).$$

4. $|\Gamma| = \mathrm{rank}(J(\Phi))$ is enough in 3.
5. There exists an $A \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ such that

$$A^* A = I_{\mathcal{X}} \quad \text{and} \quad \Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A X A^*), \quad \forall X \in L(\mathcal{X}).$$

This means A is an isometry: $A \in U(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$.