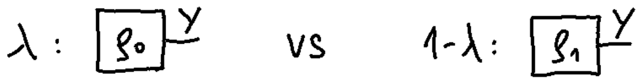
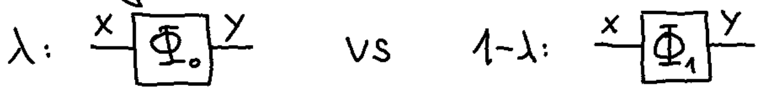
Lecture 16: Completely bounded trace norm (§ 3.3)

Recall from Holevo - Helstrom Theorem that the optimal measurement for discriminating between



succeeds with probability $\frac{1}{2} + \frac{1}{2} \| \lambda \rho_0 - (1-\lambda) \rho_1 \|_1$.

How about the more general task of discriminating channels instead of states? I.e.,



One simple strategy is to reduce this to state discrimination as follows: pick $\rho \in D(\mathcal{X})$ that maximizes the quantity

$$\| \lambda \Phi_0(\sigma) - (1-\lambda) \Phi_1(\sigma) \|_1 .$$

However, this is not the most general strategy since one could also use an additional auxiliary register W: let $\sigma \in D(\mathcal{X} \otimes \mathcal{W})$ and consider

This strategy can indeed be much better than the original one without the auxiliary register.

Example 3.36: Let $|\Sigma| = n \geq 2$, $\mathcal{X} = \mathbb{C}^\Sigma$ and $\Phi_0, \Phi_1 \in C(\mathcal{X})$ be two quantum channels defined as

$$\Phi_0(X) = \frac{1}{n+1}\left((\mathrm{Tr}\, X)\, I + X^T\right),$$

$$\Phi_1(X) = \frac{1}{n-1}\left((\mathrm{Tr}\, X)\, I - X^T\right).$$

They are clearly trace-preserving. They are completely positive since

$$J(\Phi_0) = \frac{I \otimes I + W}{n+1} \quad \text{and} \quad J(\Phi_1) = \frac{I \otimes I - W}{n-1},$$

where $W(|u\rangle \otimes |v\rangle) = |v\rangle \otimes |u\rangle$, for all $|u\rangle, |v\rangle \in S(\mathcal{X})$, i.e., W is the swap operator. Let

$$\lambda = \frac{n+1}{2n} \quad \text{and so} \quad 1-\lambda = \frac{n-1}{2n}.$$

Note that

$$\lambda\, \Phi_0(X) - (1-\lambda)\, \Phi_1(X)$$
$$= \frac{1}{2n}\left((\mathrm{Tr}\, X) I + X^T\right) - \frac{1}{2n}\left((\mathrm{Tr}\, X) I - X^T\right) = \frac{1}{n} X^T,$$

so $\|\lambda\, \Phi_0(X) - (1-\lambda)\, \Phi_1(X)\|_1 = \frac{1}{n}$ for any $X \in D(\mathcal{X})$. Hence the success probability is $\frac{1}{2} + \frac{1}{2n}$ which is $\sim \frac{1}{2}$. Intuitively, $\Phi_0(X)$ and $\Phi_1(X)$ are both close to the maximally mixed state I/n and hence hard to distinguish.

On the other hand, consider an auxiliary register and apply the channel to a half of a maximally entangled state

$$\tau = \frac{1}{n} \sum_{a,b \in \Sigma} |a\rangle\langle b| \otimes |a\rangle\langle b|.$$

Let $T \in T(\mathcal{X})$ be the transpose map: $T(X) = X^T$. Then

$$(T \otimes \mathcal{I})(\tau) = \sum_{a,b\in\Sigma} (|a\rangle\langle b|)^T \otimes |a\rangle\langle b|$$
$$= \sum_{a,b\in\Sigma} |b\rangle\langle a| \otimes |a\rangle\langle b|$$
$$= \sum_{a,b\in\Sigma} (|b\rangle \otimes |a\rangle) \cdot (\langle a| \otimes \langle b|) = W,$$

the swap operation. Hence,

$$(\Phi_0 \otimes \mathcal{I})(\tau) = \frac{I\otimes I + W}{n(n+1)} = \frac{1}{n} J(\Phi_0),$$

$$(\Phi_1 \otimes \mathcal{I})(\tau) = \frac{I\otimes I - W}{n(n-1)} = \frac{1}{n} J(\Phi_1).$$

