Reduced states, purifications, fidelity

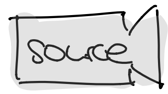
Last week: States, measurements, a glance at channels.

Recall: States on X are operators $\rho \in \mathrm{Pos}(\mathcal{X})$, $\mathrm{tr}[\rho]=1$.

Pure if $\rho = |\psi\rangle\langle\psi|$. Otherwise: mixed. How do they arise?

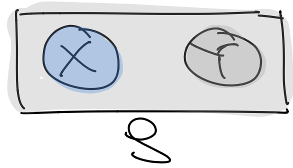
* classical states: $\rho = \sum_{x\in\Sigma} \underbrace{p(x)}_{\text{prob. dist.}} |x\rangle\langle x|$ if $\mathcal{X} = \mathbb{C}^\Sigma$

* ensembles: $\{p_i, \rho_i\}_{i\in I} \rightsquigarrow$ average state $\rho = \sum_i p_i \rho_i$



or ensembles in statistical physics

* subsystems! Given state $\rho \in D(\mathcal{X}\otimes\mathcal{Y})$, how to describe state on X ?



Clear if $\rho = \rho_X \otimes \rho_Y$. In general?

$\boxed{\text{To measure } \mu: \Omega \to \mathrm{Pos}(\mathcal{X}) \text{ on subsystem: Use } \mu\otimes I_Y: \Omega \to \mathrm{Pos}(\mathcal{X}\otimes\mathcal{Y}),\ \omega \mapsto \mu(\omega)\otimes I_Y \leftarrow \text{measurement on XY}}$

If measure in state ρ:

$$\Pr(\text{outcome } \omega) = \mathrm{tr}[\rho(\mu(\omega)\otimes I_Y)]$$

evaluate in $\otimes$ ONB

$$= \sum_{x,y} \underbrace{(\langle x|\otimes\langle y|)\rho(\mu(\omega)\otimes I_Y)(|x\rangle\otimes|y\rangle)}_{\langle x|(I_X\otimes\langle y|)\rho(I_X\otimes|y\rangle)\mu(\omega)|x\rangle}$$

$$= \mathrm{tr}\Big[\underbrace{\sum_y (I_X\otimes\langle y|)\rho(I_X\otimes|y\rangle)}_{\text{operator on } \mathcal{X}!}\ \mu(\omega)\Big]$$

$\boxed{\begin{array}{l}\text{The partial trace is defined as:} \\ \mathrm{tr}_Y: L(\mathcal{X}\otimes\mathcal{Y}) \to L(\mathcal{X}),\ \Pi \mapsto \sum_y (I_X\otimes\langle y|)\Pi(I_X\otimes|y\rangle) \quad \text{ANY ONB of } \mathcal{Y}\end{array}}$

Fact: $\rho \in D(\mathcal{X}\otimes\mathcal{Y}) \Rightarrow \mathrm{tr}_Y[\rho] \in D(\mathcal{X})$ "reduced state"

NOTATION: $\rho_X := \mathrm{tr}_Y[\rho]$, and even $\rho_{XY} := \rho$

$\Rightarrow$ For every meas. μ on X: $\boxed{\Pr(\text{outcome } \omega) = \operatorname{tr}[\rho_X \, \mu(\omega)]}$

Rules:

* $\operatorname{tr}_Y[A \otimes B] = A \cdot \operatorname{tr}[B]$ $\quad \forall A \in L(\mathcal{X}), B \in L(\mathcal{Y})$ "partial trace" ☺

* $\operatorname{tr}[M(A \otimes I_Y)] = \operatorname{tr}[\operatorname{tr}_Y[M] A]$ $\quad \forall M \in L(\mathcal{X} \otimes \mathcal{Y}), A \in L(\mathcal{X})$

↳ EX CLASS & HW $\qquad$ proved as above!

Example: $\mathcal{X} = \mathcal{Y} = \mathbb{C}^2$, $|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|0,0\rangle + |1,1\rangle)$ max. entangled state

$$\Rightarrow \rho = |\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\Big(\underbrace{|0,0\rangle\langle 0,0|}_{= |0\rangle\langle 0| \otimes |0\rangle\langle 0|} + \underbrace{|0,0\rangle\langle 1,1|}_{= |0\rangle\langle 1| \otimes |0\rangle\langle 1|} + |1,1\rangle\langle 0,0| + |1,1\rangle\langle 1,1|\Big)$$

$$\Rightarrow \rho_X = \operatorname{tr}_Y[\rho] = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

NB: Even if ρ pure, ρ_X can be mixed. IN FACT:

> Any $\sigma \in D(\mathcal{X})$ has a purification: $\exists \mathcal{Y}, |\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$ s.th.
> $\operatorname{tr}_Y[|\Psi\rangle\langle\Psi|] = \sigma$

↳ mixed states = subsystems of pure states = CHURCH OF LARGER H.SPACE!

