Lecture 12: Distillable entanglement and entanglement cost

Nielsen's theorem provides a way to compare states and thus to compare the amount of entanglement in those states. However, not every pair of states is comparable, and also we don't get a single number from these comparisons that would measure the amount of entanglement in the state. We would like to just have a single number that is easy to compute and which also has some intuitive interpretation.

A convenient way to measure the amount of entanglement of a given bi-partite state is to choose a „golden standard" state and ask how many of these states can be obtained from the given state by LOCC. The canonical maximally entangled two-qubit state $|\tau\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a natural choice of such „golden standard".

According to Nielsen's theorem, any pure two-qubit state can be obtained exactly from $|\tau\rangle$ by LOCC. But is the conversion ratio one to one, or could we potentially obtain more of the target states, especially if they are not very entangled?

One can also ask the opposite question – given many copies of some state, how many maximally entangled two-qubit states can we distill? Note that generally such operations do not need to be exact.

One has to take care of having terms in the right order in tensor products:

$$(\mathcal{X}\otimes\mathcal{Y})^{\otimes n} = (\mathcal{X}\otimes\mathcal{Y})\otimes(\mathcal{X}\otimes\mathcal{Y})\otimes\cdots\otimes(\mathcal{X}\otimes\mathcal{Y})$$

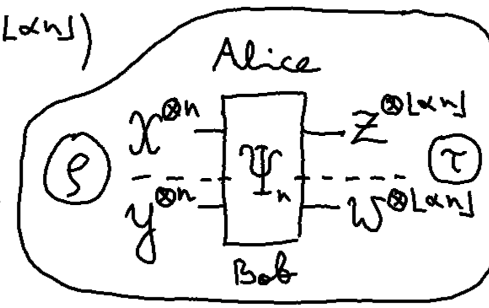
$$\Big\downarrow U_n$$

$$\mathcal{X}^{\otimes n}\otimes\mathcal{Y}^{\otimes n} = \underbrace{(\mathcal{X}\otimes\mathcal{X}\otimes\cdots\otimes\mathcal{X})}_{\text{Alice}}\otimes\underbrace{(\mathcal{Y}\otimes\mathcal{Y}\otimes\cdots\otimes\mathcal{Y})}_{\text{Bob}}$$

But we will not explicitly do this.

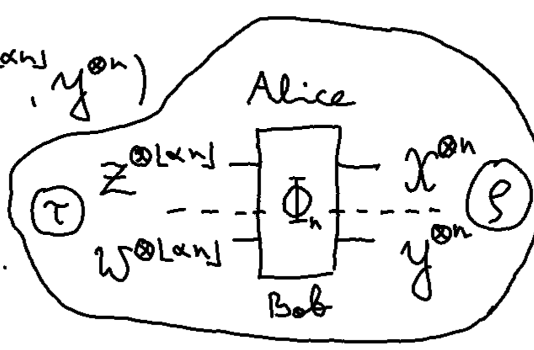
<u>Def 6.35</u>: The <u>distillable entanglement</u> $E_D(\rho)$ of $\rho\in D(\mathcal{X}\otimes\mathcal{Y})$ is the supremum over all $\alpha\geq 0$ for which there exists a sequence of LOCC channels $\Psi_n\in LOCC(\mathcal{X}^{\otimes n}, \mathcal{Z}^{\otimes\lfloor\alpha n\rfloor} : \mathcal{Y}^{\otimes n}, \mathcal{W}^{\otimes\lfloor\alpha n\rfloor})$ such that

$$\lim_{n\to\infty} F(\Psi_n(\rho^{\otimes n}), \tau^{\otimes\lfloor\alpha n\rfloor}) = 1.$$



<u>Def 6.36:</u> The <u>entanglement cost</u> $E_C(\rho)$ of $\rho\in D(\mathcal{X}:\mathcal{Y})$ is the infimum of all $\alpha\geq 0$ for which there exists a sequence of LOCC channels $\Phi_n\in LOCC(\mathcal{Z}^{\otimes\lfloor\alpha n\rfloor}, \mathcal{X}^{\otimes n} : \mathcal{W}^{\otimes\lfloor\alpha n\rfloor}, \mathcal{Y}^{\otimes n})$ such that

$$\lim_{n\to\infty} F(\Phi_n(\tau^{\otimes\lfloor\alpha n\rfloor}), \rho^{\otimes n}) = 1.$$



It is intuitive that one should not be able to repeatedly distill and create some state by LOCC and obtain more and more entanglement. I.e., you can get out at most what you put in.

Proving this formally however is not totally trivial.

<u>Prop. 6.37.</u> For any $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$, $E_D(\rho) \leq E_C(\rho)$.