These density matrices happen to be orthogonal:

$$\langle I\otimes I + W, I\otimes I - W\rangle = \mathrm{Tr}(I\otimes I + W - W - W^2) = 0$$

since $W^2 = I \otimes I$. Hence, for any $\lambda \in [0,1]$,

$$\| \lambda (\Phi_0 \otimes \mathcal{I})(\tau) - (1-\lambda)(\Phi_1 \otimes \mathcal{I})(\tau) \|_1 = 1$$

and the two channels can be discriminated perfectly when making use of an auxiliary register — the success probability is

$$\frac{1}{2} + \frac{1}{2} \left\| \left((\lambda \Phi_0 - (1-\lambda)\Phi_1) \otimes \mathcal{I} \right)(\tau) \right\|_1 = 1.$$

This example motivates the definition of a new norm that incorporates the possibility of using an auxiliary register in channel discrimination.

The completely bounded trace norm (§3.2.2)

We would like understand what norm is relevant to channel discrimination. Let us first define the following induced norm:

Def 3.37: The induced trace norm of $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is

$$\|\Phi\|_1 = \max\{\|\Phi(X)\|_1 : X \in L(\mathcal{X}), \|X\|_1 \leq 1\}.$$

Prop. 3.38: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$. Then

$$\|\Phi\|_1 = \max\{\|\Phi(|u\rangle\langle v|)\|_1 : |u\rangle, |v\rangle \in S(\mathcal{X})\}.$$

Proof: Let $X \in L(\mathcal{X})$ be a maximizer in the definition of $\|\Phi\|_1$ and let $X = \sum_i s_i |u_i\rangle\langle v_i|$ be its spectral decomposition, where $s_i \geq 0$, $\sum_i s_i \leq 1$ and $|u_i\rangle, |v_i\rangle \in S(\mathcal{X})$. By linearity and triangle inequality,

$$\|\Phi\|_1 = \|\sum_i s_i \Phi(|u_i\rangle\langle v_i|)\|_1 \leq \sum_i s_i \|\Phi(|u_i\rangle\langle v_i|)\|_1$$
$$\leq \max_i \|\Phi(|u_i\rangle\langle v_i|)\|_1$$

which establishes "$\leq$". The other inequality "$\geq$" follows trivially since $\||u\rangle\langle v|\|_1 = 1$ for any $|u\rangle, |v\rangle \in S(\mathcal{X})$. $\square$

For the purpose of channel discrimination, the relevant norm involves an auxiliary register:

Def 3.43: The completely bounded trace norm of $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is

$$|||\Phi|||_1 = \|\Phi \otimes \mathcal{I}_{\mathcal{X}}\|_1$$

where the second register has the same dimension as the input space $\mathcal{X}$.

<u>Note</u>: it is often denoted by $\|\cdot\|_\diamond$ and called the "diamond norm".

One immediate question is why does the second register has the same dimension as the input?

<u>Lemma 3.45</u>: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$. For every choice of a complex Euclidean space $\mathcal{Z}$ and unit vectors $|x\rangle, |y\rangle \in S(\mathcal{X} \otimes \mathcal{Z})$, there exist unit vectors $|u\rangle, |v\rangle \in S(\mathcal{X} \otimes \mathcal{X})$ such that

$$\|(\Phi \otimes I_Z)(|x\rangle\langle y|)\|_1 = \|(\Phi \otimes I_X)(|u\rangle\langle v|)\|_1,$$

$$\|(\Phi \otimes I_Z)(|x\rangle\langle x|)\|_1 = \|(\Phi \otimes I_X)(|u\rangle\langle u|)\|_1.$$

<u>Proof</u>: If $\dim(\mathcal{Z}) \leq \dim(\mathcal{X})$, we can take $|u\rangle = (I_X \otimes U)|x\rangle$ and $|v\rangle = (I_X \otimes U)|y\rangle$, for any isometry $U \in U(\mathcal{Z}, \mathcal{X})$, since $\|\cdot\|_1$ is invariant under isometries.