* Existence: standard purification:

square root of Herm. op.

$$|\Psi\rangle = (\sqrt{\sigma} \otimes I) \sum_x \underbrace{|x\rangle \otimes |x\rangle}_{\text{any pair of ONB o.k.}} \in \mathcal{X} \otimes \mathcal{X}$$

In general, $\operatorname{rk}(\sigma) \leq \dim \mathcal{Y}$ necessary + sufficient:

$$\underbrace{\sigma = \sum_{i=1}^{\operatorname{rk}(\sigma)} p_i |\psi_i\rangle\langle\psi_i|}_{\text{Spectral decomp.}} \quad \rightsquigarrow \quad |\Psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes |i\rangle$$

any basis

* Uniqueness: $|\Psi\rangle, |\tilde{\Psi}\rangle \in \mathcal{X} \otimes \mathcal{Y}$ purifications

$\Rightarrow |\Psi\rangle = (I_X \otimes U) |\tilde{\Psi}\rangle$ for unitary U on $\mathcal{Y}$

Schmidt decomposition: Any $|\psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$ can be written as

$$|\psi\rangle = \sum_i \underbrace{s_i}_{>0} \underbrace{|e_i\rangle \otimes |f_i\rangle}_{\text{orthonormal}}$$

reduced states

$$\rho_X = \sum_i s_i^2 |e_i\rangle\langle e_i| \qquad \rho_Y = \sum_i s_i^2 |f_i\rangle\langle f_i|$$

eigenvalues are the same

* For $\rho = |\psi\rangle\langle\psi|$: ρ product $\Longleftrightarrow$ ρ_X pure $\Longleftrightarrow$ ρ_Y pure $\rightsquigarrow$ $\boxed{\text{HW}}$
* Existence? Follows from... $\rightarrow$ $\boxed{\text{EX CLASS}}$

Singular value decomposition: For all $A \in L(\mathcal{X}, \mathcal{Y})$, there exist $s_i > 0$ and orthonormal $|e_i\rangle \in \mathcal{X}$, $|f_i\rangle \in \mathcal{Y}$ s.th.

$$A = \sum_i s_i |f_i\rangle\langle e_i|$$

$A = A^*$: $|f_i\rangle = \pm |e_i\rangle$ since want $s_i > 0$

* How to find? $\{s_i^2\}$ = nonzero eigenvalues of A^*A (or AA^*)
  $|e_i\rangle :=$ corresponding orthonormal eigenvectors, $|f_i\rangle = \frac{A|e_i\rangle}{s_i}$ or the other way

Operator norms: For arbitrary $A \in L(\mathcal{X}, \mathcal{Y})$, define

* trace norm: $\|A\|_1 = \sum_i s_i = \operatorname{tr}\left[\sqrt{A^*A}\right]$ ← square root of Hermitian matrix
* Frobenius norm: $\|A\|_2 = \sqrt{\sum_i s_i^2} = \sqrt{\operatorname{tr}[A^*A]}$
* operator norm: $\|A\|_\infty = \max_i s_i = \max_{\|\phi\|=1} \|A|\phi\rangle\|$

USEFUL:

$$|\operatorname{tr}[AB]| \leq \begin{cases} \|A\|_2 \cdot \|B\|_2 & \text{Cauchy-Schwarz ieq} \\ \|A\|_1 \cdot \|B\|_\infty & \text{Hölder ieq} \end{cases}$$

$$\boxed{\|A\|_1 = \max_{\|B\|_\infty \leq 1} |\operatorname{tr}[AB]| = \max_{U \text{ unitary}} |\operatorname{tr}[AU]| \text{ for } A \in L(\mathcal{X})}$$

($\geq$) Hölder
($\leq$) $U|f_i\rangle := |e_i\rangle$

Fidelity between $\rho, \sigma \in D(\mathcal{H})$:

$$F(\rho,\sigma) := \|\underbrace{\sqrt{\rho}\sqrt{\sigma}}_{\text{not Herm!}}\|_1 = \operatorname{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$$

* If $\rho = |\Psi\rangle\langle\Psi|$ pure: $F(\rho,\sigma) = \sqrt{\langle\Psi|\sigma|\Psi\rangle} = |\langle\Psi|\Phi\rangle|$ ← if also $\sigma = |\Phi\rangle\langle\Phi|$ pure

  $\sqrt{\rho} = \rho$

* $F(\rho,\sigma) = F(\sigma,\rho)$

* $0 \le F(\rho,\sigma) \le 1$, $F(\rho,\sigma) = \begin{cases} 1 & \text{if } \rho = \sigma \\ 0 & \text{if } \rho\sigma = 0 \end{cases}$ Similarity measure!