<u>Proof</u>: Let's try to approximately implement by LOCC the map $\Psi_n \circ \Phi_n$ such that

$$\tau^{\otimes m} \xrightarrow{\Phi_n} \rho^{\otimes n} \xrightarrow{\Psi_n} \tau^{\otimes k}$$

$\boxed{|\tau\rangle^{\otimes m} = \frac{1}{\sqrt{2}} \operatorname{vec}(I^{\otimes m}) \quad \operatorname{rank}(I^{\otimes m}) = 2^m}$

for some integers $m, n, k \geq 0$, where $\tau = |\tau\rangle\langle\tau|$.
Since $\tau^{\otimes m}$ has entanglement rank 2^m and $\Psi_n \circ \Phi_n$ is LOCC (and thus separable), the output $(\Psi_n \circ \Phi_n)(\tau^{\otimes m})$ has ent. rank at most 2^m by Thm 6.23. So from home work,

$$F\left((\Psi_n \circ \Phi_n)(\tau^{\otimes m}), \tau^{\otimes k}\right)^2 \leq \overbrace{2^m}^{\text{ent. rank}} / \overbrace{2^k}^{\text{dimension}} = 2^{m-k}.$$

By definition of entanglement cost and distillable entanglement, for all $\varepsilon > 0$ there exists n s.t.

$$F(\Phi_n(\tau^{\otimes m}), \rho^{\otimes n}) > 1 - \varepsilon,$$
$$F(\Psi_n(\rho^{\otimes n}), \tau^{\otimes k}) > 1 - \varepsilon.$$

By Exercise 2, $F((\Psi_n \circ \Phi_n)(\tau^{\otimes m}), \tau^{\otimes k}) > 1 - 4\varepsilon$. Taking $\varepsilon < 1/16$, $F((\Psi_n \circ \Phi_n)(\tau^{\otimes m}), \tau^{\otimes k})^2 > \frac{1}{2}$ and thus $m \geq k$. Since $m = \lfloor \alpha n \rfloor$ and $k = \lfloor \beta n \rfloor$, $\alpha \geq \beta$ and thus $E_D(\rho) \leq E_C(\rho)$. $\square$

For *pure* states, distillable entanglement and entanglement cost agree and are equal to entanglement entropy.

Theorem 6.38: For any *pure* state $\rho_{XY} = |u\rangle\langle u|_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$,

$$E_D(\rho_{XY}) = H(\rho_X) = H(\rho_Y) = E_C(\rho_{XY}).$$

Proof: We already know that $E_D(\rho) \leq E_C(\rho)$, and we know from Schmidt decomposition that

$$H(\rho_X) = H(\rho_Y) = H(p),$$

where $\sqrt{p}$ are the Schmidt coefficients of $\rho = |u\rangle\langle u|$:

$$|u\rangle_{XY} = \sum_{a \in \Sigma} \sqrt{p(a)}\, |x_a\rangle_X \otimes |y_a\rangle_Y.$$

Let us show that $E_C(\rho) \leq H(p)$. Recall that for any $n \geq 1$ and $\varepsilon > 0$, the set of ε-typical strings with respect to p contains those $a_1 \dots a_n \in \Sigma^n$ for which

$$2^{-n(H(p)+\varepsilon)} < p(a_1) \dots p(a_n) < 2^{-n(H(p)-\varepsilon)}.$$

Define a vector

$$|v_{n,\varepsilon}\rangle = \sum_{a_1 \dots a_n \in T_{n,\varepsilon}} \sqrt{p(a_1) \dots p(a_n)}\, \big(|x_1\rangle \otimes \dots \otimes |x_n\rangle\big) \otimes \big(|y_1\rangle \otimes \dots \otimes |y_n\rangle\big).$$

Note that $\| |v_{n,\varepsilon}\rangle \| \leq \| |u\rangle^{\otimes n} \| = 1$, so let $|w_{n,\varepsilon}\rangle = |v_{n,\varepsilon}\rangle / \| |v_{n,\varepsilon}\rangle \|$ be the normalized version of it. The eigenvalues of the reduced state of $|w_{n,\varepsilon}\rangle$ satisfy, for any $j = 1, \dots, |T_{n,\varepsilon}|$,

$$\frac{2^{-n(H(p)+\varepsilon)}}{\| |v_{n,\varepsilon}\rangle \|^2} < \lambda_j\left(\mathrm{Tr}_{Y_1 \dots Y_n}\left[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right]\right) < \frac{2^{-n(H(p)-\varepsilon)}}{\| |v_{n,\varepsilon}\rangle \|^2},$$

while the remaining eigenvalues are zero.