If $\dim(\mathcal{Z}) > \dim(\mathcal{X})$, consider the Schmidt decompositions

$$|x\rangle = \sum_{k=1}^{n} \sqrt{p_k}\, |x_k\rangle \otimes |z_k\rangle \quad \text{and} \quad |y\rangle = \sum_{k=1}^{n} \sqrt{q_k}\, |y_k\rangle \otimes |w_k\rangle$$

where $n = \dim(\mathcal{X})$. Choose $|u\rangle$ and $|v\rangle$ as follows:

$$|u\rangle = \sum_{k=1}^{n} \sqrt{p_k}\, |x_k\rangle \otimes |x_k\rangle \quad \text{and} \quad |v\rangle = \sum_{k=1}^{n} \sqrt{q_k}\, |y_k\rangle \otimes |y_k\rangle.$$

Define isometries $U, V \in U(\mathcal{X}, \mathcal{Z})$ as follows:

$$U = \sum_{k=1}^{n} |z_k\rangle\langle x_k| \quad \text{and} \quad V = \sum_{k=1}^{n} |y_k\rangle\langle w_k|.$$

Then $|x\rangle = (I_X \otimes U)|u\rangle$ and $|y\rangle = (I_X \otimes V)|v\rangle$ and

$$\|(\Phi \otimes I_Z)(|x\rangle\langle y|)\|_1 = \|(\Phi \otimes I_Z)\big((I_X \otimes U)|u\rangle\langle v|(I_X \otimes V^*)\big)\|_1$$

$$= \|(I_Y \otimes U) \cdot (\Phi \otimes I_X)(|u\rangle\langle v|) \cdot (I_Y \otimes V^*)\|_1$$

$$= \|(\Phi \otimes I_X)(|u\rangle\langle v|)\|_1$$

and similarly for $|x\rangle\langle x|$. $\square$

The following result explains the choice of the auxiliary system size in the definition of $|||\cdot|||_1$:

<u>Theorem 3.46</u>: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$. Then

$$\|\Phi \otimes I_{\mathcal{Z}}\|_1 \leq |||\Phi|||_1$$

for any complex Euclidean space $\mathcal{Z}$, with equality when $\dim(\mathcal{Z}) \geq \dim(\mathcal{X})$.

<u>Proof</u>: By Prop. 3.38, there exist unit vectors $|x\rangle, |y\rangle \in S(\mathcal{X} \otimes \mathcal{Z})$ such that

$$\|\Phi \otimes I_{\mathcal{Z}}\|_1 = \|(\Phi \otimes I_{\mathcal{Z}})(|x\rangle\langle y|)\|_1 .$$

By Lemma 3.45, there exist unit vectors $|u\rangle, |v\rangle \in S(\mathcal{X} \otimes \mathcal{X})$ such that

$$\|\Phi \otimes I_{\mathcal{Z}}\|_1 = \|(\Phi \otimes I_{\mathcal{X}})(|u\rangle\langle v|)\|_1 \leq |||\Phi|||_1 ,$$

where the inequality follows from Prop. 3.38.

For proving equality, assume $\dim(\mathcal{Z}) \geq \dim(\mathcal{X})$. Then, for any $V \in U(\mathcal{X}, \mathcal{Z})$ and $X \in L(\mathcal{X} \otimes \mathcal{X})$ with $\|X\|_1 \leq 1$,

$$\begin{aligned}
\|(\Phi \otimes I_{\mathcal{X}})(X)\|_1 &= \|(I_{\mathcal{Y}} \otimes V) \cdot (\Phi \otimes I_{\mathcal{X}})(X) \cdot (I_{\mathcal{Y}} \otimes V)^*\|_1 \\
&= \|(\Phi \otimes I_{\mathcal{Z}})\big((I_{\mathcal{X}} \otimes V) \cdot X \cdot (I_{\mathcal{X}} \otimes V)^*\big)\|_1 \\
&\leq \|\Phi \otimes I_{\mathcal{Z}}\|_1 \cdot \|(I_{\mathcal{X}} \otimes V) \cdot X \cdot (I_{\mathcal{X}} \otimes V)^*\|_1 \\
&= \|\Phi \otimes I_{\mathcal{Z}}\|_1 \cdot \|X\|_1 \\
&\leq \|\Phi \otimes I_{\mathcal{Z}}\|_1 ,
\end{aligned}$$