Thm (Uhlmann): If $\rho, \sigma \in D(\mathcal{H})$ have purifications on $\mathcal{H}\otimes\mathcal{Y}$

$$F(\rho,\sigma) = \max\{|\langle\Psi|\Phi\rangle| : \mathcal{H}\otimes\mathcal{Y} \ni |\Psi\rangle, |\Phi\rangle \text{ purif. of } \rho, \sigma\}$$

NB: $\text{RHS} = \max\{|\langle\Psi_0| I_X \otimes U |\Phi_0\rangle| : U \text{ unitary on } \mathcal{Y}\}$

fixed purifications on $\mathcal{H}\otimes\mathcal{Y}$

Pf: Assume $\mathcal{H} = \mathcal{Y}$. Arbitrary purifications:

$$|\Psi\rangle = (\sqrt{\rho}\otimes U)\sum_x |x,x\rangle \quad \& \quad |\Phi\rangle = (\sqrt{\sigma}\otimes\tilde{U})\sum_x |x,x\rangle$$

$$\Rightarrow |\langle\Psi|\Phi\rangle| = \sum_{x,y}(\langle x|\otimes\langle x|)(\sqrt{\rho}\sqrt{\sigma}\otimes U^*\tilde{U})(|y\rangle\otimes|y\rangle)$$

$$= \operatorname{tr}[\sqrt{\rho}\sqrt{\sigma}\,\underbrace{(U^*\tilde{U})^T}_{\text{arbitrary unitary}}] \Rightarrow \max_{U,V}|\langle\Psi|\Phi\rangle| \overset{\text{above}}{=} \|\sqrt{\rho}\sqrt{\sigma}\|_1.$$

What if $\mathcal{H} \neq \mathcal{Y}$? Use

$$|\Psi\rangle = (\sqrt{\rho}\,V\otimes U)\sum_{x=1}^{r}|x,x\rangle \quad \& \quad |\Phi\rangle = (\sqrt{\sigma}\,W\otimes\tilde{U})\sum_{x=1}^{r}|x,x\rangle$$

where $r = \max\{\operatorname{rk}(\rho), \operatorname{rk}(\sigma)\}$, V & W partial isos onto $\operatorname{supp}(\rho), \operatorname{supp}(\sigma)$

$\le \dim\mathcal{H}, \dim\mathcal{Y}$

transpose of submatrix of $U^*\tilde{U}$

$$\Rightarrow |\langle\Psi|\Phi\rangle| = \operatorname{tr}[V^\dagger\sqrt{\rho}\sqrt{\sigma}\,W\,B] \Rightarrow \max = \|V^*\sqrt{\rho}\sqrt{\sigma}\,W\|_1$$

But: $\|\sqrt{\rho}\sqrt{\sigma}\|_1 = \|VV^*\sqrt{\rho}\sqrt{\sigma}WW^*\|_1 \leq \|V^*\sqrt{\rho}\sqrt{\sigma}W\|_1 \leq \|\sqrt{\rho}\sqrt{\sigma}\|_1$, hence "=" $\square$

Properties of the fidelity:

* Monotonicity: $F(\rho,\sigma) \leq F(\rho_X,\sigma_X) \quad \forall \rho,\sigma \in D(\mathcal{X}\otimes\mathcal{Y})$
* Joint concavity: $F(\sum_i p_i\rho_i, \sum_i p_i\sigma_i) \geq \sum_i p_i F(\rho_i,\sigma_i)$

$\boxed{HW}$

How about the trace distance from last week?

* Helstrom: $\frac{1}{2}\|\rho-\sigma\|_1 = \max_{0\leq Q\leq I} tr[Q(\rho-\sigma)] = 2\,p_{success}^{opt} - 1$

"operational" proof

* Monotonicity: $\|\rho-\sigma\|_1 \geq \|\rho_X-\sigma_X\|_1 \quad \forall \rho,\sigma\in D(\mathcal{X}\otimes\mathcal{Y})$

(Direct proof: $0\leq Q\leq I_X \Rightarrow 0\leq Q\otimes I_Y \leq I_{XY}$ $\square$)

Fidelity vs. trace distance?

* If $\rho=|\psi\rangle\langle\psi|$, $\sigma=|\phi\rangle\langle\phi|$ pure: $\frac{1}{2}\|\rho-\sigma\|_1 = \sqrt{1-|\langle\phi|\psi\rangle|^2}$ $\boxed{HW!}$
* In general: Fuchs-van de Graaf ineqs:

$$1-\frac{1}{2}\|\rho-\sigma\|_1 \leq F(\rho,\sigma) \leq \sqrt{1-\frac{1}{4}\|\rho-\sigma\|_1^2}$$

BUT WE DID NOT DISCUSS THIS IN THE LECTURE!