Let us now bound the entanglement cost of $|u\rangle_{XY}$. Take any $\alpha > H(p)$ and let $\varepsilon > 0$ be sufficiently small so that $\alpha > H(p) + 2\varepsilon$. Let $n > 1/\varepsilon$, so that $n\varepsilon > 1$. Then

$$m = \lfloor \alpha n \rfloor \geq \lfloor n(H(p)+\varepsilon) + n\varepsilon \rfloor > n(H(p)+\varepsilon).$$

We want to create copies of $|u\rangle$ from copies of $|\tau\rangle$. Note that

$$\lambda_j\left(\mathrm{Tr}_{W_1\dots W_m}[\tau^{\otimes m}]\right) = 2^{-m},$$

for $j = 1, \dots, 2^m$. Since $2^{-m} \leq 2^{-n(H(p)+\varepsilon)} \leq \dfrac{2^{-n(H(p)+\varepsilon)}}{\| |w_{n,\varepsilon}\rangle \|^2}$, as $\| |w_{n,\varepsilon}\rangle \| \leq 1$, we get the right majorization relation:

$$\sum_{j=1}^{k} \lambda_j\left(\mathrm{Tr}_{W_1\dots W_m}[\tau^{\otimes m}]\right) \leq \underbrace{\sum_{j=1}^{k}}_{\text{this sum hits 1 when } k = |T_{n,\varepsilon}|} \lambda_j\left(\mathrm{Tr}_{Y_1\dots Y_m}[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|]\right),$$

for every $k \in \{1, \dots, 2^m\}$. By Nielsen's theorem (Cor. 6.34), there exists an LOCC channel Φ_n s.t.

$$\Phi_n(\tau^{\otimes m}) = |w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|.$$

The fidelity between this and the actual target state is

$$F(|u\rangle\langle u|^{\otimes n}, |w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|)^2 = \sum_{a_1\dots a_n \in T_{n,\varepsilon}} p(a_1)\dots p(a_n),$$

which approaches 1 as $n \to \infty$. Hence $E_c(\rho) \leq \alpha$. Since this works for any $\alpha > H(p)$, we get $E_c(\rho) \leq H(p)$.

It remains to show that $E_D(\rho) \geq H(p)$. The proof is similar. Let $\alpha < H(p)$ and $\varepsilon \in (0,1)$ be small enough so that $\alpha < H(p) - 2\varepsilon$. Let $n \geq -\log(1-\varepsilon)/\varepsilon$ and $m = \lfloor \alpha n \rfloor$. Then

$$m \leq n(H(p)-\varepsilon) + \log(1-\varepsilon) \quad \text{and} \quad \frac{2^{-n(H(p)-\varepsilon)}}{1-\varepsilon} \leq 2^{-m}.$$

Since $\| |v_{n,\varepsilon}\rangle \| \to 1$ as $n \to \infty$, $\dfrac{2^{-n(H(p)-\varepsilon)}}{\| |v_{n,\varepsilon}\rangle \|^2} \leq 2^{-m}$ for all sufficiently large n.

Hence,

$$\lambda_j\left(\mathrm{Tr}_{Y_1\ldots Y_n}\left[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right]\right)$$
$$< \frac{2^{-n(H(p)-\varepsilon)}}{\||w_{n,\varepsilon}\rangle\|^2} \leq 2^{-m} = \lambda_j\left(\mathrm{Tr}_{W_1\ldots W_m}\left[\tau^{\otimes m}\right]\right)$$

for all $j=1,\ldots,2^m$ (note that for distillation we are in the regime $|T_{n,\varepsilon}| \geq 2^m$). This implies the majorization relation

$$\sum_{j=1}^{k} \lambda_j\left(\mathrm{Tr}_{Y_1\ldots Y_n}\left[|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right]\right) \leq \sum_{j=1}^{k} \lambda_j\left(\mathrm{Tr}_{W_1\ldots W_m}\left[\tau^{\otimes m}\right]\right)$$

this sum hits 1 at $k=2^m$

for all k. By Nielsen's theorem, there exists an LOCC channel

$$\Phi_n \in \mathrm{LOCC}\left(\mathcal{X}^{\otimes n}, \mathcal{Z}^{\otimes m} : \mathcal{Y}^{\otimes n}, \mathcal{W}^{\otimes m}\right)$$

such that

$$\Phi_n\left(|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right) = \tau^{\otimes m}.$$

By monotonicity of fidelity,

$$F\left(\Phi_n(|u\rangle\langle u|), \tau^{\otimes m}\right)^2 = F\left(\Phi_n(|u\rangle\langle u|), \Phi_n(|w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|)\right)^2$$

$$\geq F\left(|u\rangle\langle u|, |w_{n,\varepsilon}\rangle\langle w_{n,\varepsilon}|\right)^2$$

$$= \sum_{a_1\ldots a_n \in T_{n,\varepsilon}} p(a_1)\ldots p(a_n) \longrightarrow 1 \text{ as } n \to \infty.$$

Hence, $E_D(\rho) \geq H(p)$, which concludes the proof. $\square$