where we used the invariance of $\|\cdot\|_1$ under isometries and the simple property $\|\Phi(A)\|_1 \leq \|\Phi\|_1 \cdot \|A\|_1$, for any $A \in L(\mathcal{X})$. We conclude that $|||\Phi|||_1 \leq \|\Phi \otimes I_{\mathcal{Z}}\|_1$ when $\dim(\mathcal{Z}) \geq \dim(\mathcal{X})$. $\square$

Corollary 3.47: For any $\Phi \in T(\mathcal{X},\mathcal{Y})$ and $\mathcal{Z}$, $|||\Phi \otimes I_{\mathcal{Z}}|||_1 = |||\Phi|||_1$.

More generally, one can show (see Theorem 3.49) that

$$|||\Phi_0 \otimes \Phi_1|||_1 = |||\Phi_0|||_1 \cdot |||\Phi_1|||_1,$$

for any $\Phi_0, \Phi_1 \in T(\mathcal{X},\mathcal{Y})$, showing that $|||\cdot|||_1$ behaves in a nicer way than the induced trace norm $\|\cdot\|_1$.

Recall from Prop. 3.38 that, for any $\Phi \in T(\mathcal{X},\mathcal{Y})$,

$$|||\Phi|||_1 = \|(\Phi \otimes I_{\mathcal{X}})(|u\rangle\langle v|)\|_1$$

for some choice of $|u\rangle, |v\rangle \in S(\mathcal{X}\otimes\mathcal{X})$. For Hermitian-preserving maps one can take $|u\rangle = |v\rangle$. To show this, we need the following Lemma:

Lemma 3.50: Let $\Phi \in T(\mathcal{X},\mathcal{Y})$ be Hermitian-preserving. Let $\mathcal{Z}$ be a complex Euclidean space with $\dim(\mathcal{Z}) \geq 2$. Then there exists $|u\rangle \in S(\mathcal{X}\otimes\mathcal{Z})$ such that

$$\|(\Phi \otimes I_{\mathcal{Z}})(|u\rangle\langle u|)\|_1 \geq \|\Phi\|_1.$$

Proof: Let $X \in L(\mathcal{X})$ be such that $\|X\|_1 = 1$ and $\|\Phi(X)\|_1 = \|\Phi\|_1$. Let $|z_0\rangle, |z_1\rangle \in S(\mathcal{Z})$ be any two orthogonal states and define $H \in \mathrm{Herm}(\mathcal{X}\otimes\mathcal{Z})$ as follows:

$$H = \frac{1}{2} X \otimes |z_0\rangle\langle z_1| + \frac{1}{2} X^* \otimes |z_1\rangle\langle z_0|.$$

Note that $\|H\|_1 = \mathrm{Tr}\sqrt{H^*H} = \frac{1}{2}\mathrm{Tr}\sqrt{XX^* \otimes |0\rangle\langle 0| + X^*X \otimes |1\rangle\langle 1|} =$
$= \frac{1}{2}\left(\mathrm{Tr}\sqrt{XX^*} + \mathrm{Tr}\sqrt{X^*X}\right) = \|X\|_1$. Moreover,

$$(\Phi \otimes I_{\mathcal{Z}})(H) = \frac{1}{2}\,\Phi(X) \otimes |z_0\rangle\langle z_1| + \frac{1}{2}\,\underbrace{\Phi(X^*)}_{=\,\Phi(X)^*} \otimes |z_1\rangle\langle z_0|$$

by Theorem 2.25 and Φ Hermitian-preserving. Note that $\|(\Phi\otimes I_{\mathcal{Z}})(H)\|_1 = \|\Phi(X)\|_1 = \|\Phi\|_1$. If $H = \sum_k \lambda_k |u_k\rangle\langle u_k|$ is the spectral decomposition of H then

$$\|\Phi\|_1 = \|(\Phi\otimes I_{\mathcal{Z}})(H)\|_1 \leq \|(\Phi\otimes I_{\mathcal{Z}})(|u_k\rangle\langle u_k|)\|_1$$

for some k, by triangle inequality. $\square$

Theorem 3.51: Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$ be Hermitian-preserving. Then

$$|||\Phi|||_1 = \max_{|u\rangle \in S(\mathcal{X}\otimes\mathcal{X})} \|(\Phi \otimes \mathcal{I}_X)(|u\rangle\langle u|)\|_1 .$$

Proof: For any unit vector $|u\rangle \in S(\mathcal{X}\otimes\mathcal{X})$,

$$\|(\Phi\otimes\mathcal{I}_X)(|u\rangle\langle u|)\|_1 \leq \|\Phi\otimes\mathcal{I}_X\|_1 = |||\Phi|||_1 ,$$

hence it suffices to prove that there exists $|u\rangle \in S(\mathcal{X}\otimes\mathcal{X})$ such that

$$\|(\Phi\otimes\mathcal{I}_X)(|u\rangle\langle u|)\|_1 \geq \|\Phi\otimes\mathcal{I}_X\|_1 = |||\Phi|||_1 .$$

Let $\mathcal{Z} = \mathbb{C}^2$. By Lemma 3.50, there exists $|x\rangle \in S(\mathcal{X}\otimes\mathcal{X}\otimes\mathcal{Z})$ such that

$$\|(\Phi\otimes\mathcal{I}_X\otimes\mathcal{I}_Z)(|x\rangle\langle x|)\|_1 \geq \|\Phi\otimes\mathcal{I}_X\|_1 ,$$

and by Lemma 3.45 there exists $|u\rangle \in S(\mathcal{X}\otimes\mathcal{X})$ such that

$$\begin{aligned}\|(\Phi\otimes\mathcal{I}_X)(|u\rangle\langle u|)\|_1 &= \|(\Phi\otimes\mathcal{I}_X\otimes\mathcal{I}_Z)(|x\rangle\langle x|)\|_1 \\ &\geq \|\Phi\otimes\mathcal{I}_X\|_1 = |||\Phi|||_1\end{aligned}$$

which completes the proof. $\square$

Channel analogue of the Holevo-Helstrom theorem

The following result is a channel analogue of Theorem 3.4 and it gives an operational meaning to the norm $|||\cdot|||_1$:

Thm 3.52: Let $\Phi_0, \Phi_1 \in C(\mathcal{X}, \mathcal{Y})$ and let $\lambda \in [0,1]$. For any $\mathcal{Z}$, measurement $\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{Y}\otimes\mathcal{Z})$, and state $\sigma \in D(\mathcal{X}\otimes\mathcal{Z})$,

$$\begin{aligned}&\lambda \langle \mu(0), (\Phi_0\otimes\mathcal{I}_X)(\sigma)\rangle + (1-\lambda)\langle \mu(1), (\Phi_1\otimes\mathcal{I}_X)(\sigma)\rangle \\ &\leq \frac{1}{2} + \frac{1}{2} |||\lambda\Phi_0 - (1-\lambda)\Phi_1|||_1 .\end{aligned}$$

If $\dim(\mathcal{Z}) \geq \dim(\mathcal{X})$, equality is achieved for some projective measurement μ and pure state σ.

Proof: By Holevo-Helstrom theorem,

$$\cdots \leqslant \frac{1}{2} + \frac{1}{2} \| \lambda (\Phi_0 \otimes \mathbb{I}_Z)(\sigma) - (1-\lambda)(\Phi_1 \otimes \mathbb{I}_Z)(\sigma) \|_1$$

$$\leqslant \frac{1}{2} + \frac{1}{2} \| (\lambda \Phi_0 - (1-\lambda) \Phi_1) \otimes \mathbb{I}_Z \|_1$$

$$\leqslant \frac{1}{2} + \frac{1}{2} ||| \lambda \Phi_0 - (1-\lambda) \Phi_1 |||_1$$

where we used Theorem 3.46 in the last line.
To derive the condition for equality, note that $\lambda \Phi_0 + (1-\lambda) \Phi_1$ is Hermitian preserving since Φ are and λ is real. By Theorem 3.51, there exists $|u\rangle \in S(\mathcal{X} \otimes \mathcal{X})$ such that

$$||| \lambda \Phi_0 - (1-\lambda) \Phi_1 |||_1 = \| \lambda (\Phi_0 \otimes \mathbb{I}_X)(|u\rangle\langle u|) - (1-\lambda) (\Phi_1 \otimes \mathbb{I}_X)(|u\rangle\langle u|) \|_1 .$$

When $\dim(\mathcal{Z}) \geqslant \dim(\mathcal{X})$, we can take

$$\sigma = (\mathbb{I}_X \otimes V) |u\rangle\langle u| (\mathbb{I}_X \otimes V)^*$$

for any isometry $V \in \mathrm{U}(\mathcal{X}, \mathcal{Z})$, and we still have

$$||| \lambda \Phi_0 - (1-\lambda) \Phi_1 |||_1 = \| \lambda (\Phi_0 \otimes \mathbb{I}_Z)(\sigma) - (1-\lambda)(\Phi_1 \otimes \mathbb{I}_Z)(\sigma) \|_1 .$$

By Holevo-Helstrom theorem, there exists a projective measurement $\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Z})$ such that

$$\lambda \langle \mu(0), (\Phi_0 \otimes \mathbb{I}_Z)(\sigma) \rangle + (1-\lambda) \langle \mu(1), (\Phi_1 \otimes \mathbb{I}_Z)(\sigma) \rangle$$

$$= \frac{1}{2} + \frac{1}{2} \| \lambda (\Phi_0 \otimes \mathbb{I}_Z)(\sigma) - (1-\lambda)(\Phi_1 \otimes \mathbb{I}_Z)(\sigma) \|_1$$

$$= \frac{1}{2} + \frac{1}{2} ||| \lambda \Phi_0 + (1-\lambda) \Phi_1 |||_1$$

which completes the proof. $\square$

SDP for the completely bounded trace norm

While the trace norm $\|\cdot\|_1$ can be computed directly from the singular value decomposition of the matrix, computing the completely bounded trace norm $|||\cdot|||_1$ is not as straightforward. Luckily it can still be described by a semidefinite program. We will not derive it here (see §3.3.4).

Let $\Phi \in T(\mathcal{X}, \mathcal{Y})$ be an arbitrary map and let

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A_0 X A_1^*)$$

be its Stinespring representation, for some isometries $A_0, A_1 \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$. Define completely positive maps $\Psi_0, \Psi_1 \in CP(\mathcal{X}, \mathcal{Z})$ as follows:

$$\Psi_0(X) = \mathrm{Tr}_{\mathcal{Y}}(A_0 X A_0^*),$$

$$\Psi_1(X) = \mathrm{Tr}_{\mathcal{Y}}(A_1 X A_1^*).$$

Then $|||\Phi|||_1$ is given by

Primal

maximize:

$$\frac{1}{2}\mathrm{Tr}\, Y + \frac{1}{2}\mathrm{Tr}\, Y^*$$

subject to:

$$\begin{pmatrix} \Psi_0(\rho_0) & Y \\ Y^* & \Psi_1(\rho_1) \end{pmatrix} \geq 0$$

$$\rho_0, \rho_1 \in D(\mathcal{X}), \quad Y \in L(\mathcal{Z})$$

Dual

minimize:

$$\frac{1}{2}\|\Psi_0^*(Z_0)\| + \frac{1}{2}\|\Psi_1^*(Z_1)\|$$

subject to:

$$\begin{pmatrix} Z_0 & -I_{\mathcal{Z}} \\ -I_{\mathcal{Z}} & Z_1 \end{pmatrix} \geq 0$$

$$Z_0, Z_1 \in Pd(\mathcal{X})$$